NON-OBTUSE TRIANGULATION OF A POLYGON*

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Abstract. We show how to triangulate a polygon without using any obtuse triangles. Such triangulations can be used to discretize partial differential equations in a way that guarantees the resulting matrix is Stieltjes, a desirable property both for computation and for theoretical analysis.

A simple divide-and-conquer approach would fail because adjacent subproblems cannot be solved independently, but this can be overcome by careful subdivision. Overlay a square grid on the polygon, preferably with the polygon vertices at grid points. Choose boundary cells so they can be triangulated without propagating irregular points to adjacent cells. The remaining interior is rectangular and easily triangulated. Small angles can also be avoided in these constructions.

1. Introduction. Can a polygon be triangulated without using any obtuse angles? This problem has been known for some time and solved manually in particular cases. For example, in an early paper [7] on discretizations of partial differential equations MacNeal says in an aside,

> "The network should be planar and none of the interior angles of the triangles should be obtuse. It may be necessary to insert a few additional points in order to fulfill the last condition."

A literature search (by looking for the keyword "triangulation" in online indices) and asking experts did not uncover any algorithms guaranteed to produce a non-obtuse triangulation. Indeed, there was some doubt whether such triangulations were even possible in general.

It turns out that elementary constructions suffice. Exactly how complicated the algorithm is depends on how many extra conditions are imposed dealing with small angles and interfaces, but no tools beyond high school geometry and trigonometry are needed.

To see why this problem is important, imagine solving $\Delta u = f$ on a domain P. The finite element method chooses some approximating space A, say piecewise linear functions on a triangulation of P, and finds the element $u \in A$ such that for all $v \in A$, $\int_P v \Delta u = \int_P vf$. This leads to a matrix with elements of the form $\int_P \nabla\phi_i \nabla\phi_j$. It is known [11,p.78] that if there are no obtuse angles in the triangulation then for $i \neq j$ these integrals are negative and consequently the matrix is Stieltjes. Recall that a Stieltjes matrix is a symmetric positive definite matrix whose off-diagonal entries are all nonpositive. This property is important in the analysis of iterative methods for solving the linear system; for example, it implies that block Gauss-Seidel has a better asymptotic rate of convergence than point Gauss-Seidel. Other discretizations such as the "box method" [12,p.191] also benefit from non-obtuse triangulations.

If all the vertices of the triangles are preassigned, as in scattered data interpolation, then excellent triangulation algorithms are available [2, 4, 5, 6, 10]. Several of these algorithms compute the Voronoi tesselation, which partitions the plane into polygonal regions by labelling an arbitrary point in the plane according to the closest vertex. Connecting vertices in adjacent regions gives the Delaunay triangulation. Actually the "no obtuse angles" condition is only a sufficient for the matrix to be

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Stieltjes. The necessary condition is that when two triangles adjoin in a side, the two angles opposite the side sum to at most 180°; the Delaunay triangulation achieves this weaker condition. But obtuse angles may occur and then the boundary of the Voronoi region about a vertex extends outside the star of triangles attached to that vertex. This causes problems for the box method with linear elements on triangles. So we still seek a nonobtuse triangulaton.

In this paper we give two solutions of increasing complexity. The first assumes that the vertices of P lie on a square grid. The second removes this hypothesis and moreover avoids any angles smaller than 13°.

2. The Problem. Given a simple polygon P with vertices $\{v_1, v_2, \cdots, v_n\}$, add points $\{v_{n+1}, \cdots, v_m\}$ inside P or on its boundary and connect the points with straight line segments to triangulate P. No resulting triangle should contain an obtuse angle. By a *triangulation* we mean a set of triangular regions such that the union is P; any two distinct triangles intersect along one full side, in a single point, or not at all; and the set of vertices of all the triangles is exactly $\{v_i\}_{1 \le i \le m}$. Figure 2.1 illustrates a legal triangulation, while figure 2.2 shows two triangulations that are illegal because of a point on a side and an obtuse angle, respectively.

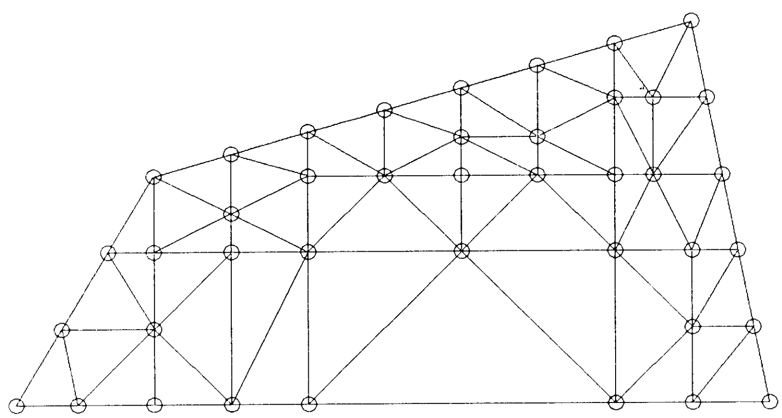


FIG. 2.1. *A legal triangulation of a quadrilateral.*

3. Solution 1. A natural approach is to partition the polygon. The trick is to divide in such a way that each subproblem can be solved independently and to prove for each subproblem that all cases have been considered.

LEMMA 1. *If the vertices of the polygon lie on a square grid and if none of the interior angles of the polygon are acute, then a non-obtuse triangulation exists.*

Proof. Refine the grid until the cell diagonals are smaller than the minimum distance between non-intersecting boundary segments. Introduce points v_i at the grid intersection points in the interior of P and everywhere that a grid line intersects the boundary of P. Each square cell in the interior of P is triangulated by adding a diagonal, leaving only cells intersecting the boundary to be dealt with. We will introduce some further points inside such cells and on the boundary of P, but not on the sides of the cells. Thus each cell is independently triangulated without propagating points

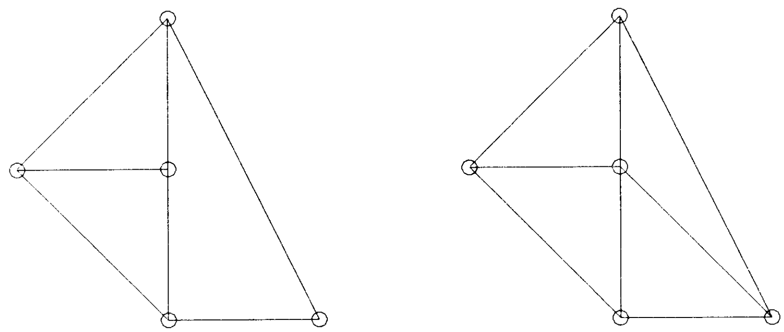


FIG. 2.2. *Illegal triangulations.*

from one cell to a neighbor.

If more than one boundary segment passes through a cell, the segments must be adjacent in order not to violate the refinement criterion. But they cannot have an acute interior angle. Therefore, they have an acute exterior angle, and the regions of P bounded by these segments and the cell boundaries are disjoint. The triangulation strategy below can be applied independently to the two regions, each of which has only one boundary segment within the cell.

So consider a cell with only one boundary segment extending into its interior. By reflection, rotation, and scaling we may assume without loss of generality that the upper right corner of the cell lies inside P and the sides of the cell have length 1. The boundary of P will be indicated by a dashed line. Figure 3.1 illustrates that no further points are needed if the boundary hits the top and right sides of the cell.

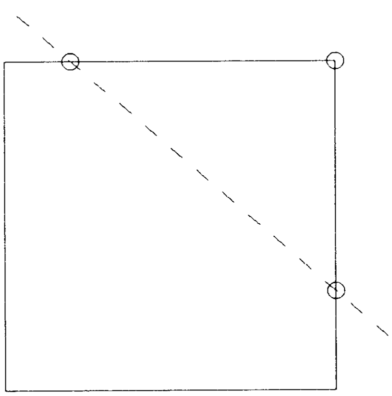


FIG. 3.1. *An easy case.*

Figure 3.2 shows how to deal with the next case, in which the top and bottom sides are hit. Without loss of generality, we may assume that $\beta \leq 90°$. The angle α is acute because its vertex lies outside the semicircle drawn on the opposite side of the triangle.

Now we come to the key step, when the left and bottom sides are hit. Split the analysis into subcases based on the location of the point q determined by vertical and horizontal lines extending from intersections of the boundary and the cell sides.

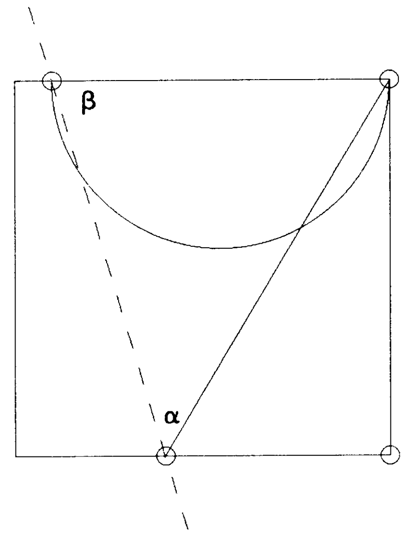


FIG. 3.2. *The semicircle rule.*

Without loss of generality, we may assume that q lies on or below the diagonal from the upper right corner to the lower left corner of the cell. See figure 3.3.

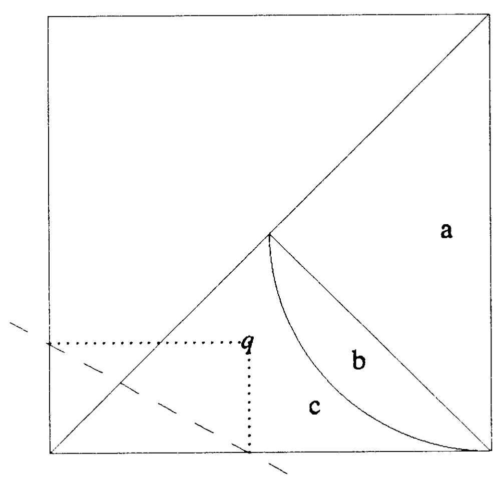


FIG. 3.3. *Crucial subcases.*

Draw a semicircle based on the right side of the cell and another diagonal to form three regions a, b, and c in which q can lie.

In subcase a, illustrated in figure 3.4, we have $x \le y$ and hence $\alpha \le \beta$. But $\beta + \gamma + (90° - \alpha) = 180°$ so $\gamma = 90° + \alpha - \beta \le 90°$. To show $\delta \le 90°$, base a semicircle on the opposite side. Since $x \le 1/2$, the radius of the semicircle is at most $\sqrt{5}/4$. But the center of the semicircle is at most 1/4 away from the right side of the cell and $1/4 + \sqrt{5}/4 < 1$. Therefore, the vertex at δ lies outside the semicircle, implying δ is acute.

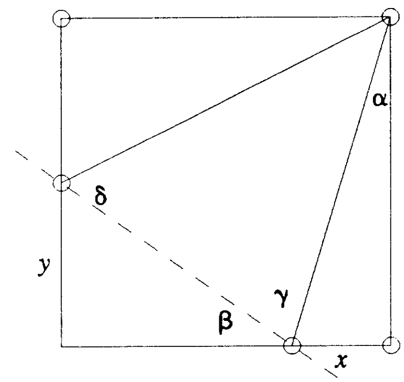


FIG. 3.4. *subcase* a.

Subcase b is illustrated in figure 3.5. In contrast to the previous figure, $\gamma \geq 90°$. Introduce a point in the interior of the cell at the intersection of lines drawn to form five right triangles. (This construction is the only one that introduces any points on the boundary other than at grid lines.) The angle δ is acute because $y < 1/2$.

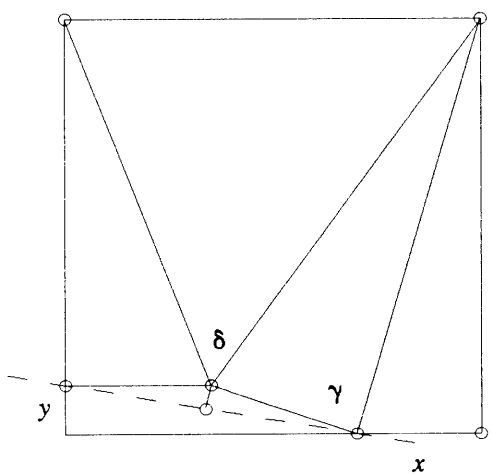


FIG. 3.5. *subcase* b.

The same argument shows that the two non-right triangles in figure 3.6 are acute, so that introducing the point q solves subcase c. □

THEOREM 1. *If the vertices of the polygon lie on a square grid, then there exists a non-obtuse triangulation.*

Proof. For each vertex v_i with an acute interior angle, cut off a corner by adding new vertices v_i' and v_i'' so that the triangle $\{v_i, v_i', v_i''\}$ being removed is acute and does not contain any other vertices and so that the newly generated interior angles are obtuse. See figure 3.7.
The only question is how to pick v_i' and v_i''. If they did not need to be on grid points, we could pick two points that form an isosceles triangle with v_i. We could thus cut off an acute triangle and leave obtuse interior angles, and by making the

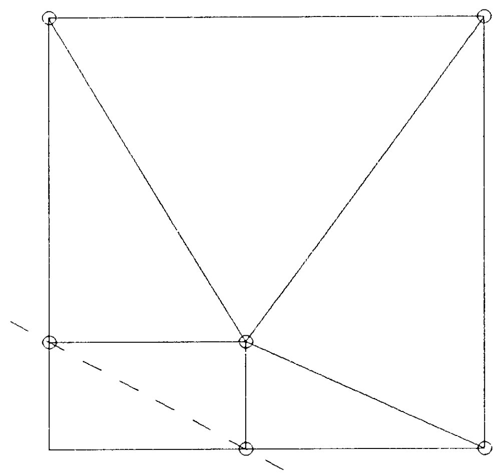


FIG. 3.6. *subcase* c.

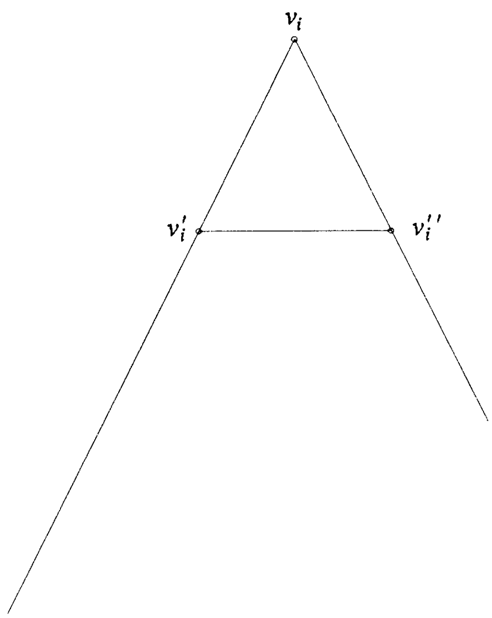


FIG. 3.7. *Cutting off an isosceles triangle.*

triangle small enough we could also guarantee that it does not contain other vertices. To obtain v_i' and v_i'' on grid points, we will approximate this solution as follows. By hypothesis, the grid is fine enough that there are grid points on the line between v_i and each adjacent vertex such that the triangle formed by v_i and these two grid points does not include any other vertices. If this triangle is satisfactory, let v_i' and v_i'' be its two new vertices. Otherwise, let v_i' be the new vertex that is closer to v_i. (See figure 3.8.)
The angle at v_i' within the current triangle must be obtuse, or the triangle would be

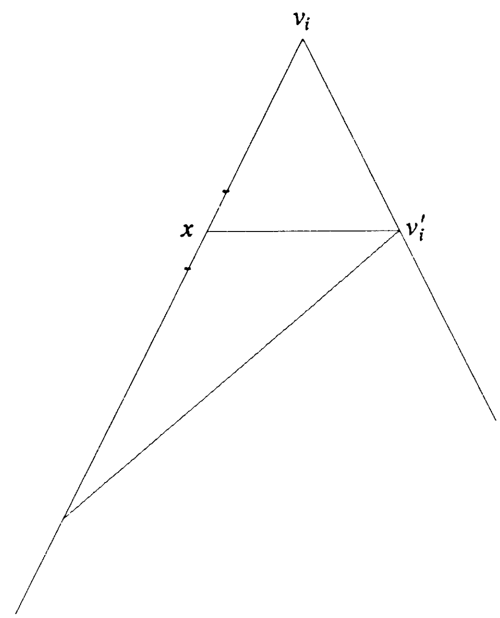


FIG. 3.8. *Approximating an isosceles triangle on grid points.*

satisfactory. Let x be the point on the other side that is the same distance from v_i as v_i'. For some ϵ, picking any point within ϵ of x and within the current triangle will give a new acute triangle that does not contain any other vertices. Refining the grid sufficiently guarantees that a grid point will lie within this interval. Letting it be v_i'' gives a satisfactory triangle with vertices that are grid points.

Apply the lemma above to the new polygon, which does not have acute interior angles. This may introduce points on the artificial boundary segment. Figure 3.9 shows how to introduce orthogonal lines emanating from such points that partition the removed triangle into right triangles and rectangles, which of course can also be divided into right triangles. □

The references to a "sufficiently fine grid" might suggest that many triangles are produced. But it is possible to refine the grid locally, using the trick illustrated in figure 3.10 to triangulate near the boundary between the coarse and fine cells. One way to do this is to use quadtrees [9, 8].

4. Solution 2. In numerical applications [3], very small angles may lead to ill-conditioned matrices. (It used to be thought that small angles also prevented convergence, but [1] showed that it suffices to avoid large angles.) We have devised another triangulation algorithm guarantees that no angle in the triangulation is less than $\tan^{-1}(1/3) \approx 18°$ or the minimum internal angle in the boundary, whichever is smaller. Moreover, if all interior angles of the polygon are at least 54°, then triangulations of two abutting polygons are consistent. However, this solution still requires that the polygon vertices lie on a square grid. In the interest of saving space, we omit this intermediate solution and move on to a more elaborate analysis which frees the vertices to lie in arbitrary position. Define a line to be *nearly horizontal* if its slope is at least -1 and at most 1, and *nearly vertical* otherwise. A vertical grid line

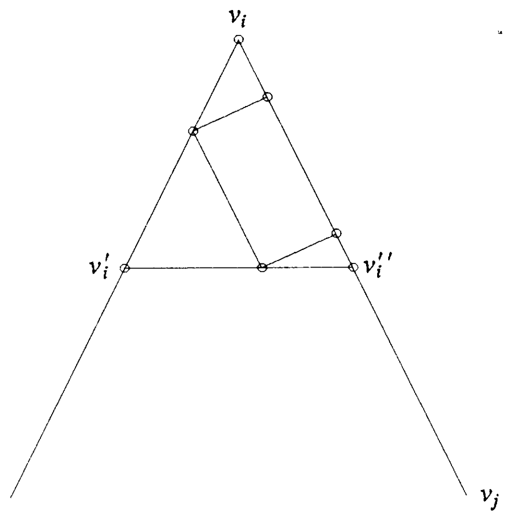


FIG. 3.9. *Triangulating the cut off corner.*

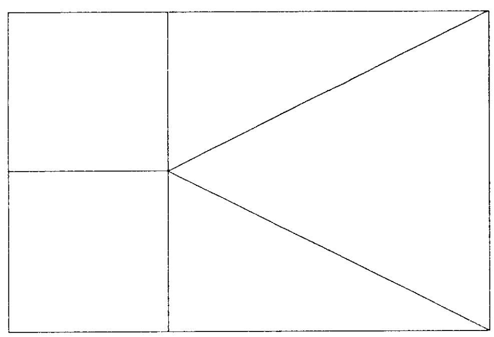


FIG. 3.10. *Triangulation at a jump in refinement.*

is *nearly perpendicular* to a nearly horizontal line; a horizontal grid line is *nearly perpendicular* to a nearly vertical line.

Define a triangulation of a polygon to be *good* if it uses no obtuse angles and no angles less than $\tan^{-1}(1/4)$ or the smallest angle in the polygon, whichever is smaller. A good triangulation *well-triangulates* the region.

We begin by showing how to well-triangulate a polygon of a particularly simple form.

LEMMA 4.1. *Let R be a simple polygon overlaid with a unit grid. Suppose each edge of R is of one of the following forms:*
(1) *cell diagonal*
(2) *cell side*
(3) *gridline segment forming the sides of two adjacent cells.*
Then R can be well-triangulated without adding any extra points on its boundary.

Proof. The following procedure triangulates R. Add a vertex at every interior gridpoint. Add edges on cell sides to connect previously unconnected vertices one unit apart. Add edges on cell diagonals where possible to do so without crossing or lying on top of another cell diagonal. Add a diagonal to each rectangle still lacking a diagonal.

The proof that this procedure well-triangulates R is by showing that every original edge borders a good triangle and every new edge borders a good triangle on each side, and is straightforward. □

Let R be a simple polygon. The triangulation strategy will be to well-triangulate in the vicinity of each vertex of R, and then to well-triangulate the remaining region R'. Figure 4.1 shows how R might be divided into regions around each vertex and R'. In triangulating the region around a vertex, points are introduced on the common boundary with R'. No new points can be added on these common boundaries while triangulating R'. (Points added on a common boundary would invalidate the triangulation already done in the adjoining region around the vertex.) The key to the proof is to restrict the edges occurring in the boundaries of the regions around the vertices so that the remaining region is easily well-triangulated.

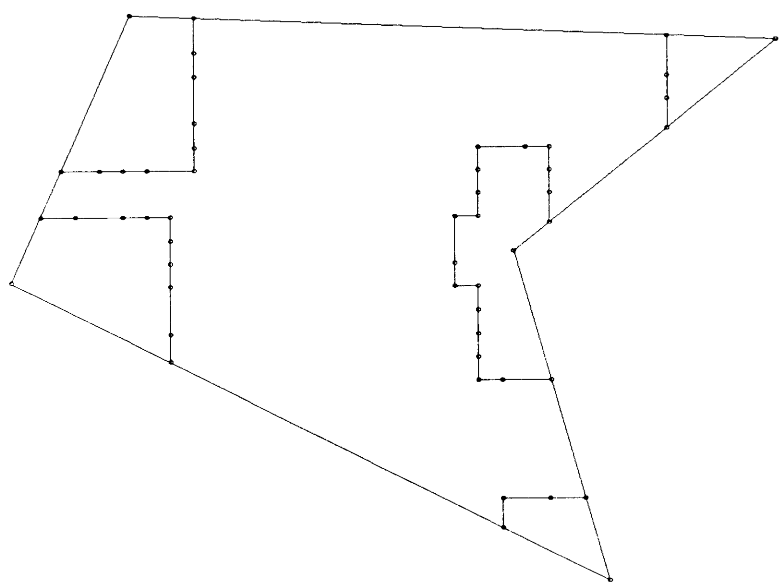


FIG. 4.1. *Vertex cells.*

If e is a side of R, and g is a gridline nearly perpendicular to R, intersecting e at A, there is a unique gridpoint on g whose distance from A is at least one and less than two and which lies on the interior side of e. This gridpoint is called the *neighbor gridpoint* of A. It is also called a neighbor gridpoint of e. Figure 4.2 illustrates a point on a side of R and its neighbor gridpoint.

Let e_1 be a side of R, and let A be an intersection point of e_1 with a nearly perpendicular gridline. Let e_2 be an adjacent side of R, and let B be an intersection point of e_2 with a nearly perpendicular gridline. A sequence of edges from A to B is a *satisfactory path* from A to B if it lies in the interior of R (except for A and B), the edges are pairwise non-intersecting (except for the point between two successive edges), and each edge is of one of the following forms:

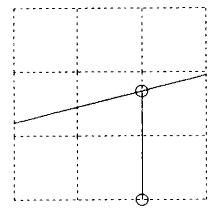


FIG. 4.2. *A neighbor gridpoint.*

(1) a cell side or diagonal
(2) a gridline segment forming the sides of two adjacent cells
(3) AG, where G is a neighbor gridpoint of A, or BG, where G is a neighbor gridpoint of B.

Satisfactory paths are illustrated in Figure 4.1. The region bounded by the path, e_1, and e_2 has a *satisfactory boundary*. An edge is satisfactory if it is of a form satisfying one of (1)-(3).

LEMMA 4.2. *Let R be a simple polygon overlaid with a unit grid, such that no vertex of R lies within four units of a non-adjacent edge. For each edge $e=(A,B)$, let e_A and e_B be points lying on gridlines nearly perpendicular to e, with e_A closer than e_B to A. Designate each gridpoint lying less than one unit from e along a nearly perpendicular gridline between those of A and B as a* forbidden point. *For each vertex V with incident edges e and f, let P_V be a satisfactory path from e_V to f_V, and let R_V be the region bounded by P_V, e_V, and e_f. If these paths are pairwise non-intersecting and no path touches a forbidden point, then the region*

$$R - \bigcup_V R_V$$

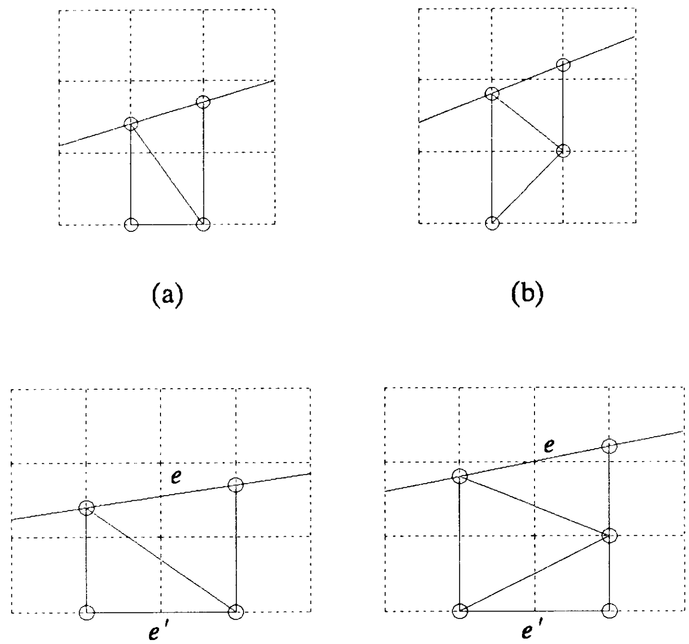
can be well-triangulated, with no new vertices introduced on any path P_V.

Proof. For each edge $e=(A,B)$, introduce a vertex on each nearly perpendicular gridline lying between e_A and e_B. On each such gridline, also add a vertex at its neighbor gridpoint of e, unless that gridpoint is occupied by some path P_V.

Wherever two successive gridlines have vertices at neighbor points of e, connect them with an edge. This forms quadrilaterals with two parallel edges. (See Figure 4.3(a) and (b).) The side connecting the two grid points is either a horizontal or vertical edge of length one, or a cell diagonal. In either case, a diagonal of the quadrilateral well-triangulates it as desired. Since none of the satisfactory paths occupy either neigbor point or any forbidden points, and no vertex not adjacent to e lies within four units of e, the new edges do not conflict with any previous edges.

Now, suppose a neighbor gridpoint of e is occupied by a satisfactory path. If it does not lie on an edge of length 2 nearly parallel to e, it can form part of a quadrilateral formed as above with an adjacent neighbor gridpoint of e. If it lies on an edge e' of length 2 nearly parallel to e, then e' can form part of a well-triangulated quadrilateral as shown in Figure 4.3(c) or d, according to whether both endpoints of e' are neighbor gridpoints of e. Again, the new edges cannot conflict with any previous edges.

After triangulating along each original edge of R as above, the region remaining to be triangulated can be well-triangulated by Lemma 4.1. □



(a) (b)

(c) (d)

FIG. 4.3. *Quadrilaterals along a nearly horizontal edge.*

Thus, we need only show that we can triangulate around each vertex of R so as to satisfy Lemma 4.2. By making the grid sufficiently fine, we will ensure that no two satisfactory paths intersect.

Vertices of R need not be on grid points. For each vertex A, define the grid cell containing A to be any grid cell for which A is in the interior or on the boundary. Number the octants at each vertex counterclockwise as shown in Figure 4.4. Similarly, number the quadrants counterclockwise.

The next two lemmas will be helpful in triangulating around each vertex so as to obtain a satisfactory path along the boundary.

LEMMA 4.3. *Let e be an edge of R that is at an angle of ρ to the vertical, $\rho<45°$, as shown in Fig. 4.5 (a)-(c), with the interior of R to the left of e. Let A be at the intersection of a horizontal gridline with e. Let G be the gridpoint that lies at least 2 and less than 3 units to the left of A. Then one of the regions shown in Figure 4.5 can be well-triangulated as shown, depending on the value of $A_x-G_x-2\tan\rho$. The boundary from G to B is satisfactory for each region.*

Proof. Let $d=A_x-G_x-2\tan\rho$. The bounds on ρ imply $d\in[0,3)$.

If $d\in[1,2)$, then triangulate as shown in Figure 4.5(a). Note that the boundary from G to B is satisfactory. The right triangle is obviously good. Since $B_x=G_x+d$, angle AGB is non-obtuse; since $G_y-B_y=2$, angle $AGB\geq 45°$. Angle $GAB=\pi/2-\rho\in[45°,90°]$. Angle GBA is non-obtuse because $|GA|$ is less than twice the height of the triangle, and at least $\tan^{-1}(1/2)$ because dropping a perpendicular from B to GA gives at least one subangle with a tangent of at least 1/2. Therefore,

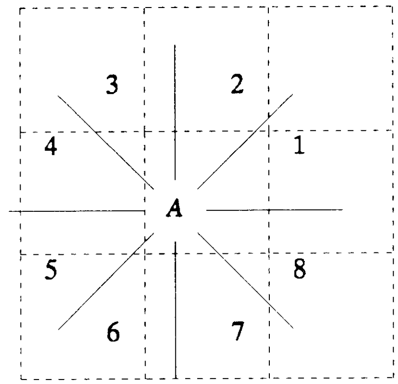


FIG. 4.4. *Octants.*

triangle *GAB* is good.

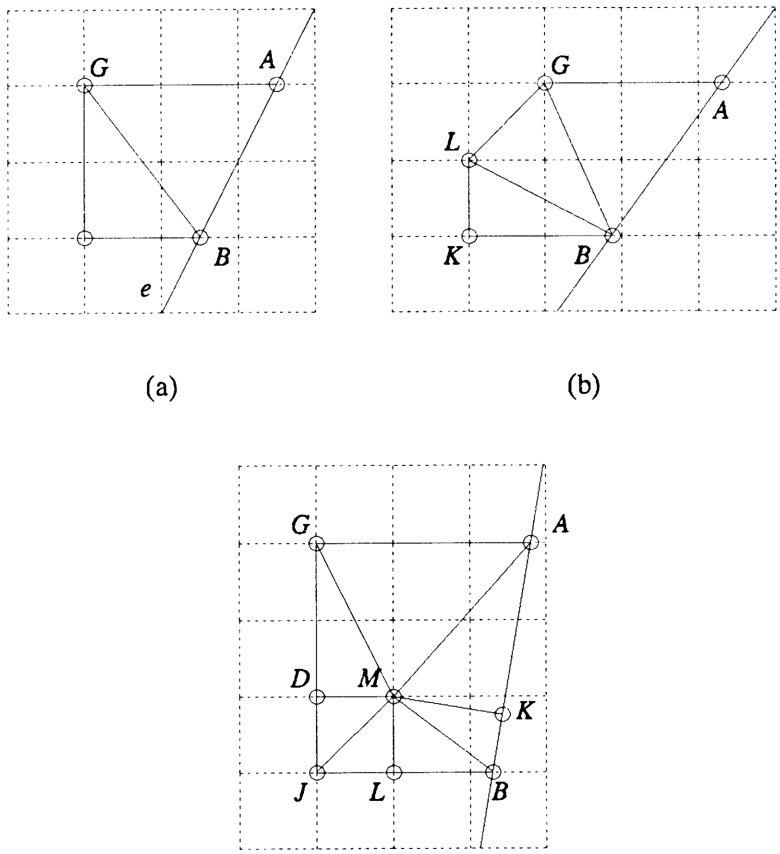
If $d \in [0,1)$, then triangulate as shown in Figure 4.4(b). Clearly, triangle *LKB* is good. Triangle *GAB* is good by the same reasons as in Figure 4.4(a). Let θ be the angle of *GB* with the vertical. Since $d<1$, $\theta \le \tan^{-1}(1/2)$. Obviously, angle $LGB = 45° + \theta \in [45°, 90°)$. Angle *GBL* is obviously acute; since angle $LBK \le 45°$ and $\theta \le \tan^{-1}(1/2)$, angle $GBL \ge 45° - \tan^{-1}(1/2) > 18°$. Since angle $GLB = 135° -$ angle *KLB* and $KLB \in [45°, \tan^{-1}(2)]$, angle $GLB \in [45°, 90°]$. Therefore, triangle *LGB* is good.

Finally, suppose $d \in [2,3)$. Then $\rho < \tan^{-1}(1/2)$. Let $d' = d - \tan\rho$. Then $d' \in [1.5,3)$. If $d' \in [1.5,2)$, triangulate as shown in Figure 4.4(c), without the vertex *L* and the edge *ML*. If $d' \in [2,3)$, triangulate as shown in Figure 4.4(c), with the vertex *L* and the edge *ML*. The edge *MK* is drawn perpendicular to *AB*. Note that the boundary *GJB* is satisfactory in either case. In the former case, triangle *JMB* is good because $|JB| = d' < 2$, and in the latter case, right triangles *JML* and *MLB* are obviously good. Triangle *GAM* is good for the same reasons as in triangle *GAB* in Figure 4.4(a). Triangles *GMD* and *JDM* are obviously good. Since $\rho < \tan^{-1}(1/2)$, $|AK| \le \sqrt{5}$. Also, $|MK| \ge 1$ because $d \ge 2$. Therefore, angle $MAK \ge \tan^{-1}(1/\sqrt{5}) > \tan^{-1}(1/4)$. Angle $AMK \ge 45°$ because it is at least as large as angle *GAM*, which is at least 45°. This implies that triangle *AMK* is good. Angle $KBM = \rho + 90° -$ angle *MBJ*. If $d' \in [1.5,2)$, angle $MBJ \in [45°, \tan^{-1}(2)]$, and angle $KBM \in [\tan^{-1}(1/2), 45° + \tan^{-1}(1/2)] \subset [\tan^{-1}(1/2), 72°]$. If $d' \in [2,3)$, then $\rho \le \tan^{-1}(1/3)$ and angle $MBJ \in [\tan^{-1}(1/2), 45°]$, implying that angle $KBM \in [45°, \tan^{-1}(1/3) + \tan^{-1}(2) \subset [45°, 82°]$. In either case, right triangle *MKB* is good. □

LEMMA 4.4. *Let e be an edge of R that is at an angle of* ρ *to the vertical,* $\rho \le 45°$, *as shown in Figure 4.6(a), with the interior of R to the right of e. Let A be at the intersection of a horizontal gridline with e. Let G be a gridpoint that lies at least 2 and less than 4 units to the right of A. Then a quadrilateral AGHJ (as shown in Figure 4.6(a)) can be triangulated, where GH has length either 2 or 4, with vertices added on GH and HJ to make these sides satisfactory.*

Proof. Let $x_1 = |AG|$, and let $x_2 = x_1 + 2\tan\rho$. Then $x_2 \in [2,6)$.

If $x_2 \in [2,4)$, then triangulate as in Figure 4.6(b), where *K* is the neighbor vertex



(c)

FIG. 4.5. *Lemma 4.3.*

of J. Consequently, $|KH|$ is either 1 or 2. Obviously, the triangles are all good.

If $x_1 \in [2,3)$ and $x_2 \in [4,5)$, or if $x_1 \in [3,4)$ and $x_2 \in [5,6)$, triangulate as in Figure 4.6(c), where K is the neighbor vertex of J, KM is perpendicular to AJ, and $|LH|=2$. Consequently, $|KL|$ is either 1 or 2. The relative sizes of x_1 and x_2 imply $\tan\rho \geq 1/2$. Obviously, triangle LGH is good. The relative sizes of the height and bases of triangles AKL and AGL force those triangles to be good. The angle $MJK = 90° - \rho$. Therefore, right triangle JMK is good. Since $|JK| \geq 1$ and angle $MJK \geq 45°$, $|MK| \geq 1/\sqrt{2}$. Note that the length of AK is at most $\sqrt{5}$. Since $|MK| \geq 1/\sqrt{2}$ and $|AK| \leq \sqrt{5}$, angle $MAK \geq \tan^{-1}(1/\sqrt{10}) > \tan^{-1}(1/4)$. Also, angle $MAK \leq \rho \leq 45°$. Consequently, right triangle AMK is good.

The only remaining case is when $x_1 \in [3,4)$ and $x_2 \in [4,5)$. In this case, triangulate as in Figure 4.6(d), where $|KN| \in [2,3)$ and region $KNMJ$ is to be filled in according to Figure 4.6(b) or (c). All the triangles are obviously good, and sides JH and GH are made satisfactory. □

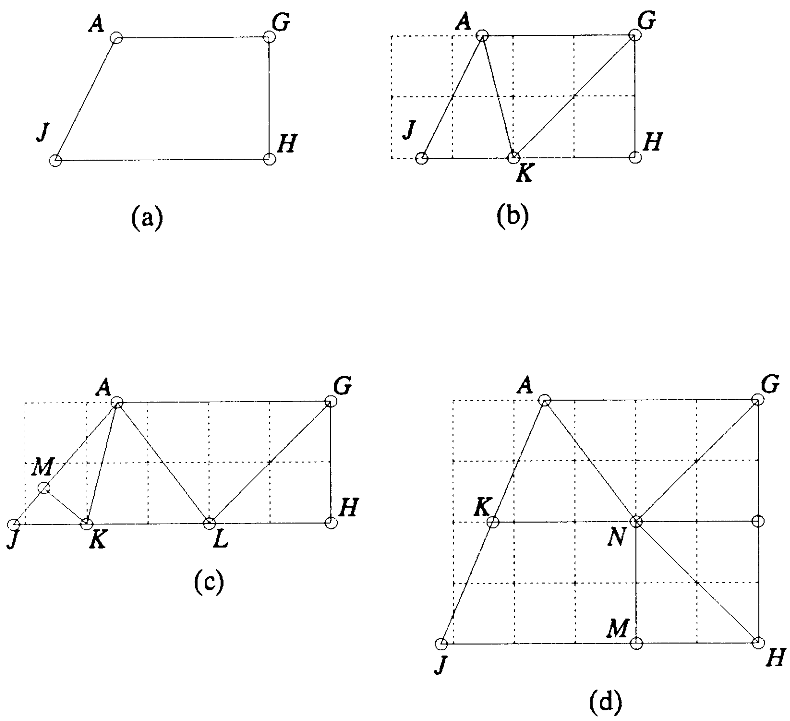


FIG. 4.6. *Lemma 4.4.*

LEMMA 4.5. *Let A be the vertex of an acute angle of R. Then there is a region around A that can be well-triangulated with a satisfactory path along its boundary.*

Proof. Let α be the angle at A. Without loss of generality, we may assume that one edge L_1 at A lies in octant 6. Since α is acute, the clockwise edge L_2 at A lies in one of octants 4-8. We consider cases according to which octant L_2 lies in. It is sufficient to consider L_2 in octants 5-8, since the case for L_2 in octant 4 can be handled by rotation from the case for L_2 in octant 8.

Let ρ, $0 \le \rho \le 45°$, be the angle of L_1 with the vertical.

Case (1). Suppose L_2 is in octant 5. Let σ, $0 \le \sigma \le 45°$, be the angle of L_2 with the horizontal.

First, we show that for any positive constant c, there is a neighbor gridpoint G of L_2 that also lies to the left of L_1 by a horizontal distance in the range $[c, c+1+\tan\rho)$. Form a parallelogram P by two lines parallel to L_1 at horizontal distances of c and $c+1+\tan\rho$ from L_1 and by two lines parallel to L_2 at vertical distances of 1 and 2 from L_2. Label vertices Q and R as shown. Then $R_x - Q_x \ge 1$ by Figure 4.7(a). Therefore, a vertical gridline lies between Q and R or passes through R. Since length 1 of this gridline lies in P, there is a gridpoint inside P or on its right or upper boundary. This gridpoint satisfies the requirement.

By symmetry, for any positive constant c, there is a neighbor gridpoint of L_1 that lies below L_2 by a vertical distance in the range $[c, c+1+\tan\sigma)$.

There are two subcases according to the values of ρ and σ.

First, suppose $\rho \le \tan^{-1}(1/4)$ and $\sigma \le \tan^{-1}(1/4)$. We begin by showing the existence of a gridpoint H that is a neighbor gridpoint of both L_1 and L_2. Let G be the neighbor gridpoint of L_2 found by the above argument with $c=1$, so that G lies to the left of L_1 by a distance in $[1, 2+\tan\rho)$. If the latter distance is less than 2, we are done. Otherwise, move right one unit to a gridpoint G_1. G_1 lies to the left of L_1 by a distance in $[1, 1+\tan\rho)$ and below L_2 by a distance in $[1+\tan\sigma, 2+\tan\sigma)$. If the latter distance is less than 2, we are done. Otherwise, move up one unit to a gridpoint G_2. G_2 lies below L_1 by a distance in $[1, 1+\tan\sigma)$ and to the left of L_2 by a distance in $[1+\tan\rho, 1+2\tan\rho)$. Since $\tan\sigma, \tan\rho \le 1/4$, we are done.

Triangulate as shown in Figure 4.7(b). The path RHQ is satisfactory because of the bounds on H. Obviously right triangle RHQ is good. Now, angle $AQR = 90° + \rho -$ angle RQH. Since angle $RQH \in [\tan^{-1}(1/2), \tan^{-1}(2)]$ and $\rho \le \tan^{-1}(1/4)$, angle $AQR \in [\tan^{-1}(1/2), 90°]$. Similarly, angle $ARQ = \pi/2 + \sigma -$ angle HRQ, and $\tan^{-1}(1/2) \le$ angle $ARQ \le 90°$. Therefore, triangle RAQ is also good.

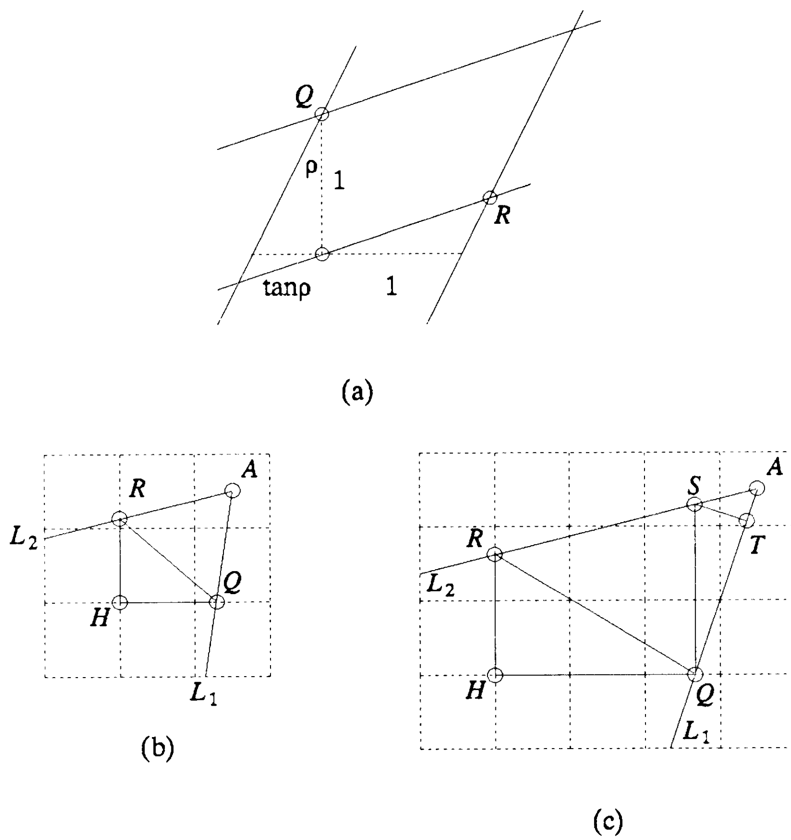


FIG. 4.7. *Lemma 4.5, case (1).*

Now, suppose $\rho > \tan^{-1}(1/4)$. From the earlier argument with $c=2$, there is a neighbor gridpoint H of L_2 that lies to the left of L_1 by a distance in $[2, 3+\tan\rho)$. Triangulate as shown in Figure 4.7(c), where R and Q lie on the gridlines through H, QS is vertical, and ST is perpendicular to L_1.

Since $\alpha \leq \pi/2 - \rho < \tan^{-1}(2)$, triangle SAT is good. Since angle $SQT = \rho$, right triangle SQT is good. Triangle RHQ is good because $|RH| \in [1,2)$ and $|HQ| \in [2,4)$. Now, angle $ARQ = \pi/2 + \sigma -$ angle HRQ. Since $\sigma \leq 45°$ and angle $HRQ > 45°$, angle ARQ is acute. Also, angle $ARQ \geq$ angle $HQR > \tan^{-1}(1/4)$. Since RH and SQ are vertical, angle $SQR =$ angle HRQ. Angle $RSQ = \pi/2 - \sigma$. Therefore, triangle RSQ is good.

Unfortunately, H is not the neighbor vertex of Q. Therefore, RHQ is not a satisfactory path. If $|HQ| \in [2,3)$, apply Lemma 4.3. Otherwise, form a quadrilateral below HQ by going down 2 from H and then right to L_1. This quadrilateral is well-triangulated by a diagonal. The lower boundary of the quadrilateral has length in $[3 - 2\tan\rho, 3 - \tan\rho)$. If this length is in $[1,2)$, we are done. Otherwise, we apply Lemma 4.3 below HQ. The final triangulated region has a satisfactory path along its border, and does not pass through any forbidden points along L_1 or L_2.

Case (2). Suppose L_2 is in octant 6. Without loss of generality, we assume L_2 is clockwise from L_1, as shown in Figure 4.8. Find the top horizontal gridline with exactly four gridpoints between or on L_1 and L_2. Such a gridline exists since $\tan\rho \leq 45°$. Let M and R be its intersection points with L_2 and L_1, respectively, as shown in Figure 4.8. Place a vertex P at the neighbor point of M. Then $|PR| \in [2,3)$. Extend a perpendicular from P, intersecting L_2 at U. Then angle $MUP \geq \alpha$ and angle $UMP \geq 45°$, implying triangle MPU is good. Since $|PR| \geq |PM|$, angle $PUR \geq$ angle $MUP \geq \alpha$. Since $|PU| \geq 1$, angle $PRU \geq \tan^{-1}(1/3)$. Hence, triangle PRU is good. Apply Lemma 4.3 below PR to obtain a satisfactory boundary.

It remains to well-triangulate RUA. Note that angle $URA \geq$ angle PUR and that angle $AUR \geq 45°$. If both angles AUR and ARU are non-obtuse, we are done. Otherwise, assume without loss of generality that angle AUR is obtuse. Draw a perpendicular from U to L_1, intersecting L_1 at T, as shown in Figure 4.8. Since angle RUA is obtuse, angle $URA \leq \pi/2 - \alpha$. Therefore, triangle RTU is good. Clearly, triangle AUT is good, and we are done.

Case (3). Suppose L_2 is in octant 7. Let σ, $0 \leq \sigma \leq 45°$, be the angle of L_2 with the vertical. Find the highest horizontal gridline such that either 4 or 5 points lie between or on L_1 and L_2. This is possible since $\rho, \sigma \leq 45°$. Let M and R be where this gridline intersects L_1 and L_2, respectively. Let P and Q be the neighbor gridpoints of M and R, respectively. Then $MPQR$ is a satisfactory path.

Let U and V be where the gridline one unit higher intersects L_1 and L_2, respectively, as shown in Figure 4.9. Since the gridline of M and R is the highest one with at least 4 points, the gridline of U and V has at most 3 points on or between L_1 and L_2. Therefore, $|UV| < 4$. The slopes of L_1 and L_2 guarantee that $|UV| \geq 1$.

We claim there is a point C on UV such that C lies above PQ, $|UC| \leq 2$, $|CV| \leq 2$, and $|UC| \in [|UV|/3, 2|UV|/3]$. If the halfway point between U and V lies above PQ, take C to be this halfway point. Since $|UV| \leq 4$, $|UC|$ and $|CV|$ have the desired sizes. Otherwise, suppose without loss of generality that the halfway point lies too far to the right to be above PQ. Take C to be the gridpoint one unit above Q. Since $|QR| \leq 2$, $|CV| \leq 2$ as well. Also, $|UC| < |UV|/2 \leq 2$. It only remains to show that $|UC| \geq |UV|/3$. Let $x_1 = |MP|$ and $x_2 = |QR|$. Note that $|PQ| = 1$, or the halfway point between U and V would have been above PQ. Since $x_1 \geq 1$, $\tan\rho \leq 1$, and $x_2 \leq 2$, we have

$$|UC| - |UV|/3 = (x_1 + 1 - \tan\rho) - \frac{x_1 + x_2 + 1 - \tan\rho - \tan\sigma}{3}$$

$$= \frac{2}{3}x_1 + \frac{2}{3} - \frac{2}{3}\tan\rho - \frac{1}{3}x_2 + \frac{1}{3}\tan\sigma$$

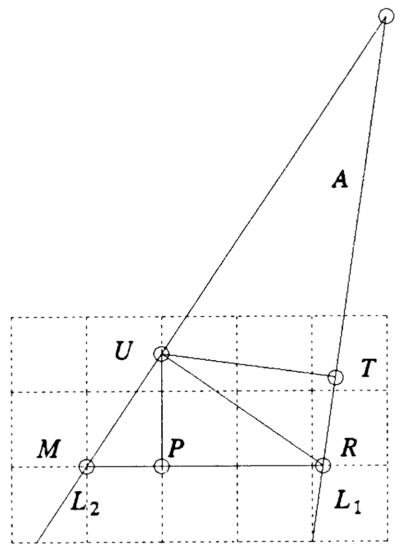


FIG. 4.8. *Lemma 4.5, case (2).*

$\geq 0.$

Therefore, C satisfies the constraints of the claim.

Triangulate the region $MRVCU$ as shown in Figure 4.9. Since C lies above PQ, angles CPQ and CQP are non-obtuse. The bounds on ρ and$C\sigma$ imply that U lies above MP and V lies above QR. Consequently, angles VQR and UPM are non-obtuse. Angles UPC, CQV, MUP, and QVR are all non-obtuse because $|UC|,|CV|\leq 2$ and their triangles have height 1. Angles UMP and VRQ are non-obtuse by assumption. We conclude that all the triangles are non-obtuse. Since $|MP|,|QR|\geq 1$, angles $MUP,QVR\geq\tan^{-1}(1/2)$. Angles $UMP,VRQ\geq 45°$ by assumption. The remaining angles except for CQV and UPC are at least $\tan^{-1}(1/2)$ because their triangles have height 1 and the relevant edges are at most 2. Angle CQV is at least $\tan^{-1}(1/4)$ because $|UV|\geq 1$ and $|CV|\geq|UV|/2\geq 1/2$. Finally, either C is the halfway point of UV, implying $|CV|=|UV|/2\geq 1/2$ and angle $UPC\geq\tan^{-1}(1/4)$, or C is the gridpoint above Q, implying $|UC|\geq 1$ and angle $UPC\geq\tan^{-1}(1/2)$. Therefore, all of these triangles are good.

It remains to triangulate above UCV. Let $t=\tan\sigma+\tan\rho$ and $z=|UV|$. Let HJ be a horizontal line connecting L_1 and L_2 at height y above UV, where

(a) if $t\geq 1$, then $y=z/3$, and

(b) if $t<1$, then $y=z/(2+t)$.

Triangulate as shown in Figure 4.9. Note that $|HJ|=z-ty\leq 2y$.

Without loss of generality, we assume that $|UC|\in[z/3,z/2]$ and $|CV|\in[z/2,2z/3]$. Angles HUC, JVC, AHJ,and AJH are between 45° and 90°. Since $y\geq z/3$ and $|UC|,|CV|\leq 2z/3$, angles UHC and CJV are non-obtuse. Since $|UC|,|CV|\geq z/3$ and $y<z/2$, angles UHC and CJV are at least $\tan^{-1}(1/3)$. In height y, JV moves horizontally by at most $y<z/2\leq|CV|$. Therefore, J is above CV and angle JCV is non-obtuse. Since $CV\leq 2z/3\leq 2y$, angle $JCV\geq\tan^{-1}(1/2)$. Angle

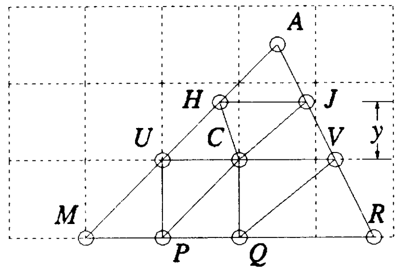


FIG. 4.9. *Triangulation for Lemma 4.5, case (3).*

$HJC = JCV$. In height y, HU moves horizontally by $y\tan\rho$. If $t \geq 1$, $y\tan\rho \leq z/3 \leq |UC|$, and H lies above UC; if $t<1$, $y\tan\rho \leq yt = zt/(2+t) \leq z/3$, and H also lies above UC. Therefore, angle JHC is non-obtuse. Since $|HJ| \leq 2y$, angle $JHC \geq \tan^{-1}(1/2)$. Angle $HCU = JHC$. Angle HCJ is non-obtuse since $HJC \leq 2y$. If $t \geq 1$, then $y = z/3$ and $|HJ| = z - ty \geq z/3 = y$, implying angle $HCJ \geq \tan^{-1}(1/2)$; if $t<1$, then $|HJ| = z - ty \geq z - y \geq z/2 > y$, implying again that angle $HCJ \geq \tan^{-1}(1/2)$. We have shown that all triangles are good.

Case (4). Suppose L_2 is in octant 8. Let σ be the angle of L_2 with the horizontal, $0 \leq \sigma \leq 45°$. We consider two subcases, according to whether $\rho \leq \tan^{-1}(2/5)$.

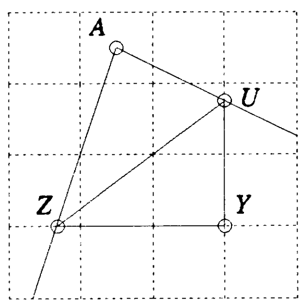


FIG. 4.10. *Triangulation for Lemma 4.5, case (4), when* $\rho \leq \tan^{-1}(2/5)$.

First, suppose $\rho \leq \tan^{-1}(2/5)$. We begin by showing the existence of a neighbor gridpoint Y of L_2 whose horizontal distance to L_1 is in the range $[2,4)$. (See Figure 4.10.) There is a vertical gridline g that lies at least 2 units and less than 3 units to the right of A. Since $\sigma \leq 45°$, L_2 drops less than 3 units before intersecting g. Let G be the neighbor point of the intersection of g with L_2. Then $A_y - G_y < 5$. Since $\rho \leq \tan^{-1}(2/5)$, L_1 moves left at most 2 in height 5, and the horizontal distance from G to L_1 is in the range $[2,5)$. If this distance is less than 4, we are done, with $Y = G$. Otherwise, let G_1 be the gridpoint one unit to the left of G. The horizontal distance from G_1 to L_1 is in $[3,4)$. The vertical distance to L_2 is in the range $[1,3)$. If this distance is less than 2, we are done, with $Y = G_1$. Otherwise, let Y be one unit above

G_1. The vertical distance from Y to L_2 is now in [1,2). The horizontal distance from Y to L_1 is in [13/5,4), and we are done.

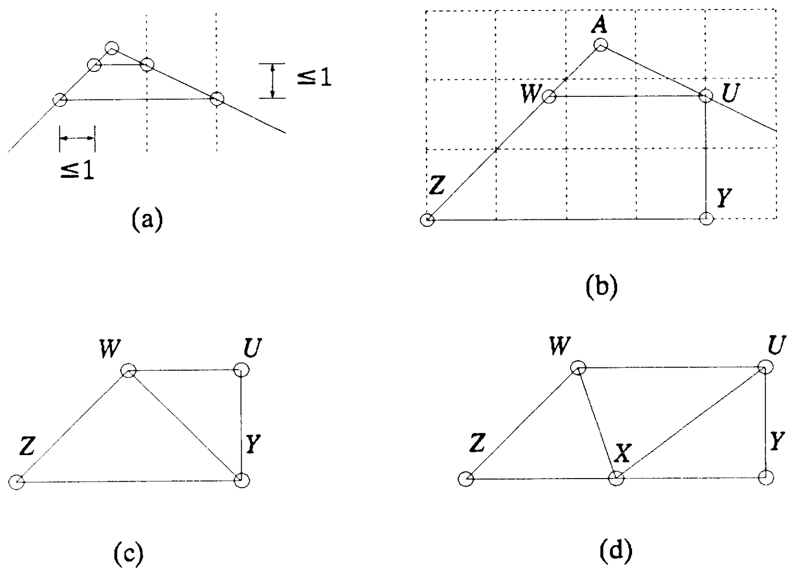


FIG. 4.11. *Triangulation for Lemma 4.5, case (4), when* $\rho > \tan^{-1}(2/5)$.

Let U be the point on L_2 above Y, and Z the point on L_1 to the left of Y. Triangulate as shown in Figure 4.10. Let $\beta =$ angle ZUY. Note that $45° < \beta < \tan^{-1}(4)$. Therefore, right triangle UYZ is good. Angle AUZ is at most $135° - \beta < 90°$, and at least $\pi/2 - \beta > \tan^{-1}(1/4)$. Angle AZU is obviously acute and is at least $\pi/2 - \rho -$ angle $UZY \geq 45° - \tan^{-1}(2/5) \geq 23°$. Therefore, triangle AZU is good.

Finally, to obtain a satisfactory boundary, triangulate below ZY as in Lemma 4.4.

Now, suppose $\rho > \tan^{-1}(2/5)$. Consider the points at which two successive vertical gridlines intersect L_2, and the horizontal lines through these intersection points from L_2 to L_1, as shown in Figure 4.11(a). The difference in length of the horizontal lines is at most 2 since $\sigma, \rho \leq 45°$. Hence, there exists a vertical gridline for which the horizontal segment WU has a length in the range [1,3), as shown in Fig. 4.11(b). Define $x_1 = |WU|$.

Let Y be the gridpoint below U by a distance in [1.5,2.5). Let Z be on L_1 to the left of Y, and let $x_2 = |YZ|$. Since $2/5 < \tan\rho \leq 1$, $x_2 \in [1.6, 5.5)$.

If Y is the neighbor point of U, UY is a satisfactory boundary edge. Otherwise, triangulate to the right of UY as in Lemma 4.3 (rotated).

Next, we show how to triangulate the quadrilateral $WUYZ$, and how to obtain a satisfactory boundary below $WUYZ$. There are two cases based on the size of x_2.

First, if $x_2 \in [1.6, 3)$, triangulate by drawing WY. Let $y = |UY|$. Since Y was chosen so that y is between $x_1/2$ and $2.5x_1$, right triangle WUY is good. Angles WZY and WYZ are obviously both non-obtuse and at least $\tan^{-1}(1/2)$. Since $x_2 < 2y$, angle ZWY is acute. Therefore, triangle ZWY is also good.

If $x_2 \in [1.6, 2)$, then ZY gives a good interface. If $x_2 \in [2, 3)$, triangulate below ZY by Lemma 4.4 to obtain a satisfactory boundary.

Next, suppose $x_2 \in [3,5.5)$. Let X be the gridpoint that lies $\lfloor x_1 \rfloor$ to the left of Y. Triangulate as shown in Figure 4.11(b). By choice of X, $|ZX|<y+1<2y$. From the relative values of $|ZX|$, x_1, and $|UY|$, it is clear that all three triangles are non-obtuse and that angles *WZX*, *WXZ*, *UWX*, *WUX*, *XUY*, and *YXU* are greater than $\tan^{-1}(1/4)$. Both x_1 and *ZX* are at least one. To see that angle *WXU* is at least $\tan^{-1}(1/4)$, draw a perpendicular from X to *WU*. Either one segment of *WU* is at least .625 and one subangle is at least $\tan^{-1}(1/4)$, or both segments are at least .375 and both subangles are at least $\tan^{-1}(.375/2.5)>.5\tan^{-1}(1/4)$. A similar argument applies to angle *ZWX*. Therefore, all triangles are good.

From above, $1\le|ZX|<y+1\le3.5$. If $|ZX|<2$, then *ZX* is a satisfactory boundary edge. Otherwise, Lemma 4.4 is applied below *ZX* to obtain a satisfactory boundary. □

To handle triangulating around obtuse and reflex angles, we make use of the following three lemmas. Each lemma involves triangulating some particular type of region at vertex A, and with rotations and reflections can be combined to handle any obtuse or reflex angle. Choose a local origin so that A is in the cell with lower left corner at (0,0).

LEMMA 4.6. *Let* $U=(2,-1)$, $V=(3,-1)$, $W=(3,1)$, $X=(3,2)$, *and* $Y=(2,2)$ *as in Figure 4.12. Then the triangulation shown of region AUVWXY is good.*

Proof. Straightforward.

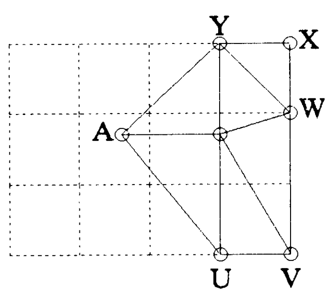


FIG. 4.12. *Triangulation for Lemma 4.4.*

LEMMA 4.7. *Let BAC be an obtuse angle such that AB is in the second octant at A, i.e. AB is nearly vertical with positive slope.*

(a) *Suppose the edge AC is in the seventh octant at A, i.e. AC is nearly vertical with negative slope, as shown in Figure 4.13(a). Let E be the point of intersection of AB and the line* $y=2$, *F be the point of intersection of AC and* $y=-1$, $U=(5,-1)$, *and* $V=(5,2)$. *Then the region AFUVE can be well-triangulated, with new points introduced as necessary to make the boundary EVUF satisfactory.*

(b) *Suppose the edge AC is in the eighth octant at A, i.e. AC is nearly horizontal with negative slope, as shown in Figure 4.13(b). Let E be the point of intersection of AB and the line* $y=3$, *and let F be the point of intersection of AC and the line* $x=7$, *and let* $W=(7,3)$. *Then the region AFWE can be well-triangulated, with new points introduced as necessary to make the boundary FWE satisfactory.*

Proof. In §7.

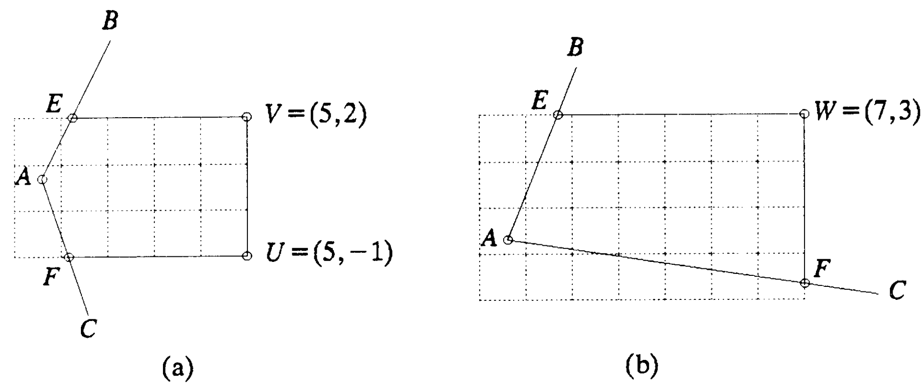


FIG. 4.13. *Triangulatable regions for Lemma 4.7.*

The division of Lemma 4.7 into cases (a) and (b) is induced by the satisfactory path requirement of an edge nearly perpendicular to AC, namely a horizontal edge in (a) and a vertical edge in (b).

LEMMA 4.8. *Let $S=(2,2)$ and let AC be in the seventh or eighth octant of A.*

(a) *Suppose AC is in the seventh octant of A, as shown in Figure 4.14(a). Let F be the point of intersection of AC and the line $y=-1$, $U=(5,-1)$, and $V=(5,2)$. Then $AFUVS$ can be well-triangulated, with new points to make sides FU, UV, and VS satisfactory, and with no points introduced on AS.*

(b) *Suppose AC is in the eighth octant of A, as shown in Figure 4.14(b). Let $R=(6,2)$, and let Q be the point of intersection of AC and $x=6$. Then $AQRS$ can be well-triangulated, with new points introduced to make sides QR and RS satisfactory, and with no points introduced on AS.*

Proof. In §7.

LEMMA 4.9. *Let A be the vertex of an obtuse or reflex angle of R. Then there is a region around A that can be well-triangulated with a satisfactory path along its boundary.*

Proof. Lemmas 4.6-4.8 are used to triangulate at A as follows. Without loss of generality we may suppose AB lies in the first quadrant of A and bounds the interior of P from above and that A lies in the cell with lower left corner at $(0,0)$. We consider cases according to the quadrant of AC.

(a) AC in the fourth quadrant at A. Since ABC is an obtuse angle, either AB is in the second octant or AC is in the seventh octant. In either case, we can apply Lemma 4.7, either directly or reflected. See Figure 4.7(a).

(b) AC in the third quadrant at A. In this case, we add a vertex $G=(2,-1)$ and an edge AG as shown in Figure 4.7(b). We apply Lemma 4.8 to CAG, and a reflection of Lemma 4.8 to BAG. The union of the two triangulated regions gives the desired triangulation at A. Note that the two regions obtained do not overlap

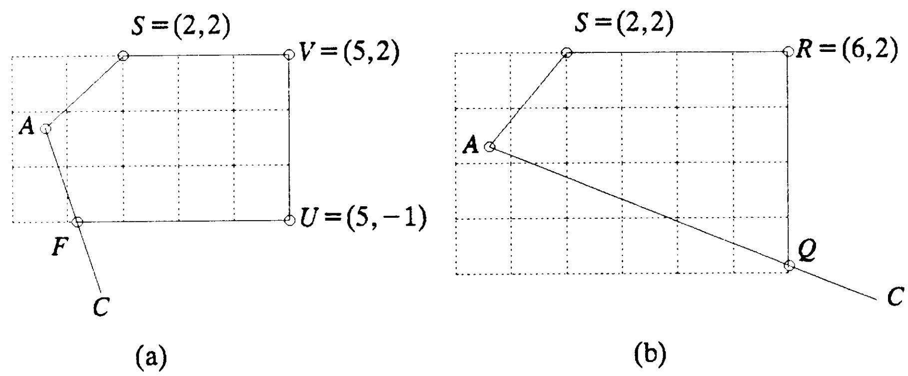


FIG. 4.14. *Triangulatable regions for Lemma 4.8.*

except along the boundary edge AG, and no points are added on AG by Lemma 4.8 or its reflection. Also, the boundaries of the triangulated region are satisfactory.

(c) AC in the second quadrant at A. In this case, we add vertices $G=(2,-1)$ and $H=(-1,-1)$ and edges AG and AH as shown in Figure 4.7(c). We apply Lemma 4.8 (rotated and reflected as necessary) to angles BAG and CAH and Lemma 4.6 (rotated) to HAG. The union of the three regions obtained is the desired triangulated region for A. The three regions overlap only along the lines AH and AG, on which no points are introduced by Lemmas 4.6 or 4.8. The boundaries of the final region for A are satisfactory.

(d) AC in the first quadrant at A. In this case, we add vertices $G=(2,-1)$, $H=(-1,-1)$, and $I=(-1,2)$ and edges AG, AH, and AI as shown in Figure 4.7(d). We apply Lemma 4.8 (reflected and rotated as necessary) to angles BAG and CAI, and Lemma 4.6 (rotated as necessary) to HAG and HAI. The union of the resulting regions is the desired triangulated region for A. As before, no points are introduced by Lemmas 4.6 or 4.8 along AG, AH, or AI, and the various regions overlap only along these lines. The boundaries of the final region for A are satisfactory.

When the above triangulations are applied at the vertices of P, the remaining region satisfies conditions (1) - (3) as desired.

□

Combining Lemmas 4.2, 4.5, and 4.8, we obtain the main result.

THEOREM 4.10. *Any polygon can be triangulated using no obtuse angles and no angles smaller than* $\tan^{-1}(1/4)$ *or the minimum angle of the polygon, whichever is smaller.*

5. Proofs of Lemmas 4.7 and 4.8. For any point P, let P_x and P_y denote the x- and y-coordinates of P, respectively. In the proofs that follow, most angles can be shown to be between $\tan^{-1}(1/4)$ and 90° by either the semicircle principle, inspection based on given constraints of position and slopes, or the inequality $\tan(\alpha_1+\alpha_2)\geq\tan(\alpha_1)+\tan(\alpha_2)$. The last formula is usually applied by dropping a

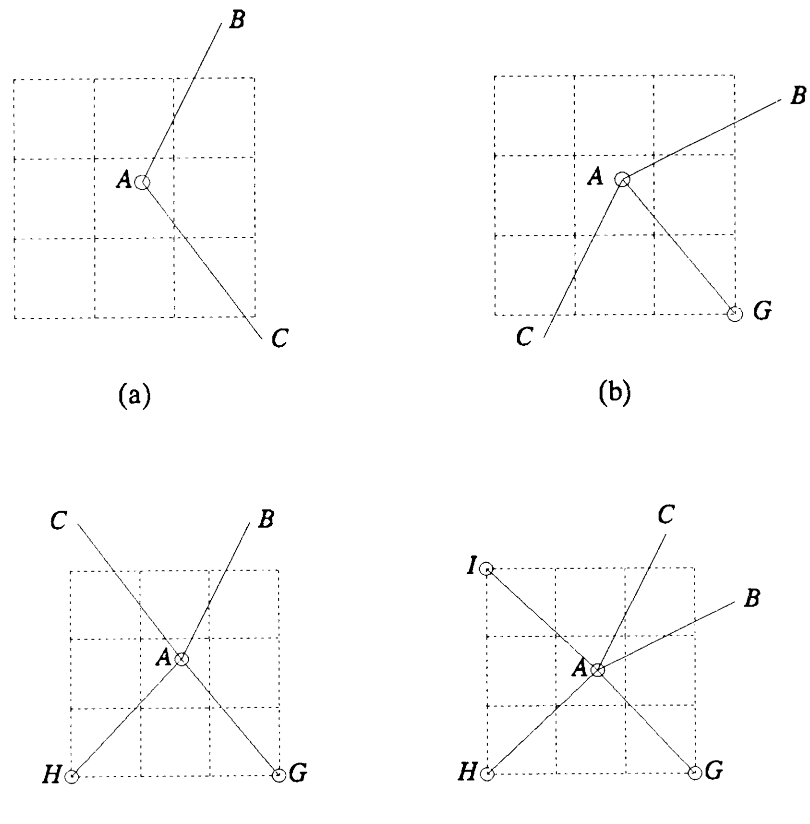


FIG. 4.7. *Cases for triangulating at A.*

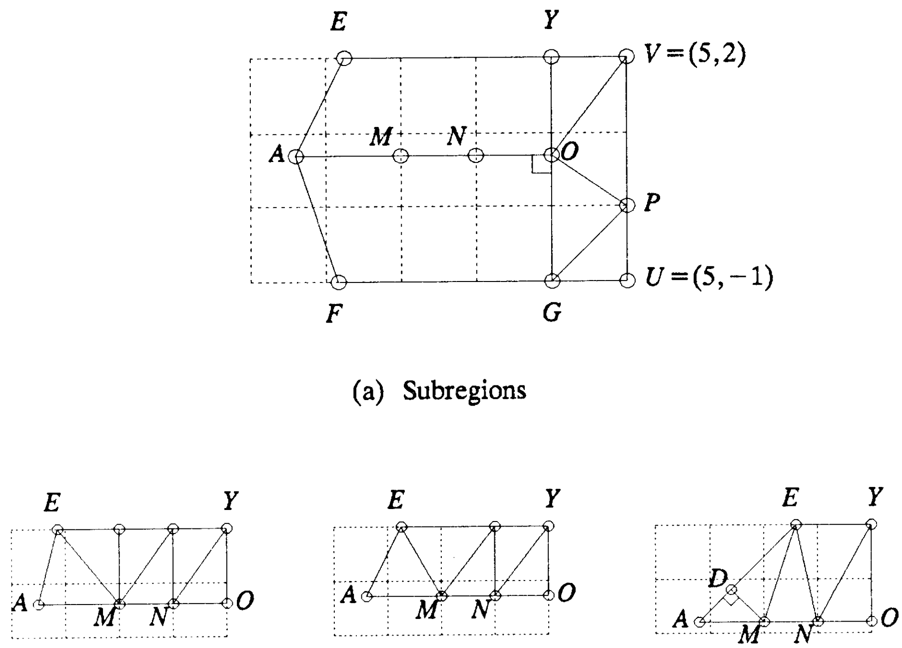
perpendicular from a point to the opposite side of the triangle. The proof will only be mentioned for an angle if it does not follow in a straightforward manner using one of these four strategies.

Proof of Lemma 4.7.

(a) Edge *AC* in the seventh octant, *i.e.* nearly vertical with negative slope. Figure 5.1(a) shows how the region *AFUVE* is subdivided into regions. The line *AO* is horizontal. The triangles in region *GUVY* are good by inspection. Region *AOYE* is triangulated as in Figure 5.1 (b)-(d), depending on where *E* lies. Since the slope of *AB* is at least one, $E_x \leq 3$, and (b)-(d) cover all possibilities. The proofs that triangles are good are straightforward except perhaps for triangle *DME* of (d). For this triangle, applying trigonometric identities and solving for minimum and maximum values of angle *DEM* according to the positions of *E* and *M* shows that the range of *DEM* is between $\tan^{-1}(1/3)$ and $\tan^{-1}(2)$, implying that the triangle is good.

Region *AFGL* is triangulated in the same fashion as *AOYE*. Note that points *M* and *N* are introduced on *AO* in each case so that the shared boundary *AO* is consistent.

(b) Edge *AC* in the eighth octant, *i.e.* nearly horizontal with negative slope. We subdivide this case further according to whether the slope of *AC* is less than or equal to



(a) Subregions

(b) $E_x \le 1$ (c) $1 < E_x \le 2$ (d) $2 < E_x \le 3$

FIG. 5.1. *Case (a) of Lemma 4.5.*

$-1/2$ or greater than $-1/2$.

First, suppose the slope of AC is less than or equal to $-1/2$. We divide the region *AFWE* as shown in Figure 5.2. Right triangle *AHG* is good because of the slope of AC. The other regions are handled as follows.

Region *AHJLM* is triangulated by Figure 5.3 (a), (b), or (c), according to whether $M_x \le 1$, $1 < M_x \le 2$, or $2 < M_x \le 3$. The slope of *AM* prevents M_x from being greater than 3. The proofs that the angles lie between $\tan^{-1}(1/4)$ and 90° are straightforward except perhaps for angles *NHJ* of (b) and *NMH* and *MHJ* of (c) . Trigonometric identities can be applied to deduce that *NMH* of (c) is between $\tan^{-1}(1/3)$ and $\tan^{-1}(2)$, and that *NHJ* of (b) and *MHJ* of (c) are between $\tan^{-1}(1/3)$ and 90°.

Region *GTKJH* of Figure 5.2 is further divided by a vertical line *JI*, with *I* at the intersection of *GT* and $x=4$. (See Figure 5.4.) Note that the slope of *AC* forces *G* to lie at least 1/2 below *H*, but at most 2 below *H*. Therefore, $1/2 \ge G_y \ge -2$. Furthermore, $G_y - 1 \ge I_y \ge G_y - 2$. If $G_y \ge 0$, let *R* be one half unit below *J*, as shown in Figure 5.4(a) and (b). If $|JI| < 2$, triangulate as in Figure 5.4(a); otherwise, triangulate as in Figure 5.4(b). If $0 > G_y > -1$, triangulate *GIJH* as in Figure 5.4(c) or (d), according to whether *I* lies one or two cells below *G*; side *TK* can be made satisfactory and region *ITKJ* can be triangulated by the method of Lemma 4.2. If $-1 \ge G_y \ge -2$, the triangulation is similar except that point *P* is moved one unit down.

Add points to make *EW* and *WF* satisfactory. The region *MLJKTFWE* now has satisfactory sides and can be well-triangulated by Lemma 4.2.

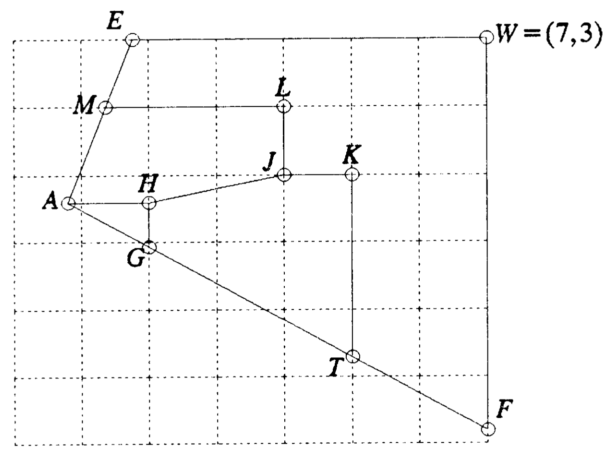


FIG. 5.2. *Lemma 4.7, case (b), when the slope of AF is less than or equal to* $-1/2$.

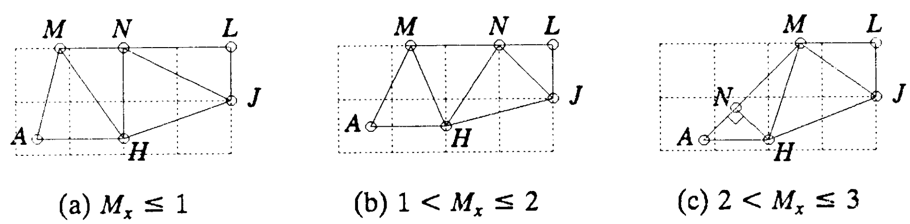


FIG. 5.3. *Triangulations for AHJLM of Figure 5.2.*

Now, suppose that the slope of AC is negative but greater than $-1/2$. Figure 5.5(a) and (b) show subdivisions of $AFWE$ according to whether AE crosses the line $y=2$ to the left of (1,2). In each case, region AHE is left as is if angle AHE is acute; otherwise a perpendicular is added from H to AE. Also, in each case, if $E_x < H_x - 1$, a line is drawn from H to a point exactly one unit above on WE. This guarantees that the angle(s) above H are not obtuse and makes ED satisfactory.

In Figure 5.5(a), the slope of AF guarantees that $G_y > -1$, so $|GH| < 3$. In Figure 5.5(b), the slope of AF forces $G_y > -3/2$, so $1 \leq |GH| < 7/2$. Add points to make sides DW and WF satisfactory. Region GFWDH can be well-triangulated by Lemmas 4.4 (rotated) and 4.2. □

Proof of Lemma 4.8.

(a) AC in the seventh octant, *i.e.* nearly vertical with negative slope. In this case, apply the same triangulation as in Case (a) of Lemma 4.7, *i.e.* with $E = S = (2,2)$ in Figure 5.1(a). Thus, Figure 5.1(c) is applied for the region $AOYE$ of Figure 5.1(a). This method well-triangulates the region even if the slope of AS is nearly horizontal, instead of nearly vertical as specified in Lemma 4.7.

(b) AC in the eighth octant, *i.e.* nearly horizontal with negative slope.

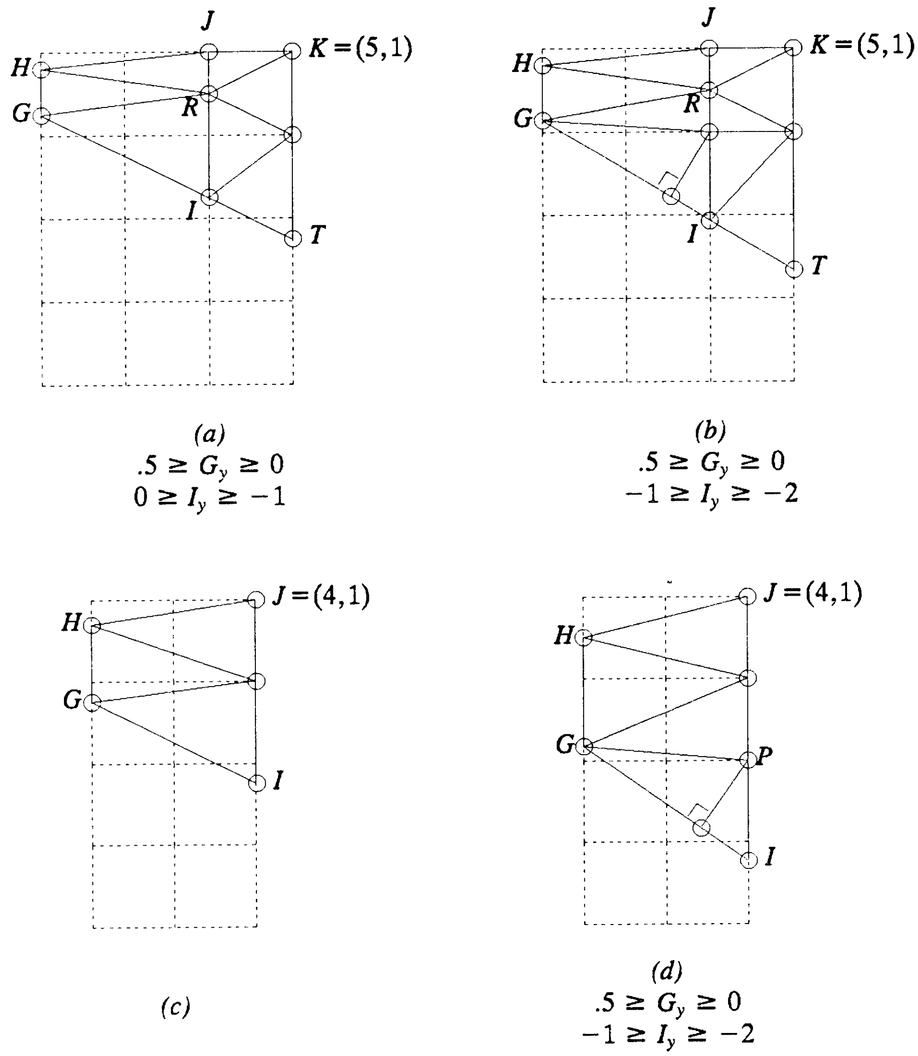


FIG. 5.4. *Further divisions of Figure 5.2.*

First, suppose *SAF* is obtuse. Then *AS* must lie in the second octant at *A*, but the position of *S* forces the slope of *AS* to be at most 2. Hence, the slope of *AC* is at most $-1/2$. We can apply the triangulation method used in case (b) of Lemma 4.7 when the slope of *AC* is at most $-1/2$ (ignoring the rectangle *MLVWE* of Figure 5.2 since it lies outside the desired region for Lemma 4.8).

Now, suppose *SAF* is acute. The region to be triangulated is shown in Figure 4.14(b). Draw a vertical line downward from *S*, intersecting *AC* at *P*. Note that $1 \leq |SP| \leq 4$. If $|SP| \geq 2$, apply Lemma 4.4 (rotated) to triangulate next to *SP*. Now, either all of *SPRW* is well-triangulated, or all but the rightmost 2 or 4 units is well-triangulated. In the latter case, make *RQ* satisfactory and triangulate the remaining region by Lemma 4.2. □

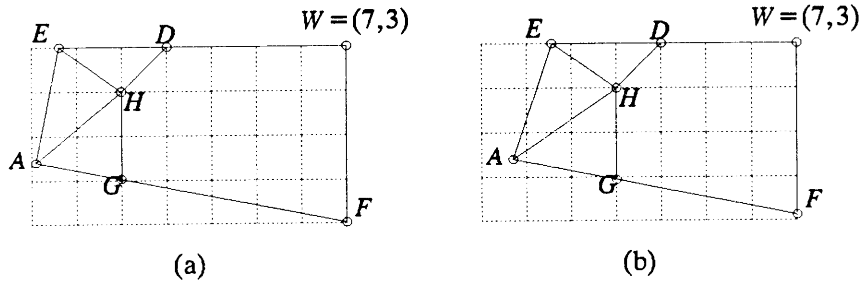


FIG. 5.5. *Lemma 4.7, case (b), when the slope of AC is greater than* $-1/2$.

6. Discussion. Our algorithms remove doubt that a polygon can be triangulated without obtuse angles. But the topic is by no means exhausted, because there are many combinations of side conditions that could be imposed resulting in simpler (or more complicated) algorithms. A hexagonal grid might be investigated. No particular effort has been made to minimize the number of triangles generated; that problem looks harder and not crucial to the applications, which require many triangles anyway. It would be interesting to see if the independent cell triangulations given here can be used to repair locally obtuse triangulations given by other algorithms.

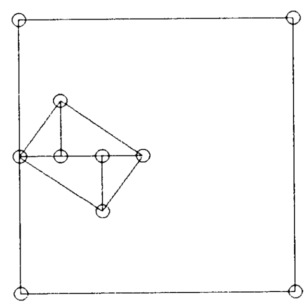


FIG. 8.1. *Two sides of a crack are independent.*

We defined our problem in terms of a *simple polygon*. More precisely, we assume that P is the closure of a bounded planar open set whose boundary is composed of finitely many straight line segments. We consider a "crack" to be made up of two line segments and allow different points on the two segments, as shown in Figure 8.1. In effect, we perturb the problem to open the crack into an infinitesimally narrow wedge.

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