THE GEOMETRY OF CORRELATION FIELDS WITH AN APPLICATION TO FUNCTIONAL CONNECTIVITY OF THE BRAIN[1]

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We introduce two new types of random field. The cross correlation field $R(\mathbf{s}, \mathbf{t})$ is the usual sample correlation coefficient for a set of pairs of Gaussian random fields, one sampled at point $\mathbf{s} \in \Re^M$, the other sampled at point $\mathbf{t} \in \Re^N$. The homologous correlation field is defined as $R(\mathbf{t}) = R(\mathbf{t}, \mathbf{t})$, that is, the 'diagonal' of the cross correlation field restricted to the same location $\mathbf{s} = \mathbf{t}$. Although the correlation coefficient can be transformed pointwise to a t-statistic, neither of the two correlation fields defined above can be transformed to a t-field as defined by Worsley (1994). For this reason, new results are derived for the geometry of the excursion set of these correlation fields that extend those of Adler (1981), and the results are used to detect functional connectivity (regions of high correlation) in 3D Positron Emission Tomography (PET) images of human brain activity.

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1 Introduction

Figure 1 illustrates the idea behind this paper. Two sets of $\nu = 20$ i.i.d. smooth time series $X_i(\mathbf{s}), Y_i(\mathbf{t})$, $i = 1, \ldots, \nu$ are shown parallel to the axes of the figure. The interior two-dimensional image is the usual sample cross correlation coefficient between the two sets of time series evaluated for data at all pairs of times $(\mathbf{s}, \mathbf{t})$ on the axes:

$$R(\mathbf{s}, \mathbf{t}) = \frac{\sum_{i=1}^{n} X_i(\mathbf{s}) Y_i(\mathbf{t})}{\sqrt{\sum_{i=1}^{n} X_i(\mathbf{s})^2 \sum_{i=1}^{n} Y_i(\mathbf{t})^2}}$$

(see Section 10.1 for more details). The set of points $(\mathbf{s}, \mathbf{t})$ where the correlation is greater than 0.5 is the interior of the contours, and local maxima of the correlations inside this region are indicated by crosses. In fact three of these correspond to true non-zero correlations (joined by dotted lines) and the rest are noise. Our main aim in this paper is to provide tests for detecting these pairs of highly correlated regions and distinguishing them from the background noise. We shall be particularly interested in the case where the time series are smooth isotropic Gaussian random fields.

The motivation for this work came from a problem in the analysis of Positron Emission Tomography (PET) images of cerebral blood flow, a measure of brain activity. In one experiment, subjects were given an arithmetic task and a mental rotation task (a task designed to measure the ability to perceive three-dimensional objects), and each was compared to a baseline or rest task (Petrides *et al.*, 1998, private communication). The hypothesis was that arithmetic ability and mental rotation ability are correlated across individuals, that is, those that are good at one task should also be good at the other. The experimenters were thus interested in detecting those areas of the brain which showed a high correlation between PET measures of brain activity during performance of the two tasks. A natural measure of this is the usual sample correlation coefficient, calculated at each voxel to produce an image of correlation coefficients. The regions of the brain where the correlation coefficient exceeded a fixed high threshold value was used to detect regions of high correlation. Their statistical significance was assessed both by the height of local maxima above the threshold, and the size of connected components of the thresholded region. The main statistical question addressed in this paper is how to find approximate null distributions for these two test statistics.

We subsequently realized that this was related to similar problems in 'functional connectivity,' a term used to describe regions of the brain whose blood flows are highly correlated, perhaps due to direct neuronal connections between the regions, or because these two regions are implicated in the same task (Friston *et al.*, 1993, Strother *et al.*, 1995). Again images are acquired either on the same subject or over different subjects, with the goal of detecting those pairs of regions that have highly correlated measures. Previous methods have studied these autocorrelations indirectly through Principal Components Analysis (Friston *et al.*, 1995), or directly through structural equations models for a small number of pre-selected regions [see Horwitz *et al.*, (1996); McIntosh *et al.*, (1994); Friston *et al.*, (1997)]. Bullmore *et al.* (1996) have looked at more extensive matrices of correlations between activated voxels, but no thresholding is attempted. In this paper we propose a direct analysis of the autocorrelations between all possible pairs of voxels. Of course we exclude autocorrelations between neighbouring voxels which can be attributed to spatial correlation. Once again the statistical question is to find the null distribution of local maximum autocorrelations, searched across all pairs of well separated voxels in the brain, and the size of connected components of voxels above a high threshold level.

Finally, we realized that both problems are related to the more general question of detecting cross correlations between two sets of images, as follows. Let $X(\mathbf{s}) = (X_1(\mathbf{s}), \ldots, X_\nu(\mathbf{s}))'$, $\mathbf{s} \in S_x \subset \Re^M$, and $Y(\mathbf{t}) = (Y_1(\mathbf{t}), \ldots, Y_\nu(\mathbf{t}))'$, $\mathbf{t} \in S_y \subset \Re^N$, be vectors of ν independent identically distributed smooth stationary Gaussian random fields with mean 0. The cross correlation field $R(\mathbf{s}, \mathbf{t})$ is defined as

$$R(\mathbf{s}, \mathbf{t}) = \frac{X(\mathbf{s})'Y(\mathbf{t})}{\sqrt{X(\mathbf{s})'X(\mathbf{s})\ Y(\mathbf{t})'Y(\mathbf{t})}}. \tag{1.1}$$

For $M = N$, $S_x = S_y = S$ say, the homologous correlation field is defined as $R(\mathbf{t}) = R(\mathbf{t}, \mathbf{t})$. If $\mathbf{s}$ and $\mathbf{t}$ are discrete parameters, then the cross correlation field can be thought of as the matrix of all pair-wise correlation coefficients, whereas the homologous correlation field is just the diagonal of this matrix (see Figure 1 and Section 10.1). If $X = Y$, then $R(\mathbf{s}, \mathbf{t})$ is the autocorrelation field that measures functional connectivity between all pairs of points. We shall not study autocorrelation fields in this paper, but if $\mathbf{s}$ and $\mathbf{t}$ are far apart, so that the spatial correlation is close to zero, then the local behaviour of the autocorrelation field is well approximated by that of the cross correlation field for independent X and Y.

At first glance, it might be felt that the study of these correlation fields is not too difficult. The sample correlation can be transformed to a t-statistic with $\nu - 1$ degrees of freedom, and the geometry of the t-field has been extensively studied in Worsley (1994) and Cao (1998). However we shall see later that although these t-transformed correlations have a t-distribution at each point, they do not form a t-field in the free parameter (in the sense of these papers, a t-field is defined as a standard Gaussian field divided by the root mean square of i.i.d. standard Gaussian fields). However note that if either $\mathbf{s}$ or $\mathbf{t}$ are fixed (in Figure 1, these are the rows and columns of $R(\mathbf{s}, \mathbf{t})$), then it is easy to see that the t-transformed correlations do form a t-field in the free parameter as defined above.

In Section 2 we shall justify the choice of local maximum correlation as a statistic for detecting positive correlation by finding a model for which this is (almost) the likelihood ratio test statistic. In Section 3 we shall show how the geometry of the excursion set provides a way to find a very good approximation to the upper tail probability of the maximum correlation. The key quantity is the expected Euler characteristic (EC) of the excursion set, which in turn depends on the distribution of the random field and its first two derivatives. Starting with the cross correlation field, the first step is to find a representation for these derivatives (Section 4) and then to find the expected EC (Section 5). Section 6 investigates the shape of the random field near local maxima and gives the null distribution of the size of the largest connected component of the excursion set. Sections 7–9 repeat the previous three sections for the homologous correlation field. Section 10 gives some simple simulations and two applications of the theory to some PET data.

2 Models and test statistics

We are interested in detecting local regions of positive correlations in a background of zero correlations, so a natural estimator is the set of local maximum correlations that are larger than a prescribed threshold. To avoid false positives, that is, detecting positive correlations when none are present, the threshold should be set so that the probability of detecting any positive correlations above the threshold, where none are present, is controlled to be say 0.05. Even if there are some positive correlations confined to a small region, this procedure is conservative, since the probability of detecting further correlations outside this region (where there are no correlations) is clearly smaller than detecting correlations in the whole region (when there are no correlations). Thus we are interested in the test statistics

$$R_{\max} = \max_{\mathbf{s}\in S_x, \mathbf{t}\in S_y} R(\mathbf{s},\mathbf{t}), \quad \tilde{R}_{\max} = \max_{\mathbf{t}\in S} R(\mathbf{t}), \tag{2.1}$$

and their upper tail probabilities under the null hypothesis of zero correlations everywhere in the region.

We shall now find a model of the correlation structure for which the test statistics (2.1) are (almost) likelihood ratio test statistics. This will suggest the local correlation structure that (2.1) is most powerful at detecting. Consider the model

$$X_i(\mathbf{s}) = \sigma_x(\mathbf{s})\epsilon_i(\mathbf{s}), \quad Y_i(\mathbf{t}) = \sigma_y(\mathbf{t})\eta_i(\mathbf{t}), \tag{2.2}$$

where $\sigma_x(\mathbf{s}), \sigma_y(\mathbf{t})$ are unknown positive parameters, and $\epsilon_i(\mathbf{s}), \eta_i(\mathbf{t})$ are stationary Gaussian random fields with mean zero, variance one, and known spatial autocorrelation functions $f_x(\mathbf{s}), f_y(\mathbf{t})$ respectively. In Figure 1 and the applications in Section 10, f_x and f_y are the Gaussian-shaped functions

$$f_x(\mathbf{s}) = \exp(-\mathbf{s}'\Lambda_x\mathbf{s}/2), \quad f_y(\mathbf{t}) = \exp(-\mathbf{t}'\Lambda_y\mathbf{t}/2), \tag{2.3}$$

where Λ_x, Λ_y are known $M \times M, N \times N$ matrices that control the spatial extent of the correlations. Suppose further that

$$\mathrm{Cor}\{X_i(\mathbf{s}), Y_i(\mathbf{t})\} = \beta f_x(\mathbf{s}-\mathbf{s}_0)f_y(\mathbf{t}-\mathbf{t}_0), \tag{2.4}$$

where β is an unknown parameter restricted to $|\beta| < 1$ to ensure that (2.4) is indeed a valid model for the correlation structure. In other words, (2.4) specifies a local correlation structure between $X_i(\mathbf{s})$ in the neighborhood of $\mathbf{s}_0$ and $Y_i(\mathbf{t})$ in the neighborhood of $\mathbf{t}_0$, whose extent matches the spatial autocorrelation of $X_i(\mathbf{s})$ and $Y_i(\mathbf{t})$. Such a correlation structure has been added to Figure 1 in three different places.

Assume initially that the locations of the correlations $\mathbf{s}_0, \mathbf{t}_0$ are known, so that the unknown parameters are $\sigma_x(\mathbf{s}), \sigma_y(\mathbf{t})$ and β. Then it can be shown that the likelihood depends on β only through the marginal likelihood of $X_i(\mathbf{s}_0), Y_i(\mathbf{t}_0)$, $i = 1, \ldots, \nu$. This can easily be demonstrated for discrete parameter random fields where $\mathbf{s}, \mathbf{t}$ take a finite set of values; moving to Hilbert spaces completes the proof for the continuous parameter case (Siegmund and Worsley, 1995). The maximum marginal likelihood estimators of $\sigma_x(\mathbf{s}_0)^2$, $\sigma_y(\mathbf{t}_0)^2$ and β are then $\sum_i X_i(\mathbf{s}_0)^2/\nu$, $\sum_i Y_i(\mathbf{t}_0)^2/\nu$ and $R(\mathbf{s}_0, \mathbf{t}_0)$ respectively. Unfortunately these are not the full m.l.e.'s because the likelihood of $\{X_i(\mathbf{s}), Y_i(\mathbf{t}) \ : \ \mathbf{s} \neq \mathbf{s}_0, \mathbf{t} \neq \mathbf{t}_0\}$ conditional on $X_i(\mathbf{s}_0), Y_i(\mathbf{t}_0)$ still depends on $\sigma_x(\mathbf{s}_0), \sigma_y(\mathbf{t}_0)$. Nevertheless, estimating $\sigma_x(\mathbf{s})^2, \sigma_y(\mathbf{t})^2$ marginally (and consistently) by $\sum_i X_i(\mathbf{s})^2/\nu$, $\sum_i Y_i(\mathbf{t})^2/\nu$ for all $\mathbf{s}, \mathbf{t}$ and substituting these into the full log(likelihood) leaves an expression that depends on $\mathbf{s}_0, \mathbf{t}_0$ only through the additive term $-(\nu/2)\log\{1 - R(\mathbf{s}_0, \mathbf{t}_0)^2\}$. Maximizing this over all $\mathbf{s}_0, \mathbf{t}_0$ is equivalent to maximizing $R(\mathbf{s}_0, \mathbf{t}_0)$, which justifies $R_{\max}$ both as an estimator of β and as a test statistic for the null hypothesis that $\beta = 0$. The same arguments apply to $\tilde{R}_{\max}$ by restricting $\mathbf{s}_0 = \mathbf{t}_0$. However more important than these optimality properties is the fact that the distribution of the random field $R(\mathbf{s}, \mathbf{t})$ does not depend on $\sigma_x(\mathbf{s}), \sigma_y(\mathbf{t})$, which makes exact inference possible.

3 Geometry of the excursion set

For the applications presented above, the main interest is to find good approximations for the null probability that the maximum correlation exceeds a high threshold value. We obtain an accurate approximation for this via the geometry of the excursion set, the set of points where the correlation field exceeds the threshold value. For high thresholds, the excursion sets consists of isolated regions containing no 'holes,' so that the Euler or Euler-Poincaré characteristic (EC) of the excursion set counts the number of connected components of the excursion set. For higher thresholds, near the maximum of the field, the EC becomes an indicator for the event that the maximum exceeds the threshold, taking the value one if the maximum is above the threshold, and zero if it is below. Thus for these high thresholds the expected EC approximates the upper tail probability of the maximum (Hasofer, 1978; Worsley, 1995). Although it is not quite what we want, the expected EC of the excursion set has several advantages over other approximations to the upper tail probability of the maximum: it is very accurate (there has been a recent breakthrough on this longstanding conjecture: Adler, 1998, has shown that the expected EC for Gaussian random fields is accurate to as many terms in its expansion); in some discrete situations, it is exact (see Naiman and Wynn, 1992); in many cases it is possible to find an exact expression for the expected EC of the excursion set for *all* threshold levels; the EC of the excursion set has inherent interest as a tool for studying the clustering behaviour of random fields and point processes, particularly in astrophysics (Torres, 1994; Vogley *et al.*, 1994; Worsley, 1995).

Before we proceed, we shall introduce some notation used throughout the paper. For a function of $\mathbf{z}$, derivatives with respect to $\mathbf{z}$ will be indicated by dot notation and second derivatives with two dots. If $\mathbf{z}$ is partitioned into two parts, i.e. $\mathbf{z} = (\mathbf{s}, \mathbf{t})$, to avoid confusion and to denote the differentiation with respect to $\mathbf{s}$ or $\mathbf{t}$ explicitly, we shall replace the dots by $\mathbf{s}$ or $\mathbf{t}$ respectively. Where appropriate, a single derivative with respect to $\mathbf{s}$ or $\mathbf{t}$ should be interpreted as a column vector, and two derivatives in a single expression, either a second derivative or a product of single derivatives, should be interpreted as a matrix. Hence if f is a scalar, we define

$$\overset{\mathbf{s}}{f} \equiv \frac{\partial f}{\partial \mathbf{s}}, \quad \overset{\mathbf{st}}{f} \equiv \frac{\partial^2 f}{\partial \mathbf{s} \partial \mathbf{t}'}, \quad \overset{\mathbf{s}}{f} \, \overset{\mathbf{t}}{f} \equiv \frac{\partial f}{\partial \mathbf{s}} \frac{\partial f}{\partial \mathbf{t}'}.$$

In more complex expressions, matrix operations take precedence over differentiation.

Wherever there is no confusion, we shall drop the dependence of a random field on its arguments. That is, we shall write $R = R(\mathbf{s}, \mathbf{t})$ for the cross correlation field or $R = R(\mathbf{t})$ for the homologous correlation field. For a vector, we shall use subscripts j and $|j$ to represent the jth and first j components. For a symmetric $n \times n$ matrix B, we shall use the subscript $|j$ to represent the sub-matrix composed of the first j rows and columns. We shall also use $\mathrm{detr}_j(B)$ to denote the sum of the determinant of all $j \times j$ principal minors of B, so that $\mathrm{detr}_n(B) = \det(B)$, $\mathrm{detr}_1(B) = \mathrm{tr}(B)$ and we define $\mathrm{detr}_0(B) = 1$. Wherever possible, we shall use lower case letters for scalars and vectors, and upper case letters for matrices; the exceptions are X, Y and R. Finally, we shall let $x^+ = x$ if $x > 0$ and 0 otherwise.

3.1 The homologous correlation field

To make the formal presentation clearer, we shall start with the homologous correlation field. We shall assume that the correlation field satisfies the regularity conditions given in Adler (1981, Theorem 5.2.2). Let $S \subset \Re^N$ be a closed compact set with a twice differentiable boundary. Let $A_r = \{\mathbf{t} \in S \,:\, R(\mathbf{t}) \geq r\}$ be the excursion set of the homologous correlation field above the threshold r, inside S. Let $\chi(A)$ be the Euler or Euler-Poincaré characteristic (EC) of a set A. We are interested in finding the expectation of the EC of the excursion set, $\mathrm{E}\{\chi(A_r)\}$.

It is possible to find a simple result for the expectation of the EC of the excursion set when the field is isotropic in $\mathbf{t}$. Define the j-dimensional *EC intensity* as

$$\begin{aligned} \rho_j(r) &= \mathrm{E}\left\{(R \geq r)\det(-\ddot{R}_{|j}) \mid \dot{R}_{|j} = 0\right\}\theta_{|j}(0) \\ &= \mathrm{E}\left\{\dot{R}_j^+ \det(-\ddot{R}_{|j-1}) \mid \dot{R}_{|j-1} = 0, R = r\right\}\phi_{|j-1}(0, r), \end{aligned} \tag{3.1}$$

where $\theta_{|j}(\cdot)$ is the density of $\dot{R}_{|j}$ and $\phi_{|j-1}(\cdot,\cdot)$ is the joint density of $\dot{R}_{|j-1}$ and R; the equivalence of the two definitions is demonstrated in Worsley (1995). The word intensity is chosen to emphasize the derivation of (3.1) from Morse theory as the expectation of a point process in $\Re^N$ taking values ± 1 at turning points of $R(\mathbf{t})$ (Adler, 1981; Worsley, 1995). Let $a_j = 2\pi^{j/2}/\Gamma(j/2)$ be the surface area of a unit $(j-1)$-sphere in $\Re^j$. Let $C(S)$ be the inside curvature matrix of S at a point $\mathbf{t}$, and for $j = 0, \ldots, N-1$ define the j-dimensional measure, proportional to the Minkowski functional, of S as

$$\mu_j(S) = \frac{1}{a_{N-j}} \int_{\partial S} \mathrm{detr}_{N-1-j}\{C(S)\} d\mathbf{t},$$

and define $\mu_N(S) = |S|$, the Lebesgue measure of S. Note that $\mu_0(S) = \chi(S)$ by the Gauss-Bonnet Theorem, and $\mu_{N-1}(S)$ is half the surface area of S. Then the expected EC is given by:

$$\mathrm{E}\{\chi(A_r)\} = \sum_{j=0}^{N} \mu_j(S)\rho_j(r), \tag{3.2}$$

where we define $\rho_0(r) = \mathrm{P}\{R \geq r\}$ (Worsley, 1995). Our main task, therefore, is to evaluate $\rho_j(r)$ from (3.1).

3.2 The cross correlation field

The result (3.2) needs to be extended to the cross correlation field $R(\mathbf{s},\mathbf{t})$. First, even if we assume that the boundaries of S_x and S_y are twice differentiable, the boundary of the search region $S_x \times S_y$ is not twice differentiable, and second, even if we assume that the field is stationary, isotropic in $\mathbf{s}$ for fixed $\mathbf{t}$, and isotropic in $\mathbf{t}$ for fixed $\mathbf{s}$, the field is not isotropic in $(\mathbf{s},\mathbf{t})$. We therefore need the following special result.

LEMMA 3.1 *Suppose $R(\mathbf{s},\mathbf{t})$ is a stationary random field as defined in (1.1), satisfying the regularity conditions of Adler (1981, Theorem 5.2.2). Define the excursion set as $A_r = \{\mathbf{s} \in S_x, \mathbf{t} \in S_y \ : \ R(\mathbf{s},\mathbf{t}) \geq r\}$. Then*

$$\mathrm{E}\{\chi(A_r)\} = \sum_{i=0}^{M}\sum_{j=0}^{N} \mu_i(S_x)\mu_j(S_y)\rho_{ij}(r),$$

where $\rho_{ij}(r)$ is the same expression as (3.1) but for the parameters of the random field restricted to the first i components of $\mathbf{s}$ and the first j components of $\mathbf{t}$.

PROOF. The proof follows closely that used to derive (3.1). This in turn follows from the expected point-set representation for $\chi(A_r)$ from Morse theory which we now present for a stationary field inside a set S with a twice differentiable boundary (Worsley, 1995). Let $\dot{R}_\perp$ be the derivative of R in the direction of the outside normal to ∂S, let $\dot{R}_\mathrm{T}$ and $\ddot{R}_\mathrm{T}$ be the first and second derivatives in the tangent plane to ∂S, let $\theta_\mathrm{T}(\cdot)$ be the density of $\dot{R}_\mathrm{T}$, and let $\theta(\cdot)$ be the density of $\dot{R}$. Then

$$\begin{aligned}\mathrm{E}\{\chi(A_r)\} &= \int_{\partial S} \mathrm{E}\left\{(R \geq r)(\dot{R}_\perp > 0)\det(-\ddot{R}_\mathrm{T} + \dot{R}_\perp C(S) \mid \dot{R}_\mathrm{T} = 0\right\}\theta_\mathrm{T}(0)d\mathbf{t} \\ &\quad + |S|\mathrm{E}\left\{(R \geq r)\det(-\ddot{R}) \mid \dot{R} = 0\right\}\theta(0), \qquad (3.3)\end{aligned}$$

where logical expressions in parentheses take the value one if true and zero if false. Extending this to $S = S_x \times S_y$, it can easily be seen that the last term of (3.3) corresponds to the $i = M$, $j = N$ term in the Lemma; the remaining terms come from the boundary.

We break up the boundary of S into three disjoint pieces: $S_x^o \times \partial S_y$, $\partial S_x \times S_y^0$ and $\partial S_x \times \partial S_y$. We shall first deal with the 'edge' $\partial S_x \times \partial S_y$ where the boundary is not twice differentiable. The derivative with respect to $\mathbf{s}$ in the direction of the outside normal to ∂S_x is denoted by $\overset{\mathbf{s}}{R}_\perp$ and the derivative with respect to $\mathbf{t}$ in the direction of the outside normal to ∂S_y is denoted by $\overset{\mathbf{t}}{R}_\perp$. Let $m = M - 1$ and $n = N - 1$. The vector of the remaining $(m+n)$ derivatives tangent to ∂S_x and ∂S_y is denoted by $\dot{R}_\mathrm{T}$, and the $(m+n) \times (m+n)$ matrix of second derivatives is denoted by $\ddot{R}_\mathrm{T}$. Let $\theta_\mathrm{T}(\cdot)$ be the density of $\dot{R}_\mathrm{T}$. Let B be an $(m+n) \times (m+n)$ block diagonal matrix with two blocks $\overset{\mathbf{s}}{R}_\perp C(S_x)$, $\overset{\mathbf{t}}{R}_\perp C(S_y)$. Then the contribution to the expected EC is

$$\int_{\partial S_x \times \partial S_y} \mathrm{E}\left\{(R \geq r)(\overset{\mathbf{s}}{R}_\perp > 0)(\overset{\mathbf{t}}{R}_\perp > 0)\det(-\ddot{R}_\mathrm{T} + B) \mid \dot{R}_\mathrm{T} = 0\right\}\theta_\mathrm{T}(0)d\mathbf{s}d\mathbf{t}. \qquad (3.4)$$

Since the field is isotropic in $\mathbf{s}$ for fixed $\mathbf{t}$, and in $\mathbf{t}$ for fixed $\mathbf{s}$, we can rotate the coordinates $\mathbf{s}$ and $\mathbf{t}$ so that the outside normals to ∂S_x and ∂S_y are aligned with the Mth and Nth coordinate axes of $\mathbf{s}$ and $\mathbf{t}$, respectively. We can further rotate S_x and S_y in their tangent planes so that B is diagonal, which by isotropy will not affect the integrand, so without loss of generality we shall assume that B is diagonal. The determinant in the integrand can then be expanded in terms of products of the determinant of each $(i+j)\times(i+j)$ principal minor of $-\ddot{R}_{\mathrm{T}}$ (i from the first m rows and columns, j from the second n rows and columns) with the determinant of the remaining $m-i$ rows and columns of $\overset{\mathbf{s}}{\dot{R}}_{\perp}C(S_x)$ and the remaining $n-j$ rows and columns of $\overset{\mathbf{t}}{\dot{R}}_{\perp}C(S_y)$ not included in the principal minor. Again by isotropy, the distribution of any principal minor of $-\ddot{R}_{\mathrm{T}}$ is the same as the distribution of $-\ddot{R}_{|ij}$, the second derivative with respect to the first i components of $\mathbf{s}$ and the first j components of $\mathbf{t}$. Thus the expectation in (3.4) becomes

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\mathrm{detr}_{m-i}\{C(S_x)\}\mathrm{detr}_{n-j}\{C(S_y)\}$$
$$\times\ \mathrm{E}\left\{(R\geq r)\left(\overset{\mathbf{s}}{\dot{R}}{}^{+}_{M}\right)^{m-i}\left(\overset{\mathbf{t}}{\dot{R}}{}^{+}_{N}\right)^{n-j}\det(-\ddot{R}_{|ij})\ \middle|\ \overset{\mathbf{s}}{\dot{R}}_{|m}=0,\overset{\mathbf{t}}{\dot{R}}_{|n}=0)\right\}.$$

The rest of the proof follows closely that of Theorem 4 of Worsley (1995), to give the terms for $0\leq i<M$, $0\leq j<N$ in the Lemma.

It is straightforward to extend this to the remaining two portions of the boundary $S_x^o\times\partial S_y$ and $\partial S_x\times S_y^o$. For example, the curvature matrix of $S_x^o\times\partial S_y$ is simply the $(M+N-1)\times(M+N-1)$ matrix formed by adding M rows and columns of zeros to $C(S_y)$. Expanding the determinant in the integrand in terms of products of the determinant of principle minors of $\ddot{R}_{\mathrm{T}}$ and the determinant of the principle minors of the curvature matrix produces the terms corresponding to $i=M$, $0\leq j<N$. Similar arguments for $\partial S_x\times S_y^o$ lead to the remaining terms for $0\leq i<M$, $j=N$.

□

4 Representations of the first two derivatives of the cross correlation field

First some notation. Let $\delta_{ij}=1$ if $i=j$ and 0 otherwise, and let I_d be the $d\times d$ identity matrix. Let $\mathrm{Normal}_d(\mu,\Sigma)$ represent the multivariate normal distribution on $\Re^d$ with mean μ and variance Σ, and if A is an $n\times m$ matrix whose elements are normally distributed we shall write $\mathrm{Normal}_{n\times m}$. Let χ^2_ν represent the χ^2 distribution with ν degrees of freedom, let $\mathrm{Wishart}_d(\Sigma,\nu)$ represent the Wishart distribution of a $d\times d$ matrix with expectation $\nu\Sigma$ and degrees of freedom ν, let $\mathrm{Beta}(n,m)$ represent the Beta distribution with parameters n,m. Finally, we shall let $\overset{D}{=}$ represent equality in law between two random variables.

LEMMA 4.1 *(Adler, 1981, page 31) Let $\xi=\xi(\mathbf{t})$ be an isotropic standard Gaussian random field on $\Re^N$ with* $\mathrm{E}(\xi)=0$, $\mathrm{Var}(\xi)=1$ *and* $\mathrm{Var}(\dot{\xi})=\Lambda$. *We shall assume that ξ satisfies the regularity conditions of Theorem 5.2.2 of Adler (1981) which ensure that realizations of ξ are sufficiently smooth. Then we can write the second derivative of ξ at a fixed point $\mathbf{t}$ in* *terms of independent random variables as follows:*

$$\ddot{\xi} \overset{D}{=} -\Lambda\xi + H,$$

where H is a symmetric $N \times N$ matrix, independent of ξ and $\dot{\xi}$. The ijth elements of H, denoted by h_{ij}, are jointly normally distributed with mean zero and covariance

$$\mathrm{Cov}(h_{ij}, h_{kl}) = \gamma(i,j,k,l) - \lambda_{ij}\lambda_{kl}, \tag{4.1}$$

where λ_{ij} is the ijth element of Λ and $\gamma(i,j,k,l)$ is symmetric in its arguments, $i,j,k,l = 1,\ldots,N$. We shall refer to this type of covariance (4.1) as $V(\Lambda)$, and write $H \sim \mathrm{Normal}_{N\times N}(0, V(\Lambda))$.

Let Λ_x and Λ_y be the variance matrices of the first derivative of the components of $X(\mathbf{s})$ and $Y(\mathbf{t})$ respectively.

LEMMA 4.2 *The first two derivatives of the cross correlation field $R(\mathbf{s},\mathbf{t})$ can be written in terms of independent random variables as follows:*

(a) $\overset{\mathbf{s}}{R} \overset{D}{=} (1-R^2)^{\frac{1}{2}} a^{-\frac{1}{2}} \Lambda_x^{\frac{1}{2}} z_x, \quad \overset{\mathbf{t}}{R} \overset{D}{=} (1-R^2)^{\frac{1}{2}} c^{-\frac{1}{2}} \Lambda_y^{\frac{1}{2}} z_y.$

(b)
$$\begin{aligned}
\overset{\mathbf{ss}}{R} &\overset{D}{=} \Lambda_x^{\frac{1}{2}}\{-Ra^{-1}z_x z_x' - Ra^{-1}Q_x - (1-R^2)^{\frac{1}{2}}a^{-1}(z_x w_x' + w_x z_x') + (1-R^2)^{\frac{1}{2}}a^{-\frac{1}{2}}H_x\}{\Lambda_x^{\frac{1}{2}}}',\\
\overset{\mathbf{tt}}{R} &\overset{D}{=} \Lambda_y^{\frac{1}{2}}\{-Rc^{-1}z_y z_y' - Rc^{-1}Q_y - (1-R^2)^{\frac{1}{2}}c^{-1}(z_y w_y' + w_y z_y') + (1-R^2)^{\frac{1}{2}}c^{-\frac{1}{2}}H_y\}{\Lambda_y^{\frac{1}{2}}}',\\
\overset{\mathbf{st}}{R} &\overset{D}{=} \Lambda_x^{\frac{1}{2}}\{-R(ac)^{-\frac{1}{2}}z_x z_y' + (ac)^{-\frac{1}{2}}Q_{xy}\}\Lambda_y^{\frac{1}{2}},
\end{aligned}$$

where $D = M + N$ and

$$\begin{aligned}
z_x, w_x &\sim \mathrm{Normal}_M(0, I_M), & z_y, w_y &\sim \mathrm{Normal}_N(0, I_N),\\
H_x &\sim \mathrm{Normal}_{M\times M}(0, V(I_M)), & H_y &\sim \mathrm{Normal}_{N\times N}(0, V(I_N)),\\
a, c &\sim \chi^2_\nu, & Q &= \begin{pmatrix} Q_x & -Q_{xy} \\ -Q_{xy}' & Q_y \end{pmatrix} \sim \mathrm{Wishart}_D(I_D, \nu-2),
\end{aligned}$$

independently, and independent of R.

PROOF. Let $a = X'X, b = X'Y$ and $c = Y'Y$. Then

$$\begin{aligned}
&\overset{\mathbf{s}}{a} = 2\overset{\mathbf{s}}{X}'X, \quad \overset{\mathbf{ss}}{a} = 2\overset{\mathbf{s}}{X}'\overset{\mathbf{s}}{X} + 2\overset{\mathbf{ss}}{X}'X, \quad \overset{\mathbf{t}}{c} = 2\overset{\mathbf{t}}{Y}'Y, \quad \overset{\mathbf{tt}}{c} = 2\overset{\mathbf{t}}{Y}'\overset{\mathbf{t}}{Y} + 2\overset{\mathbf{tt}}{Y}'Y,\\
&\overset{\mathbf{s}}{b} = \overset{\mathbf{s}}{X}'Y, \quad \overset{\mathbf{t}}{b} = \overset{\mathbf{t}}{Y}'X, \quad \overset{\mathbf{ss}}{b} = \overset{\mathbf{ss}}{X}'Y, \quad \overset{\mathbf{st}}{b} = \overset{\mathbf{s}}{X}'\overset{\mathbf{t}}{Y}, \quad \overset{\mathbf{tt}}{b} = \overset{\mathbf{tt}}{Y}'X.
\end{aligned}$$

Let $u = a^{-\frac{1}{2}}X$ and $v = c^{-\frac{1}{2}}Y$, then $R = a^{-\frac{1}{2}}c^{-\frac{1}{2}}b = u'v$. It is easy to show that a, c, u and v are independent and hence a, c and R are independent. Therefore,

$$\begin{aligned}
\overset{\mathbf{s}}{R} &= a^{-\frac{1}{2}}c^{-\frac{1}{2}}\overset{\mathbf{s}}{b} - \tfrac{1}{2}a^{-\frac{3}{2}}c^{-\frac{1}{2}}b\overset{\mathbf{s}}{a} = R\left(b^{-1}\overset{\mathbf{s}}{b} - \tfrac{1}{2}a^{-1}\overset{\mathbf{s}}{a}\right),\\
\overset{\mathbf{t}}{R} &= a^{-\frac{1}{2}}c^{-\frac{1}{2}}\overset{\mathbf{t}}{b} - \tfrac{1}{2}a^{-\frac{3}{2}}c^{-\frac{1}{2}}b\overset{\mathbf{t}}{c} = R\left(b^{-1}\overset{\mathbf{t}}{b} - \tfrac{1}{2}c^{-1}\overset{\mathbf{t}}{c}\right).
\end{aligned} \tag{4.2}$$

Let $\mathcal{R}_u = I_\nu - uu'$ and $\mathcal{R}_v = I_\nu - vv'$, and note that $Y - a^{-1}bX = \mathcal{R}_u Y$ and $X - c^{-1}bY = \mathcal{R}_v X$, then

$$\overset{\mathbf{s}}{R} = a^{-\frac{1}{2}} \overset{\mathbf{s}}{X}{}' \mathcal{R}_u v, \quad \overset{\mathbf{t}}{R} = c^{-\frac{1}{2}} \overset{\mathbf{t}}{Y}{}' \mathcal{R}_v u.$$

Let

$$z_x = (1 - R^2)^{-\frac{1}{2}} \Lambda_x{}^{-\frac{1}{2}} (\overset{\mathbf{s}}{X}{}' \mathcal{R}_u v), \quad z_y = (1 - R^2)^{-\frac{1}{2}} \Lambda_y{}^{-\frac{1}{2}} (\overset{\mathbf{t}}{Y}{}' \mathcal{R}_v u),$$

then $z_x \sim \mathrm{Normal}_M(0, I_M)$, $z_y \sim \mathrm{Normal}_N(0, I_N)$ independent of u, v and hence R. Therefore **(a)** follows.

It is easy to see that

$$\begin{aligned}
\overset{\mathbf{ss}}{R} &= R^{-1} \overset{\mathbf{s}}{R} \overset{\mathbf{s}}{R} + R \left(b^{-1} \overset{\mathbf{ss}}{b} - \tfrac{1}{2} a^{-1} \overset{\mathbf{ss}}{a} - b^{-2} \overset{\mathbf{s}}{b} \overset{\mathbf{s}}{b} + \tfrac{1}{2} a^{-2} \overset{\mathbf{s}}{a} \overset{\mathbf{s}}{a} \right), \\
\overset{\mathbf{tt}}{R} &= R^{-1} \overset{\mathbf{t}}{R} \overset{\mathbf{t}}{R} + R \left(b^{-1} \overset{\mathbf{tt}}{b} - \tfrac{1}{2} c^{-1} \overset{\mathbf{tt}}{c} - b^{-2} \overset{\mathbf{t}}{b} \overset{\mathbf{t}}{b} + \tfrac{1}{2} c^{-2} \overset{\mathbf{t}}{c} \overset{\mathbf{t}}{c} \right), \\
\overset{\mathbf{st}}{R} &= R^{-1} \overset{\mathbf{s}}{R} \overset{\mathbf{t}}{R} + R \left(b^{-1} \overset{\mathbf{st}}{b} - b^{-2} \overset{\mathbf{s}}{b} \overset{\mathbf{t}}{b} \right).
\end{aligned}$$

From Lemma 4.1, $\overset{\mathbf{ss}}{X} \overset{D}{=} -\Lambda_x X + \tilde{H}_x$ and $\overset{\mathbf{tt}}{Y} \overset{D}{=} -\Lambda_y Y + \tilde{H}_y$, where the components of $\tilde{H}_x$ and $\tilde{H}_y$ corresponding to each component of X and Y are i.i.d. $\mathrm{Normal}_{M \times M}(0, V(\Lambda_x))$ and $\mathrm{Normal}_{N \times N}(0, V(\Lambda_y))$, respectively. Note that $\mathcal{R}_u u = \mathcal{R}_v v = 0$ and so

$$\begin{aligned}
\overset{\mathbf{ss}}{R} &= R^{-1} \overset{\mathbf{s}}{R} \overset{\mathbf{s}}{R} + R(b^{-1} \overset{\mathbf{ss}}{X}{}' Y - a^{-1} \overset{\mathbf{ss}}{X}{}' X - a^{-1} \overset{\mathbf{s}}{X}{}' \overset{\mathbf{s}}{X} \\
&\quad - b^{-2} \overset{\mathbf{s}}{X}{}' Y Y' \overset{\mathbf{s}}{X} + 2a^{-2} \overset{\mathbf{s}}{X}{}' X X' \overset{\mathbf{s}}{X}) \\
&\overset{D}{=} R^{-1} \overset{\mathbf{s}}{R} \overset{\mathbf{s}}{R} + a^{-\frac{1}{2}} [\tilde{H}_x' \mathcal{R}_u v] - (Ra)^{-1} \overset{\mathbf{s}}{X}{}' (R^2 I_\nu + vv' - 2R^2 uu') \overset{\mathbf{s}}{X}, \\
\overset{\mathbf{tt}}{R} &\overset{D}{=} R^{-1} \overset{\mathbf{t}}{R} \overset{\mathbf{t}}{R} + c^{-\frac{1}{2}} [\tilde{H}_y' \mathcal{R}_v u] - (Rc)^{-1} \overset{\mathbf{t}}{Y}{}' (R^2 I_\nu + uu' - 2R^2 vv') \overset{\mathbf{t}}{Y}, \\
\overset{\mathbf{st}}{R} &= R^{-1} \overset{\mathbf{s}}{R} \overset{\mathbf{t}}{R} + b^{-1} \overset{\mathbf{s}}{X}{}' (R I_\nu - vu') \overset{\mathbf{t}}{Y},
\end{aligned} \tag{4.3}$$

where $[\tilde{H}_x' \mathcal{R}_u v]$ is understood to be the $M \times M$ matrix equal to the linear combination of the components of $\tilde{H}_x$ corresponding to each component of X with the components of $\mathcal{R}_u v$. $[\tilde{H}_y' \mathcal{R}_v u]$ is defined in a similar way. Define H_x and H_y to be $M \times M$ matrices such that

$$H_x = (1 - R^2)^{-\frac{1}{2}} \Lambda_x^{-\frac{1}{2}} [\tilde{H}_x' \mathcal{R}_u v] \Lambda_x^{-\frac{1}{2}}, \quad H_y = (1 - R^2)^{-\frac{1}{2}} \Lambda_y^{-\frac{1}{2}} [\tilde{H}_y' \mathcal{R}_v u] \Lambda_y^{-\frac{1}{2}}, \tag{4.4}$$

so that $H_x \sim \mathrm{Normal}_{M \times M}(0, V(I_M))$ and $H_y \sim \mathrm{Normal}_{N \times N}(0, V(I_N))$ independent of a, c, R. Let

$$w_x = \Lambda_x^{-\frac{1}{2}} (\overset{\mathbf{s}}{X}{}' u), \quad w_y = \Lambda_y^{-\frac{1}{2}} (\overset{\mathbf{t}}{Y}{}' v).$$

It is easy to see that conditional on R, u, v then $z_x, w_x \sim \mathrm{Normal}_N(0, I_M)$ and $z_y, w_y \sim \mathrm{Normal}_N(0, I_N)$, all independently. Let W be the $\nu \times 2$ matrix whose columns are u and v, and let $\mathcal{R} = I_\nu - W(W'W)^{-1}W'$. Define

$$Q_x = \Lambda_x^{-\frac{1}{2}} (\overset{\mathbf{s}}{X}{}' \mathcal{R} \overset{\mathbf{s}}{X}) \Lambda_x^{-\frac{1}{2}}, \quad Q_y = \Lambda_y^{-\frac{1}{2}} (\overset{\mathbf{t}}{Y}{}' \mathcal{R} \overset{\mathbf{t}}{Y}) \Lambda_y^{-\frac{1}{2}}, \quad Q_{xy} = \Lambda_x^{-\frac{1}{2}} (\overset{\mathbf{s}}{X}{}' \mathcal{R} \overset{\mathbf{t}}{Y}) \Lambda_y^{-\frac{1}{2}}.$$

Then conditional on R, u, v,

$$\begin{pmatrix} Q_x & Q_{xy} \\ Q_{xy}' & Q_y \end{pmatrix} \sim \text{Wishart}_D(I_D, \nu - 2).$$

Furthermore, since $\mathcal{R}u = \mathcal{R}\mathcal{R}_u v = \mathcal{R}v = \mathcal{R}\mathcal{R}_v u = 0$ then Q_x, Q_{xy}, Q_y are also independent of z_x, z_y, w_x, w_y conditional on R, u, v. Note that

$$\begin{aligned}
\overset{\mathbf{s}}{X}'(R^2 I_\nu - 2R^2 uu' + vv')\overset{\mathbf{s}}{X} &= \Lambda_x^{\frac{1}{2}}[R^2 Q_x + z_x z_x' + (1-R^2)^{\frac{1}{2}} R(z_x w_x' + w_x z_x')]\Lambda_x^{\frac{1}{2}}, \\
\overset{\mathbf{t}}{Y}'(R^2 I_\nu + uu' - 2R^2 vv')\overset{\mathbf{t}}{Y} &= \Lambda_y^{\frac{1}{2}}[R^2 Q_y + z_y z_y' + (1-R^2)^{\frac{1}{2}} R(z_y w_y' + w_y z_y')]\Lambda_y^{\frac{1}{2}}, \\
\overset{\mathbf{s}}{X}'(R I_\nu - vu')\overset{\mathbf{t}}{Y} &= \Lambda_x^{\frac{1}{2}}(R Q_{xy} - z_x z_y')\Lambda_y^{\frac{1}{2}}, \qquad (4.5)
\end{aligned}$$

then **(b)** follows after combining (4.3), (4.4) and (4.5). □

5 The expected Euler characteristic for the cross correlation field

We shall derive results for general Λ_x and Λ_y. The main part is to evaluate $\text{E}\{\det(\ddot{R}) \mid \dot{R} = 0, R = r\}$. Conditional on $R = r$, $\dot{R} = 0$, a, c, Lemma 4.2 has written $\ddot{R}$ as a sum of two independent random matrices $A + B$ where A is composed of Q_x, Q_y, Q_{xy} and B of H_x, H_y. Our method of attack is to break down its determinant into a sum of determinants of principal minors of A times determinants of principal minors of B. The result is given in Lemma A.4. To complete the calculation, this is combined with $\dot{R}_D$ and the density of $\dot{R}_{|D-1}$ in Theorem 5.1 and the final result is obtained by taking expectations over a and c.

THEOREM 5.1 *Let $D = M + N$. For $\nu > D$, $N > 0$, the expected EC intensity for the cross correlation field is*

$$\begin{aligned}
\rho_{M,N}(r) &= \det(\Lambda_x)^{\frac{1}{2}} \det(\Lambda_y)^{\frac{1}{2}} \frac{2^{\nu-2-D}}{\pi^{\frac{D}{2}+1}} \sum_{i=0}^{\lfloor M/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n-2j} (-1)^{i+j+k} r^{D-1-2i-2j-2k} \\
&\times (1-r^2)^{\frac{\nu-D-1}{2}+i+j+k} \frac{\Gamma(\frac{\nu-M}{2}+i)\Gamma(\frac{\nu-N}{2}+j) M! n!}{i! j! k! (\nu - D - 1 + 2i + 2j + k)! (M - 2i - k)! (n - 2j - k)!},
\end{aligned}$$

where $n = N - 1$.

PROOF. For simplicity we shall assume that $\Lambda_x = I_M$ and $\Lambda_y = I_N$. The result can be easily generalized to any Λ_x and Λ_y by a simple change of coordinates of $\mathbf{s}$ and $\mathbf{t}$. Let $d = D - 1$. From Lemma 4.2,

$$\text{E}\{\dot{R}_D^+ \mid R = r, a, c\} = (2\pi)^{-\frac{1}{2}}(1-r^2)^{\frac{1}{2}} c^{-\frac{1}{2}},$$

and conditional on a, c,

$$\phi_{|d}(0, r \mid a, c) = \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu-1}{2})} 2^{-\frac{d}{2}} \pi^{-\frac{D}{2}} (1-r^2)^{-\frac{D-\nu+2}{2}} a^{\frac{M}{2}} c^{\frac{n}{2}}.$$

From the same lemma we have conditional on $R = r, \dot{R}_{|d} = 0, a, c$,

$$\ddot{R}_{|d} = (1-r^2)^{\frac{1}{2}} \begin{pmatrix} a^{-\frac{1}{2}} H_x & 0 \\ 0 & c^{-\frac{1}{2}} H_{y_{|n}} \end{pmatrix} + \begin{pmatrix} -ra^{-1} Q_x & (ac)^{-\frac{1}{2}} Q_{xy_{|n}} \\ (ac)^{-\frac{1}{2}} Q'_{xy_{|n}} & -rc^{-1} Q_{y_{|n}} \end{pmatrix},$$

independent of $\dot{R}_D$. Applying Lemma A.4 with N replaced by n, D replaced by d, ν replaced by $\nu-2$, β_1 replaced by $(1-r^2)^{\frac{1}{2}} a^{-\frac{1}{2}}$, β_2 replaced by $(1-r^2)^{\frac{1}{2}} c^{-\frac{1}{2}}$, α_1 by $-ra^{-1}$, α_2 by $(ac)^{-\frac{1}{2}}$, α_3 by $-rc^{-1}$, and α_4 by $1-1/r^2$,

$$\begin{aligned} \mathrm{E}\left\{\det(-\ddot{R}_{|d}) \mid \dot{R}_{|d} = 0, R = r, a, c\right\} &= \sum_{i=0}^{\lfloor M/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n-2j} (-1)^{i+j+k} 2^{-(i+j)} r^{d-2i-2j-2k} (1-r^2)^{i+j+k} \\ &\quad \times a^{-M+i} c^{-n+j} \frac{(\nu-2)! M! n!}{i! j! k! (\nu-2-d+2i+2j+k)! (M-2i-k)! (n-2j-k)!}. \end{aligned}$$

Note that $\dot{R}_D$ and $\ddot{R}_{|d}$ are independent conditional on $\dot{R}_{|d} = 0, R, a, c$, hence putting these together gives:

$$\begin{aligned} &\mathrm{E}\left\{\dot{R}_D^+ \det(-\ddot{R}_{|d}) \mid \dot{R}_{|d} = 0, R = r, a, c\right\} \phi_{|d}(0, r \mid a, c) \\ &= \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu-1}{2})} 2^{-\frac{D}{2}} \pi^{-\frac{D+1}{2}} \sum_{i=0}^{\lfloor M/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n-2j} (-1)^{i+j+k} 2^{-(i+j)} r^{d-2i-2j-2k} (1-r^2)^{-\frac{D-\nu+1}{2}+i+j+k} \\ &\times a^{-\frac{M}{2}+i} c^{-\frac{N}{2}+j} \frac{(\nu-2)! M! n!}{i! j! k! (\nu-2-d+2i+2j+k)! (M-2i-k)! (n-2j-k)!}. \end{aligned}$$

Note that $\mathrm{E}\{a^l\} = \mathrm{E}\{c^l\} = 2^l \Gamma(\frac{\nu}{2}+l)/\Gamma(\frac{\nu}{2})$ and so taking expectations over a and c gives:

$$\begin{aligned} \rho_{M,N}(r) &= \mathrm{E}\left\{\dot{R}_D^+ \det(-\ddot{R}_{|d}) \mid R = r, \dot{R}_{|d} = 0\right\} \phi_{|d}(0, r) \\ &= 2^{-D} \pi^{-\frac{D+1}{2}} \sum_{i=0}^{\lfloor M/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n-2j} (-1)^{i+j+k} r^{d-2i-2j-2k} (1-r^2)^{-\frac{D-\nu+1}{2}+i+j+k} \\ &\quad \times \frac{\Gamma(\frac{\nu-M}{2}+i)\Gamma(\frac{\nu-N}{2}+j)(\nu-2)! M! n!}{\Gamma(\frac{\nu-1}{2})\Gamma(\frac{\nu}{2}) i! j! k! (\nu-2-d+2i+2j+k)! (M-2i-k)! (n-2j-k)!}. \end{aligned}$$

The result then follows by the factorization $x! = 2^x \Gamma\{(x+2)/2\}\Gamma\{(x+1)/2\}/\pi^{\frac{1}{2}}$. □

For the applications in Section 10, we need the EC intensities up to three dimensions. Evaluating the summations in Theorem 5.1 is not easy to do without making algebraic mistakes, so we give explicit expressions checked using a computer algebra package (MAPLE) in the following corollary.

Corollary 5.2 *The EC intensities $\rho_{M,N}(r)$ for $M \le 3, N \le 3$ are:*

$$\begin{aligned} \rho_{0,0}(r) &= \int_r^\infty \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}} \Gamma(\frac{\nu-1}{2})} (1-u^2)^{\frac{\nu-3}{2}} du, \\ \rho_{0,1}(r) &= \det(\Lambda_y)^{\frac{1}{2}} (2\pi)^{-1} (1-r^2)^{\frac{\nu-2}{2}}, \end{aligned}$$

$$
\begin{aligned}
\rho_{0,2}(r) &= \det(\Lambda_y)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})}{2\pi^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})} r(1-r^2)^{\frac{\nu-3}{2}}, \\
\rho_{0,3}(r) &= \det(\Lambda_y)^{\frac{1}{2}}(2\pi)^{-2}(1-r^2)^{\frac{\nu-4}{2}}\left[(\nu-1)r^2-1\right], \\
\rho_{1,1}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2\pi^{\frac{3}{2}}\Gamma(\frac{\nu-2}{2})} r(1-r^2)^{\frac{\nu-3}{2}}, \\
\rho_{1,2}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}}(2\pi)^{-2}(1-r^2)^{\frac{\nu-4}{2}}\left[(\nu-2)r^2-1\right], \\
\rho_{1,3}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-1}{2})}{2^2\pi^{\frac{5}{2}}\Gamma(\frac{\nu-2}{2})} r(1-r^2)^{\frac{\nu-5}{2}}\left[(\nu-1)r^2-3\right], \\
\rho_{2,2}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-2}{2})}{2^3\pi^{\frac{5}{2}}\Gamma(\frac{\nu-1}{2})} r(1-r^2)^{\frac{\nu-5}{2}}\left[(\nu-2)^2r^2-(3\nu-8)\right], \\
\rho_{2,3}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}}(2\pi)^{-3}(1-r^2)^{\frac{\nu-6}{2}}\left[(\nu-1)(\nu-2)r^4-3(2\nu-5)r^2+3\right], \\
\rho_{3,3}(r) &= \det(\Lambda_x)^{\frac{1}{2}}\det(\Lambda_y)^{\frac{1}{2}} \frac{\Gamma(\frac{\nu-3}{2})}{2^4\pi^{\frac{7}{2}}\Gamma(\frac{\nu-2}{2})} r(1-r^2)^{\frac{\nu-7}{2}} \\
&\quad \times\left[(\nu-1)^2(\nu-3)r^4-2(\nu-3)(5\nu-11)r^2+3(5\nu-17)\right].
\end{aligned}
$$

6 The size of the largest connected component of the excursion set for the cross correlation field

In this section, we are interested in the limiting distribution of the size or Lebesgue measure $C_{\max}$ of the largest connected component of the excursion set of the cross correlation field $R(\mathbf{s},\mathbf{t})$ above level r as $r \to 1$. As explained in the introduction of the paper, this is often used as an alternative test statistic for localized signal in a random field. We approach this by studying the distribution of the size C of individual connected components of the excursion set and then use the Poisson clumping heuristic (Aldous, 1989) to obtain an approximation of the distribution of $C_{\max}$.

To obtain the limiting distribution of C, we shall study the behaviour of $R(\mathbf{s},\mathbf{t})$ in the vicinity of high local maxima. This is done by studying the behaviour of the conditional field $R_r(\mathbf{s},\mathbf{t})$, defined as $R(\mathbf{s},\mathbf{t}) \parallel E_r$, where E_r is the event that $R(\mathbf{s},\mathbf{t})$ has a local maximum at $\mathbf{s}=0, \mathbf{t}=0$ with height r approaching 1. The essential tool for this study is the notion of horizontal window (HW) conditioning which is introduced by Kac and Slepian (1959). It is a conditioning based on the ergodic-sense Slepian model process [see Adler (1981), Section 6.5, 6.6, and Lindgren (1972) for illustrations and explanations]. To distinguish the HW conditioning from the conditioning in the usual sense, we shall use the notation $\parallel$ introduced in Aronowich and Adler (1986, 1988) to denote it.

In what follows, we shall consider the case when $M, N > 0$. For cases when $M = 0$ or $N = 0$, the cross correlation field can be transformed to a t field so that the results on C and $C_{\max}$ from Cao (1998) for a t field apply.

LEMMA 6.1 *Given that the cross correlation field $R(\mathbf{s},\mathbf{t})$ with $\nu > M, \nu > N$ has a local maximum at $\mathbf{s}=0, \mathbf{t}=0$ with height r, then as $r \to 1$, the limiting HW conditional* *distribution of* $\ddot{R}$ *at* $\mathbf{s}=0, \mathbf{t}=0$ *is*

$$\ddot{R} \parallel R=r, \dot{R}=0, \ddot{R}<0 \ \text{ converges to } \ -\begin{pmatrix} a^{-\frac{1}{2}}\Lambda_x^{\frac{1}{2}} & 0 \\ 0 & c^{-\frac{1}{2}}\Lambda_y^{\frac{1}{2}} \end{pmatrix} Q \begin{pmatrix} a^{-\frac{1}{2}}\Lambda_x^{\frac{1}{2}} & 0 \\ 0 & c^{-\frac{1}{2}}\Lambda_y^{\frac{1}{2}} \end{pmatrix},$$

where $a \sim \chi^2_{\nu-M}, c \sim \chi^2_{\nu-N}$ *and* $Q \sim \mathrm{Wishart}_D(I_D, \nu)$ *independently.*

PROOF. For simplicity we shall assume $\Lambda_x = I_M$ and $\Lambda_y = I_N$. Since $\mathbf{s}=0, \mathbf{t}=0$ is a local maximum of R of height r, then by Lemma 4.2, at $\mathbf{s}=0, \mathbf{t}=0$,

$$\ddot{R} = (1-r^2)^{\frac{1}{2}} \begin{pmatrix} a^{-\frac{1}{2}}H_x & 0 \\ 0 & c^{-\frac{1}{2}}H_y \end{pmatrix} + \begin{pmatrix} -ra^{-1}Q_x & (ac)^{-\frac{1}{2}}Q_{xy} \\ (ac)^{-\frac{1}{2}}Q'_{xy} & -rc^{-1}Q_y \end{pmatrix}.$$

Let Q be the matrix composed of $Q_x, -Q_{xy}, -Q'_{xy}, Q_y$ in the same way as in Lemma 4.2. Following the same arguments as in the proof of Theorem 6.3.1 of Adler (1981), page 134, we can see that

$$\ddot{R} \ \text{ converges to } \ -\begin{pmatrix} a^{-\frac{1}{2}}I_M & 0 \\ 0 & c^{-\frac{1}{2}}I_N \end{pmatrix} Q \begin{pmatrix} a^{-\frac{1}{2}}I_M & 0 \\ 0 & c^{-\frac{1}{2}}I_N \end{pmatrix}$$

as $r \to 1$. We now need to find the joint HW conditional distribution of a, c, Q. For any symmetric matrix A, let $A \prec 0$ denote the condition that A is negative definate, and let A^- equal A if $A \prec 0$ and zero otherwise. By applying the corollary in Section 5.1 and the results of Section 6.5 and 6.6 from Adler (1981), the joint HW conditional distribution of a, c and Q at $\mathbf{s}=0, \mathbf{t}=0$ is:

$$\begin{aligned} & f_{a,c,Q}(a,c,Q \parallel R=r, \dot{R}=0, \ddot{R} \prec 0) \\ = \ & \mu^{-1}\mathrm{E}\left(\det(-\ddot{R}^-)\Big| R=r, \dot{R}=0, a, c, Q\right) f(r,0,a,c,Q) \\ \to \ & \mu^{-1}a^{-M}c^{-N}\det(Q) f(r,0,a,c,Q), \end{aligned}$$

where $f(R, \dot{R}, a, c, Q)$ is the joint distribution of $R, \dot{R}, a, c, Q$ and μ is the derivative of the expected number of local maxima of R above s as a function of s at a level $s=r$. By Lemma 4.2,

$$f(r,0,a,c,Q) \ = \ f(r)f(a)f(c)f(Q)f(0|r,a,c),$$

where $f(\cdot|r,a,c)$ is the density of $\dot{R}$ conditional on $R=r, a, c$ that follows

$$\dot{R}|r,a,c \sim \mathrm{Normal}_D\left(0, (1-r^2)\begin{pmatrix} a^{-1}I_M & 0 \\ 0 & c^{-1}I_N \end{pmatrix}\right).$$

Hence as $r \to 1$,

$$\begin{aligned} & f_{a,c,Q}(a,c,Q \parallel R=r, \dot{R}=0, \ddot{R} \prec 0) \\ \propto \ & a^{\frac{\nu-M}{2}-1}e^{-\frac{a}{2}}c^{\frac{\nu-N}{2}-1}e^{-\frac{c}{2}}\det(Q)^{\frac{\nu-M-N-1}{2}}\exp(\mathrm{tr}(-\tfrac{1}{2}Q)), \end{aligned}$$

which proves the lemma. □

Applying Lemma 6.7.2 of Adler (1981), we can represent the finite-dimensional distribution of $R_r(\mathbf{s},\mathbf{t})$ in terms of distributions $\ddot{R}(0,0) \parallel R(0,0)=r, \dot{R}(0,0)=0, \ddot{R}(0,0) \prec 0$, and $R(\mathbf{s},\mathbf{t}) \mid R(0,0), \dot{R}(0,0), \ddot{R}(0,0)$. For the first term, we have already obtained its limiting distribution in Lemma 6.1. We shall now derive the conditional distribution in the second term and combine these results to give the limiting sample path behaviour of the conditional field $R_r(\mathbf{s},\mathbf{t})$ near high local maxima. First, we need the following lemma for smooth stationary Gaussian fields from Adler (1981) and Cao (1998).

LEMMA 6.2 *Let $X(\mathbf{s})$, $\mathbf{s} \in \Re^M$ be a smooth stationary Gaussian random field satisfying the regularity conditions, then conditional on $X(0), \dot{X}(0)$ and $\ddot{X}(\mathbf{0})$, $X(\mathbf{s})$ has the same distribution as*

$$X(0) + \dot{X}'(0)\mathbf{s} + \tfrac{1}{2}\mathbf{s}'\ddot{X}(0)\mathbf{s} + \epsilon_X(\mathbf{s}) + K_X(\mathbf{s}),$$

where $\epsilon_X(\mathbf{s}) = m \cdot o(\|\mathbf{s}\|^2)$ is deterministic with $m = \max\{|X(0)|, \|\dot{X}(0)\|, \|\ddot{X}(0)\|\}$, and $K_X(\mathbf{s})$ is normal with mean 0 and covariance of $o(\|\mathbf{s}\|^4)$ for $\mathbf{s}$ near 0.

We shall say that a sequence of surfaces Y_u converges to a surface Y^* *uniformly* as $u \to a$ if, for each $h > 0$ and $\|\mathbf{t}\| < h$, we have

$$\mathrm{P}\left\{\lim_{u\to a} \sup_{\|\mathbf{t}\|<h} |Y_u(\mathbf{t}) - Y^*(\mathbf{t})| = 0\right\} = 1.$$

THEOREM 6.3 *Given that a cross correlation field $R(\mathbf{s},\mathbf{t})$ with $\nu > M, \nu > N$ has a local maximum at $\mathbf{s}=0, \mathbf{t}=0$ with height r, then as $r \to 1$,*

$$l^{-2}(1 - R_r^2(l\mathbf{s}, l\mathbf{t})) \text{ converges uniformly to } 1 + \begin{pmatrix} a^{-\frac{1}{2}}\Lambda_x^{\frac{1}{2}}\mathbf{s} \\ c^{-\frac{1}{2}}\Lambda_y^{\frac{1}{2}}\mathbf{t} \end{pmatrix}' Q \begin{pmatrix} a^{-\frac{1}{2}}\Lambda_x^{\frac{1}{2}}\mathbf{s} \\ c^{-\frac{1}{2}}\Lambda_y^{\frac{1}{2}}\mathbf{t} \end{pmatrix},$$

where $l = \sqrt{1-r^2}$, and $a \sim \chi^2_{\nu-M}, c \sim \chi^2_{\nu-N}$, $Q \sim \mathrm{Wishart}_D(\mathbf{I}_D, \nu)$ independently.

PROOF. We shall prove this for the case $\Lambda_x = I_M$, $\Lambda_y = I_N$. Let $X_i(\mathbf{s}), Y_i(\mathbf{t}), i = 1, ..., \nu$ be the Gaussian component fields in the definition of the cross correlation field $R(\mathbf{s},\mathbf{t})$ (1.1). Let

$$a(\mathbf{s}) = X(\mathbf{s})'X(\mathbf{s}), \quad b(\mathbf{s},\mathbf{t}) = X(\mathbf{s})Y(\mathbf{t}), \quad c(\mathbf{t}) = Y'(\mathbf{t})Y(\mathbf{t}),$$

and for simplicity if $\mathbf{s}=0, \mathbf{t}=0$, we shall drop the dependence on $\mathbf{s},\mathbf{t}$, writing

$$a(0) = a, \quad b(0,0) = b, \quad c(0) = c.$$

Let $l = \sqrt{1-r^2}$. It is easy to see from Lemma 6.2 that

$$a(l\mathbf{s}) = \left\{X_i(0) + l\dot{X}_i'(0)\mathbf{s} + \frac{l^2}{2}\mathbf{s}'\ddot{X}_i(0)\mathbf{s} + \epsilon_{X_i}(l\mathbf{s}) + K_{X_i}(l\mathbf{s})\right\}^2.$$

As $r \to 1$ and hence $l \to 0$, $K_{X_i}(l\mathbf{s})/l^2$ goes to 0 uniformly on $\|\mathbf{s}\| < h_x$ for any $h_x > 0$, since K_{X_i} is twice differentiable and $\mathrm{E}(K^2_{X_i}(l\mathbf{s})) = o(l^4)$. Furthermore, given that $\mathbf{s}=0, \mathbf{t}=0$ is a local maximum of $R(\mathbf{s},\mathbf{t})$, it can be shown that the limiting HW distribution of X_i and its derivatives at $\mathbf{s}=0$ are bounded with probability one as $r \to 1$. Hence,

$$a(l\mathbf{s}) = a + l\mathbf{s}'\overset{\mathbf{s}}{a} + \frac{l^2}{2}\mathbf{s}'\overset{\mathbf{ss}}{a}\mathbf{s} + \mathrm{error}_x, \tag{6.1}$$

where error_x/l^2 converges to 0 uniformly on $\|\mathbf{s}\| < h_x$. Applying the same arguments, it can be easily shown that

$$c(l\mathbf{t}) = c + l\mathbf{t}'\overset{\mathbf{t}}{c} + \frac{l^2}{2}\mathbf{t}'\overset{\mathbf{tt}}{c}\mathbf{t} + \mathrm{error}_y, \tag{6.2}$$

$$b(l\mathbf{s}, l\mathbf{t}) = b + l\begin{pmatrix}\mathbf{s}\\ \mathbf{t}\end{pmatrix}'\dot{b} + \frac{l^2}{2}\begin{pmatrix}\mathbf{s}\\ \mathbf{t}\end{pmatrix}'\ddot{b}\begin{pmatrix}\mathbf{s}\\ \mathbf{t}\end{pmatrix} + \mathrm{error}_{xy}, \tag{6.3}$$

where error_y/l^2 and error_{xy}/l^2 converges to 0 uniformly on $\|\mathbf{s}\| < h_x, \|\mathbf{t}\| < h_y$. Following the proof of Lemma 4.2 and using the same notation as in the lemma, we have at the local maximum $\mathbf{s}=0, \mathbf{t}=0$,

$$\begin{aligned}
\overset{\mathbf{s}}{a} &= 2a^{\frac{1}{2}}w_x, \quad \overset{\mathbf{t}}{c} = 2c^{\frac{1}{2}}w_y, \quad \overset{\mathbf{s}}{b} = rc^{\frac{1}{2}}w_x, \quad \overset{\mathbf{t}}{b} = ra^{\frac{1}{2}}w_y,\\
\overset{\mathbf{ss}}{a} &= 2(Q_x + w_x w_x' - aI_M + a^{\frac{1}{2}}G_x), \quad \overset{\mathbf{ss}}{b} = -a^{\frac{1}{2}}c^{\frac{1}{2}}rI_M + c^{\frac{1}{2}}\{(1-r^2)^{\frac{1}{2}}H_x + rG_x\},\\
\overset{\mathbf{tt}}{c} &= 2(Q_y + w_y w_y' - cI_N + c^{\frac{1}{2}}G_y), \quad \overset{\mathbf{tt}}{b} = -a^{\frac{1}{2}}c^{\frac{1}{2}}rI_N + a^{\frac{1}{2}}\{(1-r^2)^{\frac{1}{2}}H_y + rG_y\},\\
\overset{\mathbf{st}}{b} &= Q_{xy} + rw_x w_y',
\end{aligned}$$

where G_x, G_y are defined as

$$G_x = [\tilde{H}_x' u], \quad G_y = [\tilde{H}_y' v].$$

Plugging the above equalities in (6.1), (6.2) and (6.3), it can be shown that

$$\frac{a(\mathbf{s})c(\mathbf{t}) - b(\mathbf{s},\mathbf{t})^2}{l^2 a(\mathbf{s})c(\mathbf{t})} \text{ converges to } 1 + a^{-1}\mathbf{s}'Q_x\mathbf{s} + c^{-1}\mathbf{t}'Q_y\mathbf{t} - 2a^{-\frac{1}{2}}c^{-\frac{1}{2}}\mathbf{s}'Q_{xy}\mathbf{t}$$

uniformly as $r \to 1$ for $\|\mathbf{s}\| < h_x, \|\mathbf{t}\| < h_y$. Define Q as in previous cases to be the matrix composed of $Q_x, -Q_{xy}, -Q_{xy}', Q_y$ (see Lemma 4.2), then the theorem follows by using the HW conditional distributions of a, c, Q derived in Lemma 6.1. □

Let C be the size of one connected component of the excursion set of $R(\mathbf{s},\mathbf{t})$ above level r. We now have the following theorem about its limiting distribution as $r \to 1$.

THEOREM 6.4 *For the cross correlation field $R(\mathbf{s},\mathbf{t})$ with $\nu > M, \nu > N$, as $r \to 1$,*

$$(1-r^2)^{-\frac{D}{2}}C \to b_D \det(\Lambda_x)^{-\frac{1}{2}}\det(\Lambda_y)^{-\frac{1}{2}}a^{\frac{M}{2}}c^{\frac{N}{2}}B^{\frac{D}{2}}\det(Q)^{-\frac{1}{2}},$$

where b_D is the Lebesgue measure of the D-dimensional unit ball, $a \sim \chi^2_{\nu-M}$, $c \sim \chi^2_{\nu-N}$, $B \sim \mathrm{Beta}(1, \frac{\nu-D-1}{2})$ and $Q \sim \mathrm{Wishart}_D(I_D, \nu)$ independently.

PROOF. Assume $\Lambda_x = I_M, \Lambda_y = I_N$. Let J_0 be the connected component of the excursion set of $R(\mathbf{s}, \mathbf{t})$ above level r that contains $\mathbf{s} = 0, \mathbf{t} = 0$ as a local maximum. We are interested in the distribution of C defined as the Lebesgue measure of J_0, i.e., $C = |J_0|$. By Theorem 6.3, with probability approaching one as $r \to 1$, $R(\mathbf{s}, \mathbf{t})$ has the following representation over J_0:

$$R^2(\mathbf{s}, \mathbf{t}) = R^2(0,0) - (a^{-\frac{1}{2}}\mathbf{s}', c^{-\frac{1}{2}}\mathbf{t}')Q(a^{-\frac{1}{2}}\mathbf{s}', c^{-\frac{1}{2}}\mathbf{t}')' + o(1 - r^2).$$

Hence, we can represent J_0 by

$$J_0 = \left\{ (\mathbf{s}, \mathbf{t}) : (1 - r^2)^{-1} \begin{pmatrix} a^{-\frac{1}{2}}\mathbf{s} \\ c^{-\frac{1}{2}}\mathbf{t} \end{pmatrix}' Q \begin{pmatrix} a^{-\frac{1}{2}}\mathbf{s} \\ c^{-\frac{1}{2}}\mathbf{t} \end{pmatrix} + o(1) \leq \frac{R(0,0)^2 - r^2}{1 - r^2} \right\}.$$

Given that $\mathbf{s} = 0, \mathbf{t} = 0$ is a local maximum with height greater than r, it can be easily shown using similar arguments as in Lemma 6.1 that the distribution of $(R(0,0)^2 - r^2)/(1 - r^2)$ converges to $\mathrm{Beta}(1, \frac{1}{2}(\nu - D - 1))$ as $r \to 1$, independent of a, c, Q. The theorem then follows from Theorem 6.3, and the fact that Lebesgue measure is continuous with respect to almost sure convergence. $\square$

Let L be the number of connected components of the excursion set of the cross correlation field $R(\mathbf{s}, \mathbf{t})$ above level r and $C_{\max}$ be the size of the largest one of them. Then the following approximations from Friston *et al.* (1994) and Cao (1998) also hold for high level excursion sets of the cross correlation field:

$$\mathrm{P}(C_{\max} \leq x) \mid L \geq 1) \;\approx\; \frac{\exp\{\mathrm{E}(L)\mathrm{P}(C \geq x)\} - \exp\{-\mathrm{E}(L)\}}{1 - \exp\{-\mathrm{E}(L)\}} \tag{6.4}$$

This approximation is obtained via the Poisson clumping heuristic (Aldous; 1989): for high threshold r, the centers of individual connected components can be seen as generated from a multidimensional Poisson process. For smooth Gaussian random fields, Adler (1981) gave a detailed treatment of this heuristic in Section 6.9 and more references will follow from there. Often, we approximate $\mathrm{E}(L)$ by $\mathrm{E}(\chi(A_r))$, where A_r is the excursion set of the field above r. For the cross correlation field, this can be easily obtained by applying Theorem 5.1 and Lemma 3.1. As in Friston *et al.* (1994), we can also correct for the mean of C to improve the overall approximation of the distribution of $C_{\max}$ using what Aldous (1989) referred as the *fundamental identity*:

$$\mathrm{E}(L)\mathrm{E}(C) = \mathrm{E}(|A_r|). \tag{6.5}$$

In the case of the cross correlation field, the right hand side of the above equality is $|S|\mathrm{P}(R \geq r) = |S|F_{\nu-1}(-\sqrt{\nu - 1}r(1 - r^2)^{-\frac{1}{2}})$, where $F_{\nu-1}(\cdot)$ is the cumulative distribution function of a t distribution with $\nu - 1$ degrees of freedom.

7 Representations of the first two derivatives of the homologous correlation field

Let Λ_x and Λ_y be the variance matrices of the first derivative of the components of $X(\mathbf{t})$ and $Y(\mathbf{t})$ respectively.

LEMMA 7.1 *R and its first two derivatives can be written in terms of independent random variables as follows:*

$$\begin{aligned}\dot R &\stackrel{D}{=} (1-R^2)^{\frac12}(a^{-\frac12}\Lambda_x^{\frac12}z_x + c^{-\frac12}\Lambda_y^{\frac12}z_y)\\ \ddot R &\stackrel{D}{=} -R(1-R^2)^{-1}\dot R\dot R' + (1-R^2)^{\frac12}(a^{-\frac12}\Lambda_x^{\frac12}H_x\Lambda_x^{\frac12} + c^{-\frac12}\Lambda_y^{\frac12}H_y\Lambda_y^{\frac12})\\ &\quad -a^{-1}\Lambda_x^{\frac12}[RQ_x + (1-R^2)^{\frac12}(z_xw_x' + w_xz_x')]\Lambda_x^{\frac12} + (ac)^{-\frac12}(\Lambda_x^{\frac12}Q_{xy}\Lambda_y^{\frac12} + \Lambda_y^{\frac12}Q_{xy}'\Lambda_x^{\frac12})\\ &\quad -c^{-1}\Lambda_y^{\frac12}[RQ_y + (1-R^2)^{\frac12}(z_yw_y' + w_yz_y')]\Lambda_y^{\frac12},\end{aligned}$$

where

$$z_x, z_y, w_x, w_y \sim \mathrm{Normal}_N(0, I_N), \qquad H_x, H_y \sim \mathrm{Normal}_{N\times N}(0, V(I_N)),$$
$$a, c \sim \chi^2_\nu, \qquad \begin{pmatrix} Q_x & Q_{xy}\\ Q_{xy}' & Q_y\end{pmatrix} \sim \mathrm{Wishart}_{2N}(I_{2N}, \nu-2),$$

independently, and independent of R.

PROOF. The result follows immediately from the more general representations for derivatives of the cross correlation field $R(\mathbf{s},\mathbf{t})$ given in Lemma 4.2, on setting $\mathbf{s}=\mathbf{t}$. □

7.1 Isotropic fields

The representations for the derivatives given by Lemma 7.1 can be simplified if the component random fields are isotropic. We shall now assume that $\Lambda_x = \lambda_x I_N$ and $\Lambda_y = \lambda_y I_N$, where λ_x and λ_y are non-negative scalars.

LEMMA 7.2

$$\begin{aligned}\dot R &\stackrel{D}{=} (1-R^2)^{\frac12}s_1^{\frac12}z_1\\ \ddot R &\stackrel{D}{=} -Rs_1z_1z_1' + (1-R^2)^{\frac12}s_3^{\frac12}H - e_1Q_1 - e_2Q_2\\ &\quad -(1-R^2)^{\frac12}\{s_2^{\frac12}(z_1w_1' + w_1z_1' + z_2w_2' + w_2z_2') + s_4(z_1w_2' + w_2z_1')\},\end{aligned}$$

where

$$s_1 = \frac{\lambda_x}{a} + \frac{\lambda_y}{c}, \qquad s_2 = \frac{\lambda_x\lambda_y}{ac}, \qquad s_3 = \frac{\lambda_x^2}{a} + \frac{\lambda_y^2}{c}, \qquad s_4 = \frac{\lambda_x}{a} - \frac{\lambda_y}{c},$$

and e_1, e_2 are the solutions of the following two equations:

$$e_1 + e_2 = Rs_1, \quad e_1e_2 = -(1-R^2)s_2,$$

and

$$z_1, z_2, w_1, w_2 \sim \mathrm{Normal}_N(0, I_N), \qquad H \sim \mathrm{Normal}_{N\times N}(0, V(I_N)),$$
$$a, c \sim \chi^2_\nu, \qquad Q_1, Q_2 \sim \mathrm{Wishart}_N(I_N, \nu-2),$$

independently, and independent of R.

PROOF. Let

$$z_1 = s_1^{-\frac{1}{2}}\left(\sqrt{\frac{\lambda_x}{a}}z_x + \sqrt{\frac{\lambda_y}{c}}z_y\right), \quad z_2 = s_1^{-\frac{1}{2}}\left(\sqrt{\frac{\lambda_y}{c}}z_x - \sqrt{\frac{\lambda_x}{a}}z_y\right),$$

$$w_1 = s_1^{-\frac{1}{2}}\left(\sqrt{\frac{\lambda_y}{c}}w_x + \sqrt{\frac{\lambda_x}{a}}w_y\right), \quad w_2 = s_1^{-\frac{1}{2}}\left(\sqrt{\frac{\lambda_x}{a}}w_x - \sqrt{\frac{\lambda_y}{c}}w_y\right).$$

Then z_1, z_2, w_1, w_2 are i.i.d. $\mathrm{Normal}_N(0, I_N)$. The result for the first derivative is immediate. For the second derivative, it can be shown that

$$\begin{aligned}
&\frac{\lambda_x}{a}(z_x w_x' + w_x z_x') + \frac{\lambda_y}{c}(z_y w_y' + w_y z_y') \\
= \;& \sqrt{\frac{\lambda_x \lambda_y}{ac}}(z_1 w_1' + w_1 z_1' + z_2 w_2' + w_2 z_2') + \left(\frac{\lambda_x}{a} - \frac{\lambda_y}{c}\right)(z_1 w_2' + w_2 z_1').
\end{aligned}$$

Note that

$$\begin{aligned}
&Ra^{-1}\Lambda_x^{\frac{1}{2}} Q_x \Lambda_x^{\frac{1}{2}} + Rc^{-1}\Lambda_y^{\frac{1}{2}} Q_y \Lambda_y^{\frac{1}{2}} - (ac)^{-\frac{1}{2}}(\Lambda_x^{\frac{1}{2}} Q_{xy} \Lambda_y^{\frac{1}{2}} + \Lambda_y^{\frac{1}{2}} Q_{xy}' \Lambda_x^{\frac{1}{2}}) \\
= \;& R\left(\frac{\lambda_x}{a}Q_x + \frac{\lambda_y}{c}Q_y\right) - \sqrt{\frac{\lambda_x\lambda_y}{ac}}(Q_{xy} + Q_{xy}') \\
= \;& e_1 Q_1 + e_2 Q_2,
\end{aligned}$$

where $Q_1, Q_2 \sim \mathrm{Wishart}_N(I_N, \nu - 2)$ independently, since e_1, e_2 are the eigenvalues of

$$\begin{pmatrix} R\frac{\lambda_x}{a} & -\sqrt{\frac{\lambda_x\lambda_y}{ac}} \\ -\sqrt{\frac{\lambda_x\lambda_y}{ac}} & R\frac{\lambda_y}{c} \end{pmatrix}.$$

Finally, we have

$$a^{-\frac{1}{2}}\Lambda_x^{\frac{1}{2}} H_x \Lambda_x^{\frac{1}{2}} + c^{-\frac{1}{2}}\Lambda_y^{\frac{1}{2}} H_y \Lambda_y^{\frac{1}{2}} = \frac{\lambda_x}{\sqrt{a}}H_x + \frac{\lambda_y}{\sqrt{c}}H_y = \left(\frac{\lambda_x^2}{a} + \frac{\lambda_y^2}{c}\right)^{\frac{1}{2}} H,$$

where $H \sim \mathrm{Normal}_{N\times N}(0, V(I_N))$. Putting these together gives the result for the second derivative. □

8 The expected Euler characteristic for the homologous correlation field

To find the expected EC, again we need to evaluate expressions of the form $\mathrm{E}\{\det(\ddot{R}) \mid \dot{R} = 0, R = r\}$. The method of attack follows closely that in Section 5, but in this case a closed form expression can only be obtained for some situations. Conditional on R, a, c and $\dot{R} = 0$, Lemma 7.2 has written $\ddot{R}$ as a sum of four independent random matrices involving Q_1, Q_2, H and $z_2 w_2' + w_2 z_2'$. Like the previous case, to find expectation of its determinant, we write the expectation of $\det(A+B)$ in terms of the expectations of $\mathrm{detr}_j(A)$ and $\mathrm{detr}_{N-j}(B)$ (Lemma A.5). This is applied to Q_1 and Q_2 (Lemma A.6), then to H and $z_2 w_2' + w_2 z_2'$ (Lemma A.8). Finally we combine these two to get an expression for $\mathrm{E}\{\det(\ddot{R}) \mid \dot{R}=0, R=r,a,c\}$ involving three nested summations (Lemma A.9). The rest of the calculation is carried out in Theorem 8.1, but unlike the cross correlation field, the final integrals over a and c cannot be done analytically. Integrating over $t=a+c$ is straightforward, but the last integral over $q=a/t$ cannot be done analytically, except when $N=2$ or when $\lambda_x=\lambda_y$. Fortunately the case $\lambda_x=\lambda_y$ is the most important for the applications presented in Section 10, and so a Corollary displays the resulting EC intensities up to three dimensions ($N \le 3$).

The algebra in this section and the preceding ones is quite heavy. Careful cross-checking was done to ensure that the results were correct. One check is to compare the results with those for a t-field given in Worsley (1994). For the case of $\Lambda_y = 0$, $Y(\mathbf{t}) = Y(0)$, and so $T(\mathbf{t}) = R(\mathbf{t})\sqrt{(\nu-1)/(1-R(\mathbf{t})^2)}$ is a t-field with $m=\nu-1$ degrees of freedom. For this case, the representations for the derivatives in Lemmas 7.1 and 7.2 were checked with results for the t-field in Lemma 5.1 of Worsley (1994) (note that there is a typographical error in Lemma 5.1(b) of this paper: the first factor of m should be $m^{1/2}$). The computer algebra software Maple was used to evaluate the expression for $\rho_N(r)$ in Theorem 8.1 and to integrate over q to get the results in the Corollary. Finally, results for $\rho_N(r)$ for the case where $\lambda_y=0$ were checked with the t-field results of Theorem 5.4 of Worsley (1994).

THEOREM 8.1 *Let $n=N-1$. Then for $\nu > N$, $N>0$, the expected EC intensity for the homologous correlation field is*

$$\begin{aligned}\rho_N(r) \;=\; & \frac{\Gamma(\frac{\nu}{2}) n!}{\pi^{\frac{n+2}{2}}\Gamma(\frac{\nu-1}{2})(\nu-1)!}\sum_{i=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor - i}\sum_{k=0}^{\lfloor n/2\rfloor - i - j}(-1)^{i+j}4^{-n+i+j}r^{n-2i-2j-2k} \\ & \times (1-r^2)^{\frac{\nu-n}{2}-1+i+j}\frac{\Gamma(\nu+i-\frac{n+3}{2})}{(1+\delta_{n-2i,2j})i!}\binom{\nu-2}{j}\binom{\nu-2}{n-2i-j}\binom{n-2i-2j}{2k} \\ & \times \mathrm{E}_q\left\{\tilde{s}_1^{\frac{n+1}{2}-2i-2j-2k}\tilde{s}_2^j\tilde{s}_3^{i-1}\left[(\nu+i-\tfrac{n+3}{2})\tilde{s}_3 + i\tilde{s}_2\right]\left[r^2\tilde{s}_1^2 + 4(1-r^2)\tilde{s}_2\right]^k\right\},\end{aligned}$$

where

$$\tilde{s}_1 = \frac{\lambda_x}{q} + \frac{\lambda_y}{1-q}, \qquad \tilde{s}_2 = \frac{\lambda_x\lambda_y}{q(1-q)}, \qquad \tilde{s}_3 = \frac{\lambda_x^2}{q} + \frac{\lambda_y^2}{1-q},$$

and $q \sim \mathrm{Beta}(\nu/2, \nu/2)$.

PROOF. Let $t=a+c$ and $q=a/(a+c)$, then $t \sim \chi^2_{2\nu}$ independently of $q \sim \mathrm{Beta}(\nu/2,\nu/2)$. From Lemma 7.2

$$\mathrm{E}\{\dot{R}_N^+ \mid R=r,q,t\} = (1-r^2)^{\frac{1}{2}}t^{-\frac{1}{2}}\tilde{s}_1^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}},$$

and conditional on q,t,

$$\phi_{|n}(0,r \mid q,t) = (1-r^2)^{-\frac{n}{2}}t^{\frac{n}{2}}\tilde{s}_1^{-\frac{n}{2}}(2\pi)^{-\frac{n}{2}} \cdot \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{\nu-1}{2})}(1-r^2)^{\frac{\nu-3}{2}}.$$

From the same lemma, we have conditional on $\dot{R}_{|n}=0, R=r,q,t$,

$$\ddot{R}_{|n} = -(1-r^2)^{\frac{1}{2}}s_2^{\frac{1}{2}}(z_{2|n}w_{2|n}' + w_{2|n}z_{2|n}') + (1-r^2)^{\frac{1}{2}}s_3^{\frac{1}{2}}H_{|n} - e_1Q_{1|n} - e_2Q_{2|n}.$$

Applying Lemma A.9 with g replaced by $-(1-r^2)^{\frac{1}{2}}s_2^{\frac{1}{2}}$, h replaced by $(1-r^2)^{\frac{1}{2}}s_3^{\frac{1}{2}}$, s replaced by rs_1, p replaced by $-(1-r^2)s_2$, ν replaced by $\nu-2$ and N by n, we get:

$$\begin{aligned}
& \mathrm{E}\{\det(\ddot{R}_{|n}) \mid \dot{R}_{|n}=0, R=r,q,t\} \\
= & \sum_{i=0}^{\lfloor N/2\rfloor}\sum_{j=0}^{\lfloor N/2\rfloor-i}\sum_{k=0}^{\lfloor N/2\rfloor-i-j} \frac{(-1)^{N+i+j}N!}{2^{(N-i-2j-1)}i!(1+\delta_{N-2i,2j})}\binom{\nu}{j}\binom{\nu}{N-2i-j}\binom{N-2i-2j}{2k} \\
& \times (1-R^2)^{i+j}(R\tilde{s}_1)^{N-2i-2j-2k}\tilde{s}_2^j\tilde{s}_3^{i-1}\left[R^2\tilde{s}_1^2+4(1-R^2)\tilde{s}_2\right]^k t^{-(N-i+1)}(\tilde{s}_3t+2i\tilde{s}_2),
\end{aligned}$$

where

$$\tilde{s}_1=s_1t, \quad \tilde{s}_2=s_2t^2, \quad \tilde{s}_3=s_3t,$$

depend (stochastically) only on q. Note from Lemma 7.2, $\dot{R}_N^+$ is independent of $\ddot{R}_{|n}$ conditional on R,q,t. Therefore, putting these results together we have:

$$\begin{aligned}
& \mathrm{E}\{\dot{R}_N^+\det(-\ddot{R}_{|n}) \mid \dot{R}_{|n}=0, R=r,q,t\}\phi_{|n}(0,r\mid q,t) \\
= & \tilde{s}_1^{-\frac{n-1}{2}}(2\pi)^{-\frac{n+1}{2}}\frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{\nu-1}{2})}(1-r^2)^{\frac{\nu-2-n}{2}} \\
& \times \sum_{i=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor-i}\sum_{k=0}^{\lfloor n/2\rfloor-i-j} \frac{(-1)^{i+j}n!}{2^{(n-i-2j-1)}i!(1+\delta_{n-2i,2j})}\binom{\nu-2}{j}\binom{\nu-2}{n-2i-j}\binom{n-2i-2j}{2k} \\
& \times (1-r^2)^{i+j}(r\tilde{s}_1)^{n-2i-2j-2k}\tilde{s}_2^j\tilde{s}_3^{i-1}\left[r^2\tilde{s}_1^2+4(1-r^2)\tilde{s}_2\right]^k t^{-\frac{n-2i+3}{2}}(\tilde{s}_3t+2i\tilde{s}_2).
\end{aligned}$$

Note that $\mathrm{E}\{t^l\}=2^l\Gamma(\nu+l)/\Gamma(\nu)$ and so taking expectations over t,

$$\begin{aligned}
& \mathrm{E}\{\dot{R}_N^+\det(-\ddot{R}_{|n}) \mid \dot{R}_{|n}=0, R=r,q\}\phi_{|n}(0,r\mid q) \\
= & \tilde{s}_1^{-\frac{n-1}{2}}\frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{n+2}{2}}\Gamma(\frac{\nu-1}{2})}(1-r^2)^{\frac{\nu-2-n}{2}}\sum_{i=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor-i}\sum_{k=0}^{\lfloor n/2\rfloor-i-j}\frac{(-1)^{i+j}n!}{4^{(n-i-j)}i!(1+\delta_{n-2i,2j})} \\
& \times \left[(\nu+i-\tfrac{n+3}{2})\tilde{s}_3+i\tilde{s}_2\right]\frac{\Gamma(\nu+i-\frac{n+3}{2})}{\Gamma(\nu)}\binom{\nu-2}{j}\binom{\nu-2}{n-2i-j}\binom{n-2i-2j}{2k} \\
& \times (1-r^2)^{i+j}(r\tilde{s}_1)^{n-2i-2j-2k}\tilde{s}_2^j\tilde{s}_3^{i-1}\left[r^2\tilde{s}_1^2+4(1-r^2)\tilde{s}_2\right]^k.
\end{aligned}$$

Finally, taking expectations over q gives $\rho_N(r)$. □

There is no simple expression for this last integral over q, except for the case $N=2$. The EC intensities for $N\leq 3$ are:

Corollary 8.2

$$\begin{aligned}
\rho_0(r) &= \int_r^\infty \frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{1}{2}}\Gamma(\frac{\nu-1}{2})}(1-u^2)^{\frac{\nu-3}{2}}du, \\
\rho_1(r) &= \frac{2^{\nu-3}\Gamma(\nu-\frac{1}{2})}{\pi^{\frac{3}{2}}\Gamma(\nu-1)}(1-r^2)^{\frac{\nu-2}{2}}\int_0^1\left(\frac{\lambda_x}{q}+\frac{\lambda_y}{1-q}\right)^{\frac{1}{2}}[q(1-q)]^{\frac{\nu-2}{2}}dq,
\end{aligned}$$

$$\rho_2(r) = \frac{\lambda_x+\lambda_y}{2}\frac{\Gamma(\frac{\nu}{2})}{\pi^{\frac{3}{2}}\Gamma(\frac{\nu-1}{2})}(1-r^2)^{\frac{\nu-3}{2}}r,$$

$$\rho_3(r) = \frac{2^{\nu-5}\Gamma(\nu-\frac{3}{2})}{\pi^{\frac{5}{2}}\Gamma(\nu-1)}(1-r^2)^{\frac{\nu-4}{2}}\int_0^1\left(\frac{\lambda_x}{q}+\frac{\lambda_y}{1-q}\right)^{-\frac{1}{2}}\left\{(\nu-2)(\nu-3)r^2\left(\frac{\lambda_x}{q}+\frac{\lambda_y}{1-q}\right)^2\right.$$
$$\left.-(1-r^2)\left[(2\nu-2)\frac{\lambda_x\lambda_y}{q(1-q)}+(2\nu-3)\left(\frac{\lambda_x^2}{q}+\frac{\lambda_y^2}{1-q}\right)\right]\right\}[(q(1-q)]^{\frac{\nu-2}{2}}\,dq.$$

For two important special cases, the integrals can be evaluated:

Corollary 8.3 *For $\lambda_x=\lambda_y=\lambda$,*

$$\rho_1(r) = \frac{\lambda^{\frac{1}{2}}\Gamma(\nu-\frac{1}{2})}{2^{\nu-1}\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2})^2}(1-r^2)^{\frac{\nu-2}{2}},$$
$$\rho_3(r) = \frac{\lambda^{\frac{3}{2}}\Gamma(\nu-\frac{3}{2})}{2^{\nu+1}\pi^{\frac{3}{2}}\Gamma(\frac{\nu}{2})^2}(1-r^2)^{\frac{\nu-4}{2}}[(4\nu^2-12\nu+11)r^2-(4\nu-5)].$$

For $\lambda_x=\lambda,\ \lambda_y=0$,

$$\rho_1(r) = \frac{\lambda^{\frac{1}{2}}}{2\pi}(1-r^2)^{\frac{\nu-2}{2}},$$
$$\rho_3(r) = \frac{\lambda^{\frac{3}{2}}}{(2\pi)^2}(1-r^2)^{\frac{\nu-4}{2}}[(\nu-1)r^2-1].$$

9 The size of the largest connected component of the excursion set of the homologous correlation field

Using the approach developed in Section 6, we can also study the behaviour of the homologous correlation field $R(\mathbf{t})$ in the vicinity of high local maxima and derive the limiting distribution of the size of the individual ($\tilde{C}$) and the largest ($\tilde{C}_{\max}$) connected component of the excursion set above level r as $r\to 1$. Since there is little to gain by giving the details of the proofs, we shall simply state these results.

Let $u\sim\chi^2_{2\nu-N}$, $\tilde{Q}\sim\mathrm{Wishart}_N(I_N,\nu)$, $v\sim\mathrm{Beta}(1,\frac{\nu-N-1}{2})$ and let the density of q be

$$f_q(q)\propto[\lambda_x/q+\lambda_y/(1-q)]^{\frac{N}{2}}q^{\frac{\nu}{2}-1}(1-q)^{\frac{\nu}{2}-1}, \tag{9.1}$$

all independently distributed. Note that if $\lambda_x=\lambda_y$, then $q\sim\mathrm{Beta}(\frac{\nu-N}{2},\frac{\nu-N}{2})$; if $\lambda_y=0$, then $q\sim\mathrm{Beta}(\frac{\nu-N}{2},\frac{\nu}{2})$. Finally, let $\tilde{s}_1=\lambda_x/q+\lambda_y/(1-q)$. Given that the homologous correlation field $R(\mathbf{t}),\mathbf{t}\in\Re^N$ has a local maximum at $\mathbf{t}=0$ with height r, then as $r\to 1$, the limiting HW conditional distribution of $\ddot{R}$ at $\mathbf{t}=0$ is

$$\ddot{R}\parallel R=r,\dot{R}=0,\ddot{R}\prec 0 \text{ converges to } -u^{-1}\tilde{s}_1\tilde{Q},$$

$$l^{-2}[1-R^2(l\mathbf{t})] \text{ converges uniformly to } 1+u^{-1}\tilde{s}_1\mathbf{t}'\tilde{Q}\mathbf{t},$$

where $l = \sqrt{1 - r^2}$, and

$$(1 - r^2)^{-\frac{N}{2}} \tilde{C} \to b_N u^{\frac{N}{2}} \tilde{s}_1^{-\frac{N}{2}} v^{\frac{N}{2}} \det(\tilde{Q})^{-\frac{1}{2}},$$

where b_N is the Lebesgue measure of the N-dimensional unit ball. Finally we note that an approximation of the distribution of $\tilde{C}_{\max}$ can be obtained in a similar way as illustrated in Section 6 using the approximation (6.4) and the correction (6.5).

10 Applications

10.1 Simulated one-dimensional field

Figure 1 illustrates these concepts for the simplest case of $M = N = 1$. It is worth explaining exactly how this was done. $\nu = 20$ random fields were simulated on $S_x = S_y = [0, 64]$ according to the model (2.2–2.4) with $\sigma_x = \sigma_y = 1$ and $\Lambda_x = \Lambda_y = 1/18$. This was achieved by first simulating ν pairs of independent white noise processes on a larger region $u_i(\mathbf{s}), v_i(\mathbf{t})$, and ν pairs of scalar normal random variables U_i, V_i with zero mean, unit variance and correlation β, independent of $u_i(\mathbf{s}), v_i(\mathbf{t}), i = 1, \ldots, \nu$. Let $\tilde{f}$ be a Gaussian function scaled so that the convolution of $\tilde{f}$ with itself is $f = f_x = f_y$ as in (2.3). Then it can be checked that

$$\begin{aligned} X_i(\mathbf{s}) &= \int \tilde{f}(\mathbf{s} - \mathbf{s}_1) u_i(\mathbf{s}_1) d\mathbf{s}_1 + f(\mathbf{s} - \mathbf{s}_0) \left(U_i - \int \tilde{f}(\mathbf{s}_1) u_i(\mathbf{s}_1) d\mathbf{s}_1 \right) \\ Y_i(\mathbf{t}) &= \int \tilde{f}(\mathbf{t} - \mathbf{t}_1) v_i(\mathbf{t}_1) d\mathbf{t}_1 + f(\mathbf{t} - \mathbf{t}_0) \left(V_i - \int \tilde{f}(\mathbf{t}_1) v_i(\mathbf{t}_1) d\mathbf{t}_1 \right) \end{aligned}$$

has the required correlation structure (2.4). In fact three such correlation 'signals' were added at points $(\mathbf{s}_0, \mathbf{t}_0) = (48, 48), (16, 32), (32, 16)$ by adding three terms to $X_i(\mathbf{s}), Y_i(\mathbf{t})$ above, instead of one, each with an independent U_i, V_i with a correlation of $\beta = 0.85$. It can be checked that provided the points are well separated then the correlation structure is the sum of three terms of the form (2.4), one for each point. The locations of the correlations were chosen so that the first one was a homologous correlation at $\mathbf{s}_0 = \mathbf{t}_0 = 48$, and the other two were cross correlations.

Contour lines at the threshold $R(\mathbf{s}, \mathbf{t}) = 0.5$ are shown. The enclosed excursion set has an Euler characteristic of 5, one less than the number of local maxima greater than 0.5 (indicated by crosses). Applying the above theory, only the two local maxima indicated by large crosses are significant at $P < 0.05$ (the critical threshold is 0.744); a third local maximum of the homologous correlation field is significant at $P < 0.05$ (not shown; critical threshold is 0.614). All three local maxima are within one pixel of the locations of the peak correlation 'signals'. Finally, the sizes of the three largest connected components of the excursion set are significant at $P < 0.05$ (critical size is 42.9 pixels^2 for the cross correlation field and 3.66 pixels for the homologous correlation field).

10.2 Homologous correlation field

There is some evidence that arithmetic ability is correlated with the ability to do mental rotation. Petrides *et al.* (1998, private communication) conducted a PET study to see if this correlation occurred at the same location in the brain. PET measures of cerebral blood flow on 9 subjects were obtained while the subjects performed four different tasks: 1) mental arithmetic, 2) no mental arithmetic (reading numbers), 3) mental rotation, 4) no mental rotation (fixation on the shapes). Blood flow values 1) minus 2) gave 9 images of arithmetic activation, 3) minus 4) gave 9 images of rotation activation. The sample correlation gives the homologous correlation field $R(\mathbf{t})$ with $\nu = 8$ (minus 1 for subtracting the mean).

An example of the excursion set above 0.75, chosen to be the uncorrected $P < 0.01$ threshold value, is shown in Figure 2. To calculate the expected EC, the images were assumed to be isotropic with λ estimated to be 1.07 cm^{-2} using methods in Worsley (1995b). The search region S was taken as the upper part of the cortex (rendered transparent in Figure 2), with Minkowski functionals $\mu_0(S) = -6$, $\mu_1(S) = 2.688\mathrm{cm}$, $\mu_2(S) = 586.9\mathrm{cm}^2$, $\mu_3(S) = 858.5\mathrm{cm}^3$. Note that the interior holes due to ventricles and white matter, visible in Figure 2, make the EC of S negative.

The observed and expected EC are plotted in Figure 3. For the threshold of $r = 0.75$ shown in Figure 2, the observed EC is 23 and the expected EC is 54.6. Note that there is overall reasonable agreement, suggesting that there is no evidence for correlations between the two tasks. This was confirmed by the the value of $\tilde{R}_{\max} = 0.965$, which gives an expected EC of 1.91, close to the value expected by chance alone. The size of the largest connected component above a threshold of 0.75 was $4.85\mathrm{cm}^3$, visible as the largest blob in the right frontal of Figure 2. In this case the P-value was 0.018, which was the only significant signal that was found at the 0.05 level using this method. Thus there seems to be some evidence for an extended region of activation in the right frontal area that was detected using its size, but not its peak value.

10.3 Cross correlation field

Paus *et al.* (1997) undertook a study to find functional connectivity between distant brain regions while performing a vigilance task (attending to an intensity drop in an audio signal). Changes in cerebral blood flow were measured by PET on 8 subjects, each scanned 6 times while performing the vigilance task. A subject mean effect was subtracted from the data before analysis, leaving $\nu = 40$ effectively independent scans. Paus *et al.* (1997) focused on two regions, the thalamus and the right ventro-lateral frontal. Correlations between these regions and all other regions was transformed to a t field (as outlined in Section 8) and previous theory for t fields was used to assess the significance of local maxima.

To illustrate the method, we focused on inter-hemispheric functional connectivity by looking at cross correlations between all points in the left and right hemispheres. The images X and Y were the blood flow changes in the left and right hemispheres, respectively. The search regions S_x and S_y were symmetric left and right regions that covered most of the brain except for a 20mm wide slice either side of the midline, to remove points whose correlations were due to spatial smoothing of the data across the midline. The Minkowski functionals of S_x and S_y were $\mu_0(S_x) = \mu_0(S_y) = 1$, $\mu_1(S_x) = \mu_1(S_y) = 32.3\mathrm{cm}$, $\mu_2(S_x) = \mu_2(S_y) = 233.3\mathrm{cm}^2$, $\mu_3(S_x) = \mu_3(S_y) = 424.4\mathrm{cm}^3$. The images were assumed to be isotropic with λ estimated to be 0.738 cm^{-2} using methods in Worsley (1995b).

All cross correlations were calculated, and Figure 4 shows the 10 local maximum correlations above 0.75 (light colours), and the 5 local minimum correlations below -0.75 (dark colours). The critical threshold for $R_{\max}$ is 0.783, and only two local maxima exceeded this threshold; both were connections in the visual cortex at the back of the brain, visible in Figure 4. It was not possible to render the excursion set of all correlations greater than 0.75 because there were too many, but the observed Euler characteristic was 9 (close to the 10 local maxima), while the expected was only 0.254, indicating some evidence that the 10 local maxima above 0.75 are not all due to chance alone. A plot of the Euler characteristic for thresholds $r > 0$ is given in Figure 5, which shows reasonable agreement between observed and expected except at the upper tail, which again indicates some evidence of non-null correlations.

A Appendix

LEMMA A.1 *(Adler, 1981, Lemma 5.3.2) Let* $H \sim \mathrm{Normal}_{N\times N}(0, V(I_N))$ *and* H_j *be any* $j \times j$ *principal minor of* H. *Then for any non-negative integer* i,

$$\mathrm{E}\{\det(H_{2i})\} = \frac{(-1)^i(2i)!}{2^i i!}, \quad \mathrm{E}\{\det(H_{2i+1})\} = 0.$$

LEMMA A.2 *Let* A *be the following* $D \times D$ *matrix:*

$$A = \begin{pmatrix} \alpha_1 Q_x & \alpha_2 Q_{xy} \\ \alpha_2 Q'_{xy} & \alpha_3 Q_y \end{pmatrix},$$

where Q_x, Q_{xy}, Q_y *are* $M \times M, M \times N, N \times N$ *matrices with*

$$Q = \begin{pmatrix} Q_x & Q_{xy} \\ Q'_{xy} & Q_y \end{pmatrix} \sim \mathrm{Wishart}_D(I_D, \nu),$$

and $\alpha_1, \alpha_2, \alpha_3$ *are fixed scalars. Then*

$$\mathrm{E}\{\det(A)\} = \sum_{k=0}^{N} \alpha_1^M \alpha_3^N \alpha_4^k \frac{\nu! M! N!}{k!(\nu - D + k)!(M-k)!(N-k)!},$$

where $\alpha_4 = 1 - \alpha_2^2/(\alpha_1\alpha_3)$ *and division by the factorial of a negative integer is treated as multiplication by zero.*

PROOF. Let $G = Q'_{xy} Q_x^{-1} Q_{xy}$ and $G^* = Q_y - G$. Since $Q \sim \mathrm{Wishart}_D(I_D, \nu)$, by standard results from multivariate statistics, $Q_x \sim \mathrm{Wishart}_M(I_M, \nu)$, $G \sim \mathrm{Wishart}_N(I_N, M)$ and $G^* \sim \mathrm{Wishart}_N(I_N, \nu - M)$, independently. Then using Lemma A.5,

$$\begin{aligned} \mathrm{E}\{\det(A)\} &= \alpha_1^M \alpha_3^N \mathrm{E}\{\det(Q_x)\} \mathrm{E}\left\{\det\left(\alpha_4 G + G^*\right)\right\} \\ &= \alpha_1^M \alpha_3^N \frac{\nu!}{(\nu - M)!} \sum_{k=0}^{N} \alpha_4^k \binom{N}{k} \frac{M!(\nu - M)!}{(M-k)!(\nu - D + k)!} \end{aligned}$$

and hence the result. □

LEMMA A.3 *Let A be the same matrix as in Lemma A.2 and B be a fixed $D \times D$ matrix as follows:*

$$B = \begin{pmatrix} \beta_1 H_x & 0 \\ 0 & \beta_3 H_y \end{pmatrix}, \tag{A.1}$$

where β_1, β_3 are fixed scalars, and H_x and H_y are fixed symmetric $M \times M$ and $N \times N$ matrices, respectively. Then

$$\begin{aligned} \mathrm{E}\{\det(A+B)\} &= \sum_{i=0}^{M}\sum_{j=0}^{N}\sum_{k=0}^{N-j} \beta_1^i \beta_3^j \alpha_1^{M-i}\alpha_3^{N-j}\alpha_4^k \mathrm{detr}_i(H_x)\mathrm{detr}_j(H_y) \\ &\quad \times \frac{\nu!(M-i)!(N-j)!}{k!(\nu-D+i+j+k)!(M-i-k)!(N-j-k)!}, \end{aligned}$$

where division by the factorial of a negative integer is treated as multiplication by zero.

PROOF. Let

$$U = \begin{pmatrix} U_x & 0 \\ 0 & U_y \end{pmatrix}$$

be an orthonormal matrix such that $B^* = U'BU$ is diagonal. It is easy to see that $U'AU$ has the same distribution as A and hence

$$\det(A+B) = \det(U'AU + B^*) \stackrel{D}{=} \det(A+B^*).$$

The determinant on the right hand side then becomes the summation of the product of the determinant of any $l \times l$ principle minor B_l^* of B^* and the determinant of the $(D-l)\times(D-l)$ principle minor A_{D-l} of A, formed from the remaining $D-l$ rows and columns not included in B_l^*. For B_l^*, let i be the number of rows or columns from the upper $M \times M$ diagonal matrix and $j = l - i$ be the number of rows or columns from the $N \times N$ lower diagonal matrix of B^*. From Lemma A.2,

$$\mathrm{E}\{\det(A_{D-l})\} = \sum_{k=0}^{N-j} \alpha_1^{M-i}\alpha_3^{N-j}\alpha_4^k \frac{\nu!(M-i)!(N-j)!}{k!(\nu-D+i+j+k)!(M-i-k)!(N-j-k)!}.$$

Summing up over all principal minors we get

$$\begin{aligned} \mathrm{E}\{\det(A+B^*)\} &= \sum_{i=0}^{M}\sum_{j=0}^{N} \beta_1^i \beta_3^j \mathrm{detr}_i(H_x)\mathrm{detr}_j(H_y) \sum_{k=0}^{N-j} \alpha_1^{M-i}\alpha_3^{N-j}\alpha_4^k \\ &\quad \times \frac{\nu!(M-i)!(N-j)!}{k!(\nu-D+i+j+k)!(M-i-k)!(N-j-k)!}. \end{aligned}$$

□

LEMMA A.4 *Let A be the $D \times D$ matrix as in Lemma A.2 and B be a random $D \times D$ matrix as in Lemma A.3 where $H_x \sim \mathrm{Normal}_{M\times M}(0, V(I_M))$ and $H_y \sim \mathrm{Normal}_{N\times N}(0, V(I_N))$ independently of A. Then*

$$\begin{aligned} \mathrm{E}\{\det(A+B)\} &= \sum_{i=0}^{\lfloor M/2\rfloor}\sum_{j=0}^{\lfloor N/2\rfloor}\sum_{k=0}^{N-2j} (-1)^{i+j} 2^{-(i+j)} \beta_1^{2i}\beta_3^{2j}\alpha_1^{M-2i}\alpha_3^{N-2j}\alpha_4^k \\ &\quad \times \frac{\nu!M!N!}{i!j!k!(\nu-D+2i+2j+k)!(M-2i-k)!(N-2j-k)!}, \end{aligned}$$

where division by the factorial of a negative integer is treated as multiplication by zero.

PROOF. Applying Lemmas A.1 and A.3,

$$\begin{aligned}\mathrm{E}\{\det(A+B)\} &= \sum_{i=0}^{\lfloor M/2\rfloor}\sum_{j=0}^{\lfloor N/2\rfloor}\sum_{k=0}^{N-2j}\beta_1^{2i}\beta_3^{2j}\binom{M}{2i}\binom{N}{2j}\frac{(-1)^{i+j}(2i)!(2j)!}{2^{i+j}i!j!}\\ &\quad\times\alpha_1^{M-2i}\alpha_3^{N-2j}\alpha_4^k\frac{\nu!(M-2i)!(N-2j)!}{k!(\nu-D+2i+2j+k)!(M-2i-k)!(N-2j-k)!}.\end{aligned}$$

The result then follows after proper simplifications. □

LEMMA A.5 *Let A, B be symmetric $N\times N$ matrices, with A fixed and $B \stackrel{D}{=} U'BU$ for any fixed orthonormal matrix U. Let B_j be any $j\times j$ principal minor of B. Then*

$$\mathrm{E}\{\det(A+B)\} = \sum_{j=0}^{N}\mathrm{detr}_j(A)\mathrm{E}\{\det(B_{N-j})\}.$$

PROOF. Write $A = ULU'$, where the columns of U are the eigenvectors of A and L is a diagonal matrix of eigenvalues. Then

$$\det(A+B) = \det(ULU'+B) = \det(L+U'BU) \stackrel{D}{=} \det(L+B).$$

Since each principal minor of B has the same distribution, then

$$\mathrm{E}\{\det(L+B)\} = \sum_{j=0}^{N}\mathrm{detr}_j(L)\mathrm{E}\{\det(B_{N-j})\}.$$

Since $\mathrm{detr}_j(L) = \mathrm{detr}_j(A)$ then the result follows. □

LEMMA A.6 *Let $Q_1, Q_2 \sim \mathrm{Wishart}_N(I_N,\nu)$ independently and e_1, e_2 be fixed scalars satisfying $e_1+e_2 = s$ and $e_1e_2 = p$. Then*

$$\begin{aligned}\mathrm{E}\{\det(e_1Q_1+e_2Q_2)\} &= \sum_{j=0}^{\lfloor N/2\rfloor}N!\binom{\nu}{j}\binom{\nu}{N-j}p^j 2^{-(N-2j-1)}(1+\delta_{N,2j})^{-1}\\ &\quad\times\sum_{k=0}^{\lfloor N/2\rfloor-j}\binom{N-2j}{2k}(s^2-4p)^k s^{N-2j-2k}.\end{aligned}$$

PROOF. Condition on Q_1 and apply Lemma A.5, then take expectations over Q_1 to get

$$\begin{aligned}\mathrm{E}\{\det(e_1Q_1+e_2Q_2)\} &= \sum_{j=0}^{N}\binom{N}{j}e_1^j e_2^{N-j}\frac{\nu!^2}{(\nu-j)!(\nu-N+j)!}\\ &= \sum_{j=0}^{\lfloor N/2\rfloor}N!\binom{\nu}{j}\binom{\nu}{N-j}(e_1e_2)^j(1+\delta_{N,2j})^{-1}(e_1^{N-2j}+e_2^{N-2j}).\end{aligned}$$

Solving the quadratic,

$$e_1 = \left(s + \sqrt{s^2 - 4p}\right)/2, \qquad e_2 = \left(s - \sqrt{s^2 - 4p}\right)/2.$$

In terms of s, p we have

$$\begin{aligned} e_1^i + e_2^i &= 2^{-i} \sum_{l=0}^{i} \binom{i}{l} (s^2 - 4p)^{\frac{l}{2}} s^{i-l} (1 + (-1)^l) \\ &= 2^{-(i-1)} \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{i}{2k} (s^2 - 4p)^k s^{i-2k}, \end{aligned}$$

which leads to the result. □

LEMMA A.7 *For* $z, w \sim \mathrm{Normal}_N(0, I_N)$ *independently,*

$$\mathrm{E}\{\mathrm{detr}_j(zw' + wz')\} = \left\{ \begin{array}{cc} 1 & j = 0 \\ 0 & j = 1 \\ -N(N-1) & j = 2 \\ 0 & j \geq 3 \end{array} \right. .$$

PROOF. For $j = 1$,

$$\mathrm{detr}_1(zw' + wz') = \mathrm{tr}(zw' + wz') = 2z'w,$$

whose expectation is zero. For $j = 2$ and $N \geq 2$, the rank of $zw' + wz'$ is 2 (with probability one), so there are 2 non-zero eigenvalues. Hence $\mathrm{detr}_2(zw' + wz')$ is the product of these two eigenvalues. Let U be the $2 \times N$ matrix whose columns are the orthonormal vectors $z/||z||$ and $\tilde{w}/||\tilde{w}||$, where $\tilde{w} = (I_N - zz'/z'z)w$ is orthogonal to z. Then

$$\mathrm{detr}_2(zw' + wz') = \det(U'(zw' + wz')U) = \det \begin{pmatrix} 2z'w & ||z||\,||\tilde{w}|| \\ ||z||\,||\tilde{w}|| & 0 \end{pmatrix} = -(z'z)(\tilde{w}'\tilde{w}).$$

Since $z'z \sim \chi^2_N$ and $\tilde{w}'\tilde{w} \sim \chi^2_{N-1}$ independently, the result follows. For $j \geq 3$ and $N \geq 3$, $\mathrm{detr}_j(zw' + wz') = 0$ with probability one. □

LEMMA A.8 *Let* $z, w \sim \mathrm{Normal}_N(0, I_N)$ *independently and* G_j *be any* $j \times j$ *principal minor of* $zw' + wz'$. *Then for any fixed scalars* g, h *and non-negative integer* i*:*

$$\mathrm{E}\{\det(gG_{2i} + hH_{2i})\} = \frac{(-1)^i (2i)!}{2^i i!}(h^{2i} + 2ig^2h^{2i-2}), \qquad \mathrm{E}\{\det(gG_{2i+1} + hH_{2i+1})\} = 0.$$

PROOF. Condition on G_j and apply Lemma A.5, then take expectations over G_j using Lemma A.7:

$$\mathrm{E}\{\det(gG_j + hH_j)\} = h^j \mathrm{E}\{\det(H_j)\} - j(j-1)g^2h^{j-2}\mathrm{E}\{\det(H_{j-2})\}.$$

The result follows on applying Lemmas A.1, noting that the above is zero unless $j = 2i$.

□

LEMMA A.9 *Let g, h, s, p be fixed scalars and e_1, e_2 be the solutions of equations $e_1 + e_2 = s$, $e_1 e_2 = p$. Let $z, w \sim \mathrm{Normal}_N(0, I_N)$, $H \sim \mathrm{Normal}_{N\times N}(0, V(I_N))$ and $Q_1, Q_2 \sim \mathrm{Wishart}_N(I_N, \nu)$ independently. Then*

$$\begin{aligned}
&\mathrm{E}\{\det(g(zw' + wz') + hH - e_1Q_1 - e_2Q_2)\} \\
= &\sum_{i=0}^{\lfloor N/2\rfloor} \sum_{j=0}^{\lfloor N/2\rfloor - i} \sum_{k=0}^{\lfloor N/2\rfloor - i - j} \binom{\nu}{j}\binom{\nu}{N-2i-j}\binom{N-2i-2j}{2k} \\
&\times \frac{(-1)^{N-i}N!}{2^{(N-i-2j-1)}(1+\delta_{N-2i,2j})i!}(h^{2i} + 2ig^2h^{2i-2})s^{N-2i-2j-2k}p^j(s^2-4p)^k.
\end{aligned}$$

PROOF. From Lemma A.5 applied to $A = g(zw' + wz') + hH$ and $B = -e_1Q_1 - e_2Q_2$ we get:

$$\begin{aligned}
&\mathrm{E}\{\det(g(zw' + wz') + hH - e_1Q_1 - e_2Q_2\} \\
= &\sum_{i=0}^{\lfloor N/2\rfloor} (-1)^{N-2i}\mathrm{E}\{\det(gG_{2i} + hH_{2i})\}\mathrm{E}\{\mathrm{detr}_{N-2i}(e_1Q_1 + e_2Q_2)\} \\
= &\sum_{i=0}^{\lfloor N/2\rfloor} (-1)^{N-2i}\binom{N}{2i}\frac{(-1)^i(2i)!}{2^i i!}(h^{2i} + 2ig^2h^{2i-2}) \\
&\times \sum_{j=0}^{\lfloor N/2\rfloor - i}(N-2i)!\binom{\nu-2}{j}\binom{\nu-2}{N-2i-j}\frac{p^j}{2^{N-2i-2j-1}(1+\delta_{N-2i,2j})}.
\end{aligned}$$

Proper simplification will lead to the result. □

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Figure 1

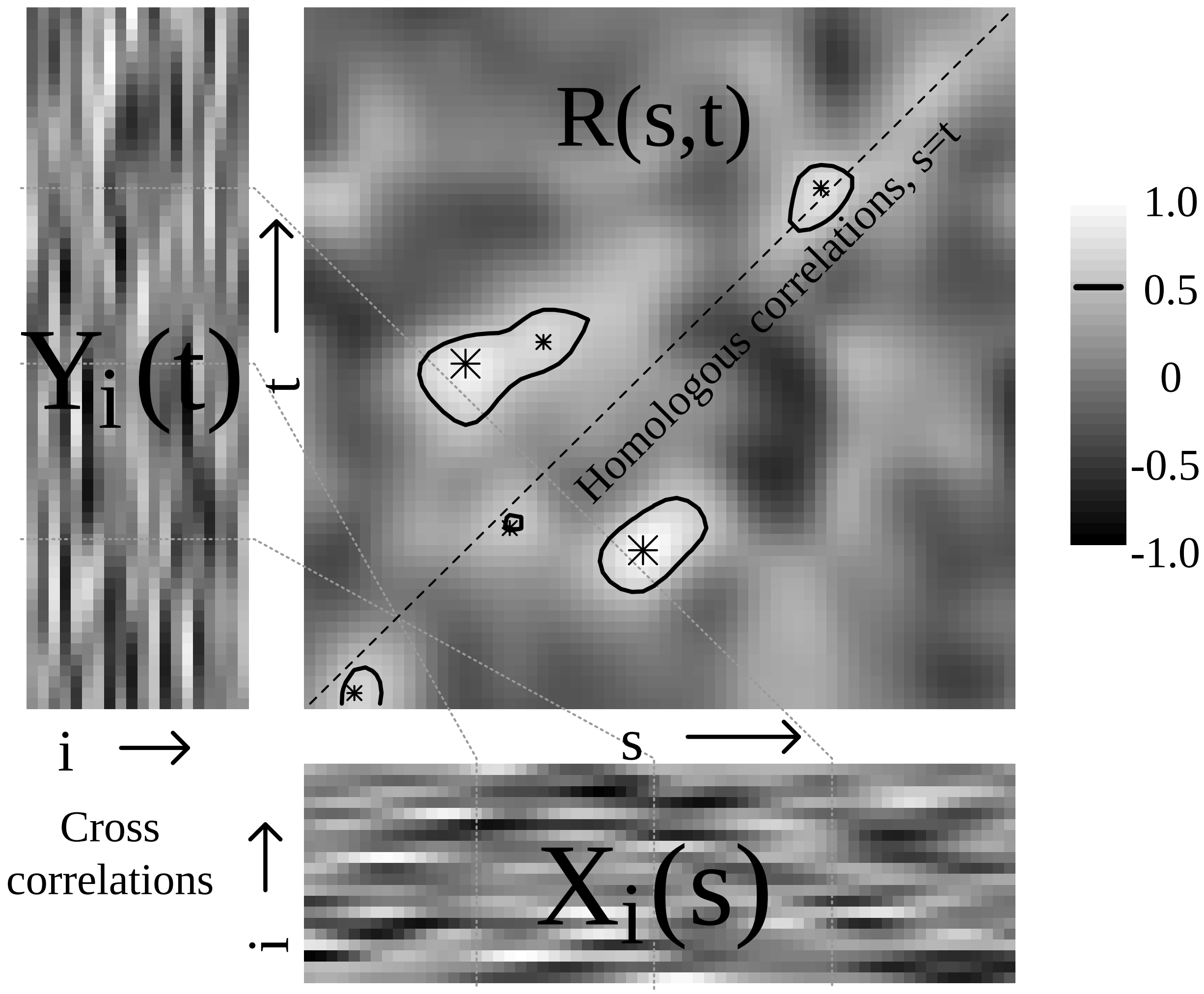

Figure 1: Illustration of a correlation field in $M = N = 1$ dimensions. Two sets of $\nu = 20$ i.i.d. smooth Gaussian time series ($X_i(\mathbf{s})$, $Y_i(\mathbf{t})$, $i = 1, \ldots, \nu$) are shown parallel to the axes. The interior two dimensional image $R(\mathbf{s}, \mathbf{t}) = \sum_i X_i(\mathbf{s}) Y_i(\mathbf{t}) / \sqrt{\sum_i X_i(\mathbf{s})^2 \sum_i Y_i(\mathbf{t})^2}$ is the cross correlation field of sample correlation coefficients between X and Y for all pairs of time points. The homologous correlation field $R(\mathbf{t}) = R(\mathbf{t}, \mathbf{t})$ is restricted to the diagonal. Three non-zero correlations were added at points linked by the dotted lines. Contours at $R = 0.5$ are shown (the interior is the excursion set) together with local maxima (crosses; large crosses are significant at $P < 0.05$) inside the excursion set (the sizes of the three largest connected components are significant at $P < 0.05$).

Figure 2. Homologous correlations > 0.75

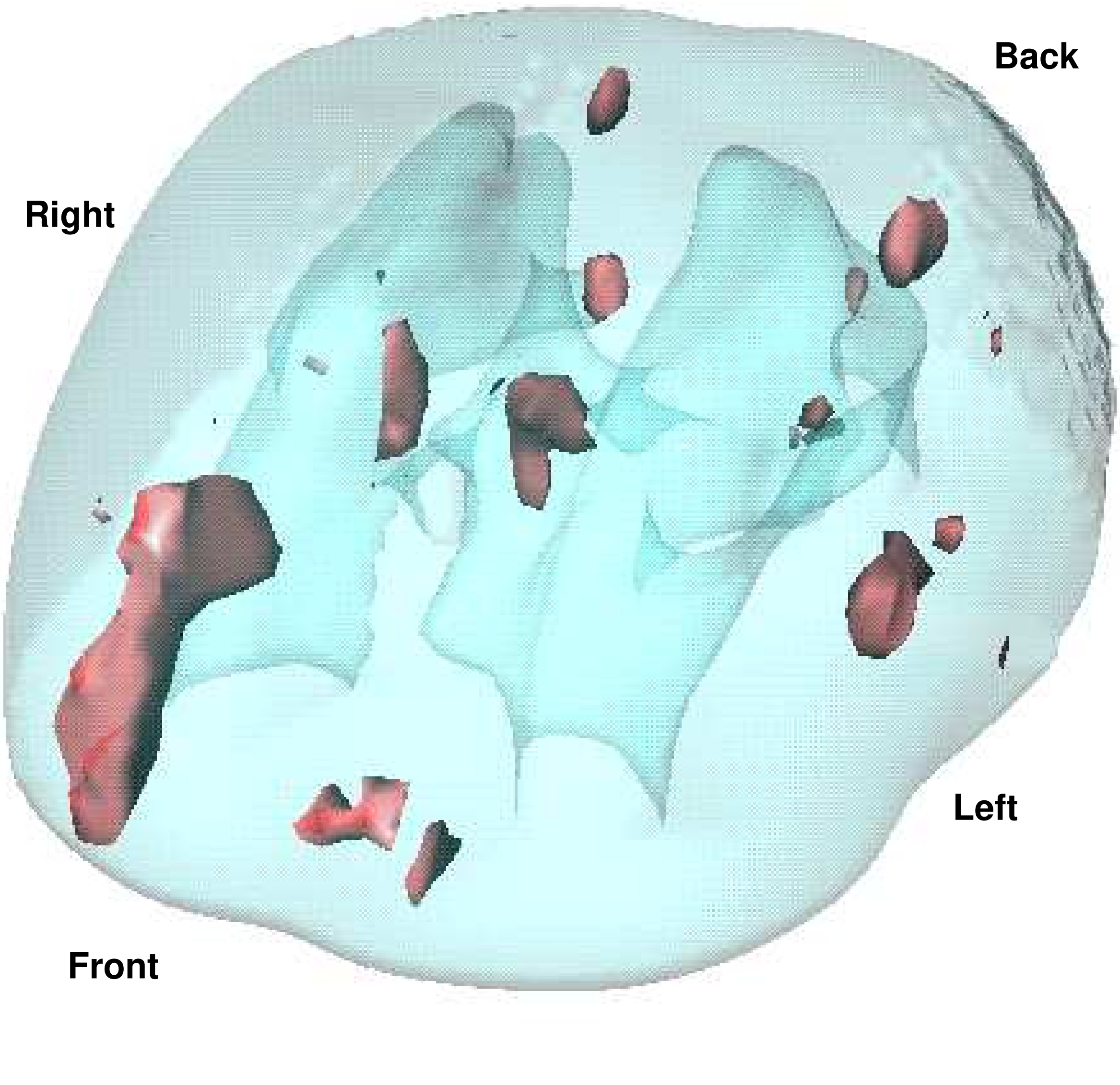

Figure 2: Homologous correlations $R(\mathbf{t})$ for functional connectivity between mental rotation and arithmetic ability. Solid regions are the excursion set above 0.75, chosen to be the uncorrected $P < 0.01$ threshold value. The transparent region is the search region S, chosen to be the cortex, excluding white matter and ventricles.

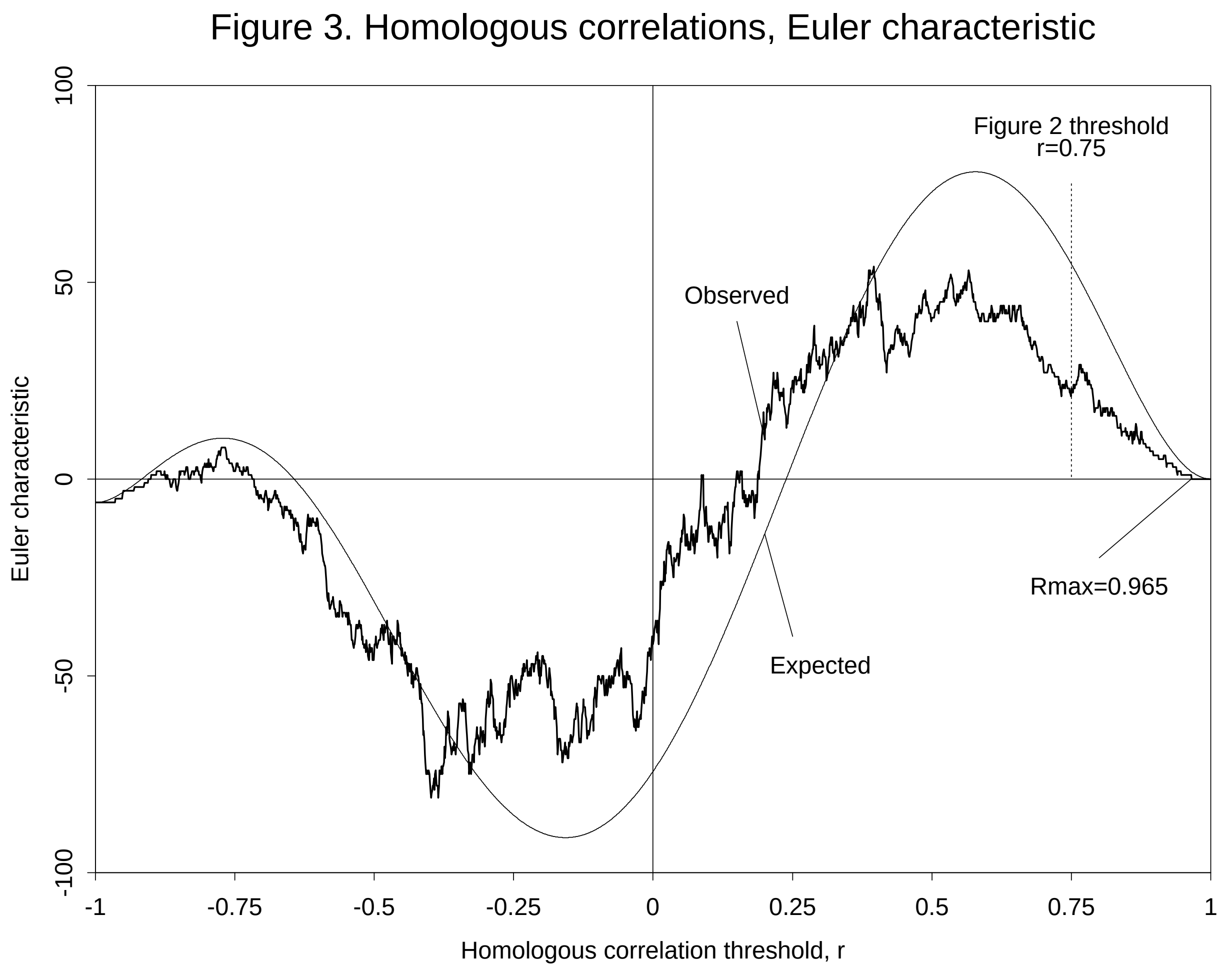

Figure 3: Observed and expected Euler characteristic of excursion sets of the homologous correlations data in Figure 2.

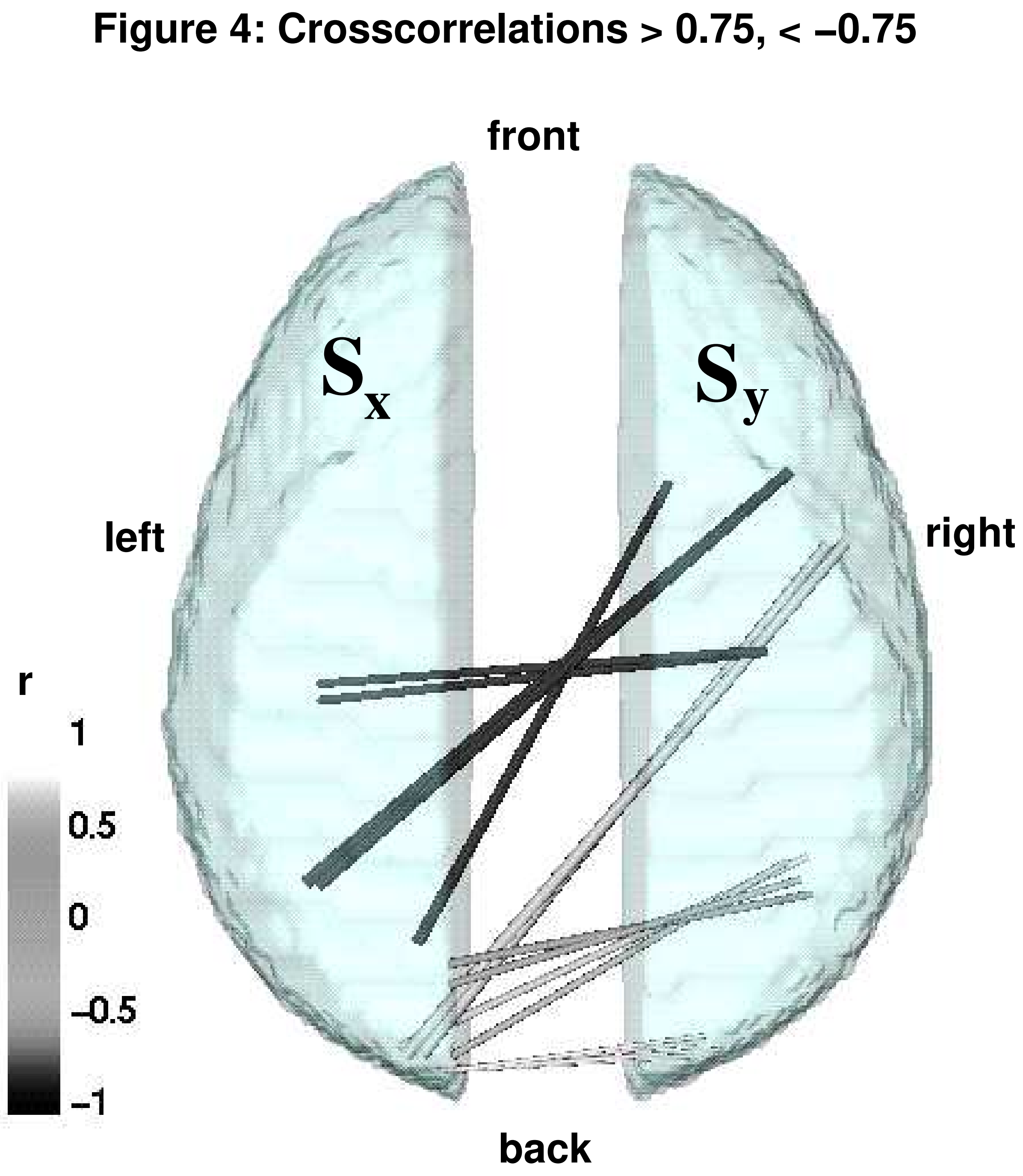

Figure 4: Cross correlations for functional connectivity between the two hemispheres while performing the vigilance task. The search regions are the left hemisphere (S_x) and the right hemisphere (S_y), minus a 20mm slice either side of the midline (top view; transparent). Rods are the local maximum correlations above 0.75 (light colours), and local minimum correlations below -0.75 (dark colours).

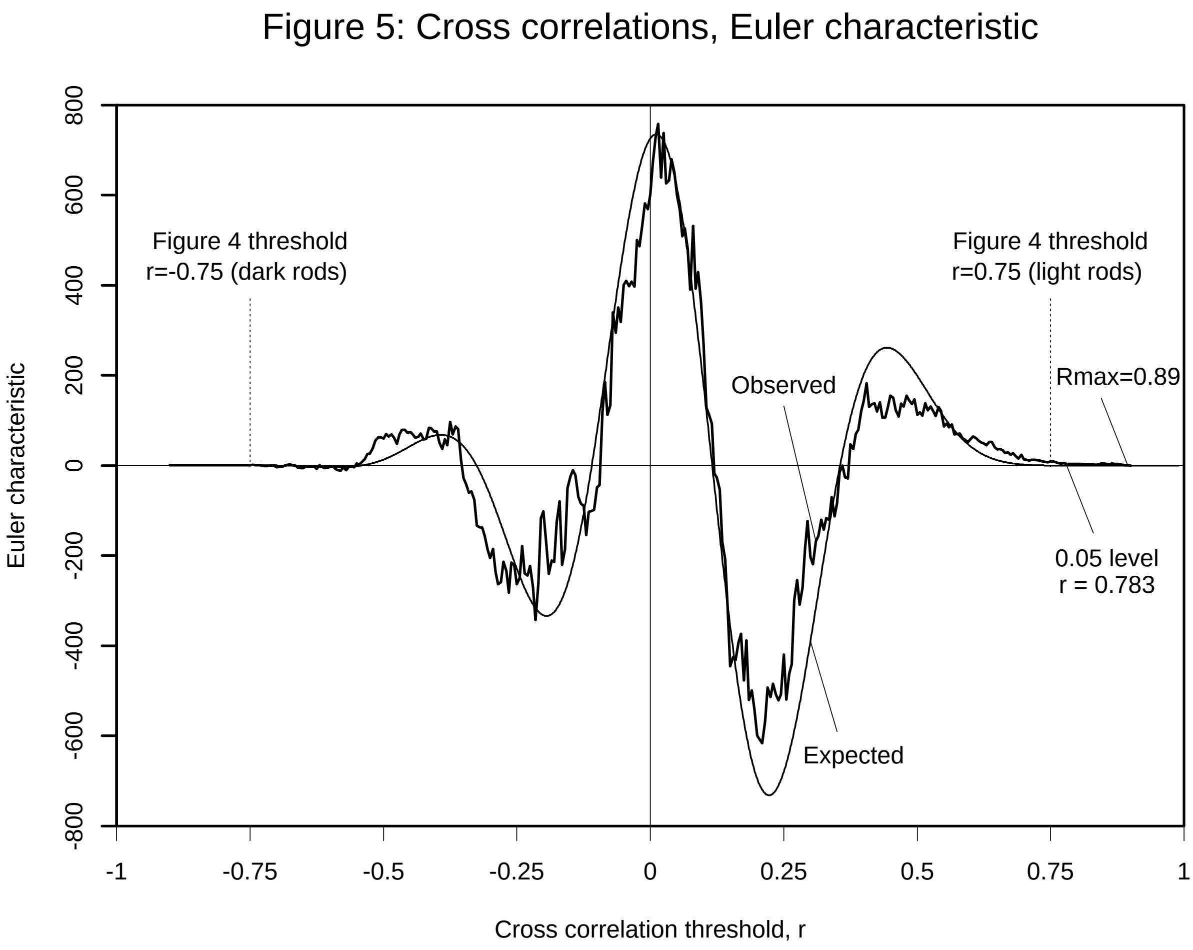

Figure 5: Observed and expected Euler characteristic of excursion sets of the cross correlations data in Figure 4.