Optimal Blocking Schemes for 2^n and 2^{n-p} Designs

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Abstract

Systematic sources of variations in factorial experiments can be effectively reduced without biasing the estimates of the treatment effects by grouping the runs into blocks. For full factorial designs optimal blocking schemes are obtained by applying the minimum aberration criterion to the block defining contrast subgroup. A related concept of order of estimability is proposed. For fractional factorial designs, because of the intrinsic difference between treatment factors and block variables, the minimum aberration approach has to be modified. A concept of admissible blocking schemes is proposed for selecting block designs based on multiple criteria. The resulting 2^n and 2^{n-p} designs are shown to have better overall properties for practical experiments than those in the literature, e.g., the National Bureau of Standards Tables (1957) and Box, Hunter and Hunter (1978).

Key Words: *Clear Main Effects, Clear 2-factor Interactions, Defining Contrast Subgroup, Minimum Aberration, Word-Length Pattern.*

1 Introduction

Systematic sources of variations in experiments, such as day-to-day environmental variation and batch-to-batch variation of raw material, can be effectively eliminated without biasing the estimates of the treatment effects by properly grouping the runs into blocks. See design textbooks like Box, Hunter and Hunter (1978) (henceforth abbreviated as BH^2) for principles and assumptions in the construction of block designs. In particular, the following two assumptions on block variables distinguish the problem of constructing block designs from that of constructing fractional factorial designs.

Assumptions:

(i) The "interaction" between two block variables has the same importance as the main effects of block variables; therefore it should be treated as a "generalized block variable".

(ii) Block variables are usually assumed to have no interaction with treatment variables.

Before we discuss the problem of constructing blocking schemes for 2^n and 2^{n-p} designs, some basic definitions and notation about these designs are briefly reviewed.

A 2^{n-p} fractional factorial (henceforth abbreviated as FF) design has n factors of 2 levels and 2^{n-p} runs and is uniquely determined by p independent *defining words*. A word consists of *letters* which are names of factors denoted by $1, 2, \ldots, n$ or $a, b, \ldots$, etc. The group formed by the p defining words can be represented by

$$I = w_1 = \cdots = w_{2^p-1} \tag{1}$$

or, the *treatment defining contrast subgroup* $G = \{I, w_1, \cdots, w_{2^p-1}\}$. The number of letters in a word is called *word-length* and the vector

$$W = (A_1, \ldots, A_n)$$

is called the *word-length pattern*, where A_i denotes the number of words of length i in G. The shortest word-length in G is called the *resolution* of the design. For two designs d_1 and d_2 let r be the smallest value such that $A_r(d_1) \neq A_r(d_2)$. We say that d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has *minimum aberration*(MA). For more details, see Fries and Hunter (1980). When $p = 0$, a 2^{n-p} design is reduced to a full factorial 2^n design.

For a full factorial design its blocking scheme is defined by the block defining contrast subgroup(see Section 2). Since this group has the same mathematical structure as the treatment defining contrast subgroup defined above, we can apply the minimum aberration criterion to the block defining contrast subgroup and use it as an optimality criterion to rank-order any blocking schemes. In Section 2 we also define a related concept of *order of estimability* as a measure of capacity of a blocking scheme for estimating factorial effects. It is related but not equivalent to the notion of resolution for FF designs. This new approach provides a satisfactory solution to the problem of defining and finding optimal blocking schemes for 2^n designs. A complete collection of minimum aberration blocking schemes are given in Table **??** for up to 128 runs.

This straightforward approach does not apply to blocking schemes for 2^{n-p} designs. The complication is caused by the intrinsic difference between the treatment defining contrast subgroup and the block defining contrast subgroup. Word-length in the latter group is simply the order of a factorial effect being confounded with a block effect while the word-length in the former group implies various aliasing relations between effects which are more complicated to study. As a consequence of these complicated relations an FF design with lower resolution or worse aberration may be better for blocking, that is, it may accommodate more block variables than designs with higher resolution or less aberration. Several such examples are given in Section 3. Instead of using a single criterion we use four criteria jointly to compare blocking schemes. A concept of *admissibility* is used to screen out poor blocking schemes. Based on these concepts and the search procedure described in Section 4 we were able to compile a list of useful blocking schemes for 2^{n-p} designs in Tables 3-7. In many situations our blocking schemes can accommodate more block variables than the tables in BH^2 and the National Bureau of Standards (1957) tables.

In the statistical literature, the classic work by the National Bureau of Standards (1957) (henceforth abbreviated as NBS) represents the most significant effort on this subject. The NBS tables included 35 two-level FF designs with run size ranging from 16 to 256 and the number of treatment factors ranging from 5 to 16. For the 35 FF designs, 125 blocking schemes were provided with the number of block variables ranging from 1 to 5. This collection of designs is a useful source for practitioners in planning factorial experiments. However, a shortcoming in this work is that the criteria for design selection were not spelled out. Furthermore, it is not known whether the designs given in the NBS tables are optimal in some sense. Another collection of blocking schemes is provided by the tables in BH^2 (1978, p. 346 and p. 408). Noting that both collections are limited in that they cover typically larger designs with high resolution, Bisgaard(1994a) provided a more comprehensive list of blocking schemes for 8 and 16 runs by using a modified definition of resolution for blocked 2^{n-p} designs given in Bisgaard(1994b). Relations between our work and his will be discussed in Sections 2 and 3.

2 Optimal Blocking Schemes for 2^n Designs

We use a simple example to illustrate our approach. Following the notation in BH^2, we denote the ith main effect by i, the two-factor interaction (2fi) between i and j by ij, the three-factor interaction (3fi) between i, j and k by ijk, and so forth. Take, for example, the 2^7 design arranged in $8(=2^3)$ blocks in Table 10B.1 of BH^2 (p. 346), which uses

$$B_1 = 123, B_2 = 456, B_3 = 167, \tag{2}$$

as the block variables. The three 3fi's in (**??**) are used to define the three independent block variables and are thus confounded with block effects. Since there are seven degrees of freedom for the block effects, the remaining four block effects are confounded with the following interactions, $B_1B_2 = 123456, B_1B_3 = 2367, B_2B_3 = 1457, B_1B_2B_3 = 23457$. For this blocking arrangement, three 3fi's, two 4fi's, one 5fi and one 6fi are confounded with the block effects. Because block effects are assumed *a priori* to be significant, these seven factorial effects are not estimable. Now we consider an alternative arrangement (see design *7-0.1/B3.1* in Table **??**) with

$$B_1 = 1234, B_2 = 1256, B_3 = 1357. \tag{3}$$

It is easy to show that the remaining four block effects for (**??**) are confounded with 4fi's. Between (**??**) and (**??**) which arrangement is preferred? To answer this question, we need to make the following hierarchical assumptions on the relative importance of effects:

(i) lower order interactions are more likely to be important than higher order interactions,

(ii) effects of the same order are equally important.

Under this assumption, the arrangement in (**??**) is preferred because it does not confound any 3fi's while the arrangement in (**??**) confounds three 3fi's.

This example motivates our general definition of optimal blocking schemes for full factorial designs. Arranging a 2^n design in 2^k blocks is equivalent to selecting k defining words for the k independent block variables $B_1, \cdots, B_k$ out of the 2^n-1 defining words which represent the factorial effects (main effects and interactions of all orders) of the 2^n design. Formally if we choose

$$B_1 = v_1, \cdots, B_k = v_k$$

then the v_i's and all their "generalized interactions" $v_1v_2, v_1v_3, \cdots, v_1 \cdots v_k$ are confounded with the 2^k-1 block effects represented notationally by the B_i's and their products B_1B_2, B_1B_3, $\cdots$, $B_1 \cdots B_k$. Here we treat B_1B_2 as a component of the block effects, not as an interaction between B_1 and B_2. As usual, we assume that the treatment-by-block interactions like $1B_1, 2B_1$ are negligible. Then the words v_i, v_iv_j, etc. and the identity element form a group of size 2^k. We call this group

$$\{I, v_1, v_2, v_1v_2, v_3, \cdots, v_1 \cdots v_k\}$$

the *block defining contrast subgroup* , which is denoted by G_b. Under the hierarchical assumptions, we would like to have as few as possible lower order treatment factor interactions confounded with block effects. For a blocking scheme b, let $g_i(b)$ be the number of words of length i (i.e., ith order factorial effects) in G_b. Since main effects of treatment factors should not be confounded with block effects, $g_1(b)$ should always be zero for meaningful blocking schemes. Therefore, we do not include $g_1(b)$ in the following definition, where the vector

$$W_b = (g_2(b), g_3(b), \cdots, g_n(b)) \tag{4}$$

is called the *word-length pattern* of the blocking scheme b. Since W_b is mathematically equivalent to the word-length pattern W for an FF design in (**??**), we can apply the minimum aberration criterion to rank-order any two blocking schemes b_1 and b_2. Let r be the smallest value such that $g_r(b_1) \neq g_r(b_2)$, we say that b_1 has less aberration than b_2 if $g_r(b_1) < g_r(b_2)$. If there is no design with less aberration than b_1, then b_1 has minimum aberration (MA).

Similar to the definitions in Wu and Chen (1992), we call a 2fi *eligible* if it is not aliased with any main effects and any block effects. An eligible 2fi is called *clear* if it is not aliased with any other 2fi's. The notion of "clearness" can also be applied to the main effects. We call a main effect *clear* if it is not aliased with any 2fi's and any block effects. We use C1, C2 to denote the numbers of clear main effects and clear 2fi's. For full factorial designs, it is easy to see that C1 equals to the number of treatment factors, and C2 $= \binom{n}{2} - g_2$, which is the number of 2fi's that are not confounded with any block effects in the blocking scheme. Note that the definitions in this paragraph apply to both full factorial and fractional factorial designs.

The mathematical equivalence to the minimum aberration criterion for FF designs enables us to obtain theoretical results on MA blocking schemes by drawing on the similar results on MA 2^{n-p} designs (e.g., Chen and Wu, 1991; Chen 1992; Tang and Wu, 1996) and to construct MA blocking schemes by using the same algorithm described by Chen, Sun and Wu (1993) (henceforth abbreviated as CSW) for the 2^{n-p} FF designs. There is, however, a major difference between the two problems. The aberration criterion has a stronger justification for the blocking problem because g_i as defined before has the natural interpretation that g_i of the ith order effects are "sacrificed" in the blocking scheme. In the FF designs, the number of defining relations of length i and its implication to the estimability of some effects under the assumption that other effects are negligible are much more complicated. A counter-example (see p. 133 of CSW) to the aberration criterion for the FF designs shows that the MA 2^{9-4} design has eight clear 2fi's while the second best 2^{9-4} design (according to the aberration criterion) has 15 clear 2fi's. Such counter examples would not occur in MA blocking schemes, since the number of clear 2fi's is $\binom{n}{2} - g_2$.

The minimum aberration criterion makes it possible to compare any two blocking schemes but does not provide a single summary measure of capacity of a blocking scheme for estimating factorial effects. We say that a blocking scheme has *estimability of order e* if e is the largest integer such that all factorial effects of order e and lower are estimable, i.e., they are not confounded with block effects. It is easy to show that $e+1$ equals the shortest word-length among the non-identity elements in G_b, i.e., the smallest r with $g_r(b) > 0$. Note that, if G_b were the defining contrast subgroup of an FF design, then $e+1$ would be identical to the resolution of the FF design. So why do we not use the term "resolution" in the blocking context ? The meaning of resolution for an FF design is quite different from the notion of estimability for a blocking scheme. For example, in a resolution III design, all main effects are estimable under the assumption that 2fi's and higher order fi's are negligible. For a resolution IV design all main effects are estimable under the *weaker* assumption that 3fi's and higher order fi's are negligible. So from the estimability point of view, resolutions III and IV differ only in the assumptions on negligible effects. They both correspond to $e = 1$ for blocking schemes. One major difference is that the latter does not need any assumption on negligible effects in defining estimability. Generally, estimability of order e for a blocking scheme corresponds to resolutions $2e+1$ and $2e+2$ for an FF design. Note that Bisgaard's(1994b) definition of resolution for a blocked 2^n design is equivalent to using $e+2$ as the resolution. As argued above the proposed notion of estimability is better in capturing what e or $e+1$ means.

To catalog the optimal blocking schemes, we use the same representation of full factorial and fractional factorial designs as in CSW (1993). We use columns to represent both treatment factors and block variables. For each run size, we arrange the candidate columns corresponding to all the factorial effects of a full factorial design in Yates order (see Table **??**). The independent columns, from which all other columns can be generated by module-2 addition, are indicated in bold face. The columns of treatment factors and block variables for each design are listed in Tables **??**- **??** under the headings of "Columns (t)" and "Columns (b)". Since the sets of independent columns are included in all designs, they are omitted in the tables to save space.

To illustrate the usage of the tables, take as an example the 16-run design *4-0.1/B2.1* in Table **??** using columns {13, 14} for the two block variables. To find the conventional representation of a blocking scheme using defining words, we need to refer to Table **??**. First, find column 13 which is used as the first blocking variable. Second, identify the letters of the rows that contain 1's in column 13. They are a, c, and d in this case. So we write $B_1 = acd$ as the defining word for the first blocking variable. The second block variable can be obtained in the same way from column 14 in Table ??: $B_2 = bcd$. Note that Table ?? only provides the column representation matrix for up to 64-run designs, and it is straightforward to extend it for 128- and 256-run designs.

In Table ??, all MA blocking schemes for full factorial designs with 8, 16, 32, 64, 128, and 256 runs are given. The blocking schemes provided by BH^2 (1978, pp. 346-347) are also included for comparison (marked by "$\triangle$"). The column representations of the designs marked by "$\triangle$" may be different from the designs in BH^2, but the designs are equivalent according to the definition given in Section ??. For instance, design *4-0.1/B2.1* in Table ?? can be represented by

$$B_1 = acd, B_2 = bcd$$

which is obviously equivalent to the design

$$B_1 = 124, B_2 = 134$$

as in BH^2. Another example: although the design *4-0.1/B3.1*

$$B_1 = ab, B_2 = ac, B_3 = abcd$$

has a different set of defining words from the design

$$B_1 = 12, B_2 = 23, B_3 = 34$$

in BH^2, the block defining contrast subgroups for these two designs are

$$G_b = \{I, ab, ac, ad, bc, bd, cd, abcd\}$$

and

$$G_b = \{I, 12, 13, 14, 23, 24, 34, 1234\},$$

which are isomorphic.

From Table ??, we find that three designs (marked by "$\times$") in BH^2 are inferior to the MA blocking schemes. Only Case 1 will be discussed here in detail.

Case 1: 2^6 design in 2^4 blocks (block size = 4)

BH^2 gives a blocking scheme defined by $B_1 = 126, B_2 = 136, B_3 = 346, B_4 = 456$ with $W_b = (4, 6, 3, 2, 0)$, while the new design *6-0.1/B4.1* in Table **??** has $W_b = (3, 8, 3, 0, 1)$, and has the additional advantage of having one more clear 2fi, i.e., its C2 value is 12 instead of 11. Recall from (**??**) that the first component of W_b is $g_2(b)$.

Case 2: 2^7 design in 2^3 blocks (block size = 16)

The BH^2 design has estimability of order 2, while the new design *7-0.1/B3.1* in Table **??** has estimability of order 3.

Case 3: 2^7 design in 2^5 blocks (block size = 4)

The new design *7-0.1/B5.1* in Table **??** has a better W_b and also has one more clear 2fi.

3 Optimal Blocking Schemes for 2^{n-p} Designs

Following the design representation in CSW (1993), a 2^{n-p} fractional factorial(FF) design is denoted by n-$p.i$ where i represents the ith 2^{n-p} design in the catalogue of CSW (1993). (Readers should be reminded that the designs in CSW (1993) are arranged from the best to worst in terms of the aberration criterion, i.e., the first design is the MA FF design.) Such designs can be represented by a subset of n columns ($n-p$ independent columns and p generated columns) out of a total of $2^{n-p}-1$ columns in design matrices similar to Table **??**. In the same fashion, a 2^{n-p} FF design arranged in 2^k blocks (i.e., k block variables) can be represented by a subset of n columns for the treatment factors and a subset of k columns for the block variables.

The problem of obtaining optimal blocking schemes for FF designs is much more complicated than that for full factorial designs because we have to deal with two groups: (i) the defining contrast subgroup which defines the fractionation scheme for treatment factors, and (ii) the block defining contrast subgroup which defines the treatment effects that are confounded with block effects. Because of the intricate relations between the two groups, a 2^{n-p} design with higher resolution may be less desirable when blocking is required. For example, it is known and easy to verify that the MA 2^{5-1} design with the defining relation $I = 12345$ cannot be arranged in 8 blocks without confounding main effect of treatment factors, while the resolution IV 2^{5-1} design with $I = 1235$ can be arranged in 8 blocks with the blocking scheme defined by

$$B_1 = 12, B_2 = 13, B_3 = 14. \tag{5}$$

Even for 2^{n-p} designs with the same resolution, a "worse" design according to the aberration criterion may be able to accommodate more block variables than a "better" FF design. For example, the MA 2^{7-2} design defined by $I = 1236 = 12457$ (with resolution IV) cannot be arranged in 16 blocks without confounding some main effects of treatment factors, while the second best 2^{7-2} design defined by $I = 1236 = 1457$ (with resolution IV) can be arranged in 16 blocks with the blocking scheme defined by

$$B_1 = 12, B_2 = 13, B_3 = 14, B_3 = 15. \tag{6}$$

Throughout the tables in our collection there are five more cases which exhibit this phenomenon. For completeness, they are described briefly below.

Case 1: 2^{8-3} design

The MA design defined by $I = abcf = abdg = acdeh$ can accommodate at most three block variables, while the second best design defined by $I = abcf = abdg = aceh$ can accommodate four block variables with $B_1 = ab, B_2 = ac, B_3 = ad, B_4 = ae$.

The former is design *8-3.1/B3.1* of Table **??** and the latter is design *8-3.2/B4.1* of Table **??**.

Case 2: 2^{9-4} design

The MA design defined by $I = abcf = abdg = abeh = acdei$ (cf. design *9-4.1/B3.1*) and the second best design defined by $I = abcf = abdg = acdh = bcdei$ (cf. design *9-4.2/B3.1*) can accommodate at most three block variables, while the third best design defined by $I = abcf = abdg = aceh = adei$ (cf. design *9-4.3/B4.1*) can accommodate four block variables with $B_1 = ab, B_2 = ac, B_3 = ad, B_4 = ae$.

Case 3: 2^{7-1} design

The MA design defined by $I = abcdefg$ (cf. design *7-1.1/B4.1*) can accommodate at most four block variables, while the second best design defined by $I = abcdeg$ (cf. design *7-1.2/B5.1*) can accommodate five block variables with $B_1 = ab, B_2 = ac, B_3 = ad, B_4 = ae, B_5 = af$.

Case 4: 2^{8-2} design

The MA design defined by $I = abcdg = abefh$ (cf. design *8-2.1/B4.1*) can accommodate at most four block variables, while the second best design defined by $I = abcg = abdefh$ (cf. design *8-2.2/B5.1*) can accommodate five block variables with $B_1 = ab, B_2 = ac, B_3 = ad, B_4 = ae, B_5 = af$.

Case 5: 2^{9-3} design

The MA design defined by $I = abcg = abdeh = acdfi$ (cf. design *9-3.1/B4.1*), the second best design defined by $I = abcg = adeh = abdfi$, and the third best design defined by $I = abcg = abdeh = abdfi$ can accommodate at most four block variables, while the fourth best design defined by $I = abcg = abdh = acdefi$ (cf. design *9-3.4/B5.1*) can accommodate five block variables with $B_1 = ab, B_2 = ac, B_3 = ad, B_4 = ae, B_5 = af$.

We now use the example in (**??**) to illustrate a general definition to be given later. Take the first block variable $B_1 = 12$. Multiplying both sides of the equation by the defining relation $I = 1235$, we have $B_1 = 12 = 35$, i.e., the aliased pair of the 2fi's 12 and 35 are confounded with a block effect. Similarly we have $B_2 = 13 = 25$, $B_3 = 14 = 2345$, $B_1B_2 = 23 = 15$, $B_1B_3 = 24 = 1345$, $B_2B_3 = 34 = 1245$, $B_1B_2B_3 = 1234 = 45$. Among the 14 effects confounded with the seven block effects, ten effects are 2fi's and four are 4fi's.

To arrange a 2^{n-p} design in 2^k blocks, we first represent the blocking scheme by the group $G_b = \{I, v_1, \cdots, v_{2^k-1}\}$ as defined in Section 2. By multiplying the treatment defining relations in (**??**) by v_i, we see that ith block effect confounds the following set of aliased effects of treatment factors ($2^p(2^k-1)$ effects in total):

$$v_i = v_i w_1 = v_i w_2 = \cdots = v_i w_{2^p-1}, \tag{7}$$

for $i = 1, \cdots, 2^k-1$. Notationally we use $G_{b\otimes t}$ to represent the $2^p(2^k-1)$ sets of treatment effects that are confounded with the block effects, i.e., the cosets of all block effects. For the example in (**??**), we have

$$G_t = \{I, 1235\}, \quad G_b = \{I, 12, 13, 14, 23, 24, 34, 1234\}, \quad \text{and}$$

$$G_{b\otimes t} = \{12, 13, 14, 23, 24, 34, 1234, 35, 25, 2345, 15, 1345, 1245, 45\}$$

To measure the impact of blocking on the treatment defining contrast subgroup G_t, let g_i denote the number of ith order interactions among the $2^p(2^k-1)$ interactions in the set $G_{b\otimes t}$, which are confounded with the block effects. The word-length pattern of $G_{b\otimes t}$ is then defined by $W_b = (g_2, g_3, \cdots)$ to indicate the influence of block variables on the treatment effects. For FF designs with resolution IV and higher, $\binom{n}{2} - g_2$ equals the number of eligible 2fi's. A smaller g_2 is better since it gives a larger number of eligible 2fi's. For G_t with resolution III this equivalence does not hold and we need to compute the number of eligible 2fi's directly. (Definitions of eligible and clear effects were given in Section 2.)

For the example in (**??**) all ten 2fi's are confounded with the block effects, leaving no eligible 2fi's, and three pairs of 3fi's are eligible: $125 = 345, 135 = 245, 145 = 235$. In this example, there are no clear main effects or clear 2fi's.

Since there is no unique measure for selecting "optimal" blocking schemes for FF designs, we propose a concept of *admissibility* of blocking schemes using multiple criteria such as the word-length pattern W_t of the treatment defining contrast subgroup, the word-length pattern W_b of the set $G_{b\otimes t}$, the number of clear main effects C1, the number of clear 2fi's C2, and so on. Any given design criterion considered in this paper can be used to rank-order two designs. In other words, for any two designs d_1 and d_2, either d_1 is strictly better than, equal to, or strictly worse than d_2 according to a given design criterion. Two types of criteria will be considered: criteria which order real-valued quantities and criteria which order vector-valued quantities. "The smaller C1 the better" is an example of a real-valued criterion, and "the smaller the aberration of W_t the better" is an example of a vector-valued criterion. Such criteria can obviously be used to rank-order any two designs.

Definition: *Admissibility of blocking scheme*

Suppose q design criteria are under consideration. A design d_1 is said to be inadmissible if there exists another design d_2 such that d_2 is better than or equal to d_1 for all the q criteria, and strictly better than d_1 for at least one of the q criteria. Otherwise, d_1 is admissible.

In Tables **??**–**??**, we use C1, C2, W_t and W_b as four criteria to define admissibility and compile a list of admissible blocking schemes for FF designs with 8, 16, 32, 64, 128 runs. The blocking schemes given in BH^2 (1978, p. 408) and NBS are also included in the tables for comparison. The BH^2 designs are indicated by "$\triangle$" and the NBS designs by "$\star$". Some of these designs are not admissible, and are marked by "$\times$" in the tables. The column representations of the BH^2 and NBS designs may be different from the original designs, but the designs are equivalent according to the definition given in Section **??**.

From Tables **??**–**??**, we find that some designs given in BH^2 or NBS are inadmissible. Because of the wide-spread use of designs in BH^2 and NBS, it is worthwhile to compare these designs and the better designs found in this paper. Only Case 1 will be considered in detail.

Case 1: 16-run 2^{7-3} design in 2 blocks (block size = 8)

NBS gives

$$I = abeg = acfg = abcd, \quad B_1 = eg$$

with

$$W_t = (0,7,0,0,0), \quad W_b = (3,0,4,0,1,0),$$

while the new design *7-3.1/B1.1* in Table **??** has

$$W_t = (0,7,0,0,0), \quad W_b = (0,7,0,0,0,1).$$

The former has three 2fi's confounded with block effects while the latter does not confound any 2fi with block effects and therefore is preferred.

Case 2: 32-run 2^{7-2} design in 2 blocks (block size = 16)

The new design *7-2.1/B1.1* in Table **??** has a better W_b than the NBS design and, unlike the latter, does not confound any 2fi with block effects.

Case 3: 32-run 2^{8-3} design in 2 blocks (block size = 16)

The new design *8-3.1/B1.1* in Table **??** has a better W_b than the NBS design and, unlike the latter, does not confound any 2fi with block effects.

Case 4: 64-run 2^{8-2} design in 2^3 blocks (block size = 8)

The new design *8-2.1/B3.1* in Table **??** has a better W_b than the NBS design.

Case 5: 64-run 2^{9-3} design in 2 blocks (block size = 32)

The new design *9-3.1/B1.1* in Table **??** has a better W_b than the NBS design and, unlike the latter, does not confound any 2fi with block effects.

Case 6: 64-run 2^{9-3} design in 2^2 blocks (block size = 16)

The new design *9-3.1/B2.1* in Table **??** has a better W_b than the NBS design.

Case 7: 128-run 2^{9-2} design in 2^3 blocks (block size = 16)

The new design *9-2.1/B3.1* in Table **??** has a better W_b than the NBS and BH^2 designs.

For 2^{5-1}, 2^{7-3}, 2^{8-4} designs (16-run), NBS has no blocking schemes for three block variables; for 2^{6-1}, 2^{7-2}, 2^{8-3}, 2^{8-4} designs (32-run), NBS has no blocking schemes for four block variables; for 2^{7-1}, 2^{8-2}, 2^{9-3} designs (64-run), NBS has no blocking schemes for five block variables; and for 2^{9-2}, NBS has no blocking schemes for five and six block variables. Thanks to more powerful computing and the conceptual framework developed in this paper, we were able in all these cases to find blocking schemes to accommodate more block variables than the NBS tables. See Tables 4 to 6 for details.

In spite of the problems just mentioned, many NBS designs have desirable properties in terms of the number of clear main effects and clear 2fi's (denoted by C1 and C2 respectively in the tables). We observe that many NBS blocking schemes have larger C2 values than the corresponding MA schemes. This interesting property about the NBS designs has not been discussed in the literature. It may have something to do with the implicit criteria used in constructing the NBS tables.

Our list also contains all the 8- and 16-run designs for less than 10 factors in Bisgaard's(1994a) tables. It is interesting to note that, unlike the designs in the NBS tables, the designs in Bisgaard's tables always allow for the largest number of block variables.

To conclude this section we discuss Bisgaard's(1994b) modified definition of resolution for blocked 2^{n-p} designs and its relation to our approach. Consider the example from his paper, a 2^{6-2} design in 4 blocks with the defining relations:

$$
\begin{aligned}
I &= 1235 = 2346 = 1456 = B_1134 = B_1245 = B_1126 = B_1356 \\
&= B_2345 = B_2136 = B_2256 = B_2124 \\
&= B_1B_223 = B_1B_215 = B_1B_246 = B_1B_2123456. \qquad (8)
\end{aligned}
$$

Bisgaard correctly pointed out that B_1B_2 should be counted as of length one (as we also argued in Assumption (i) of Section 1). Then he proceeded to amend the usual definition of resolution (based on the shortest word-length among the defining relations) by adding the rule that all "interactions" between the block variables be counted as having length one. This modified definition provides a conservative estimate of the "resolution" of a blocked design as he indicated in the conclusion of the paper. Here we provide an explanation on why it is conservative. For the particular example the blocked design in (8) has resolution III according to his definition. To measure how well it estimates the main effects and the 2fi's, we compute C1 and C2 for the design. Since the only words of "length" three involve block variables and the interactions between blocks and treatment factors are assumed negligible (see Assumption(ii) of Section 1), all the main effects are clear of 2fi's and block effects, so C1=6. This clearly indicates that the design is better than a resolution III design since the latter would imply that at least one main effect is confounded with a 2fi or block effect. On the other hand, this design is not as good as a resolution IV design because three of its 2fi's, namely, 15, 23, 46 cannot be estimated due to confounding with block effects. Recall that in a resolution IV design any 2fi can still be estimated if the aliased 2fi's are assumed negligible. Because block effects are potentially large, it is not realistic to assume block effects are negligible in order to be able to estimate 2fi's. So the design can be considered to lie somewhere between resolution III and IV. Because his definition does not distinguish words involving B_i from those not involving B_i in the defining relations, it does not capture the subtlety between resolution III and IV.

4 Method for the Construction of Blocking Schemes

In this section we briefly describe how the blocking schemes in the tables are constructed.

4.1 Equivalence of blocking schemes

For given n and p, there are many different ways of selecting a 2^{n-p} FF design (see CSW, 1993). Similarly, for a given 2^{n-p} design, there are many different ways of choosing k block variables. However, among all the possible choices of blocking schemes, many are equivalent in the following sense. As we pointed out in Section **??**, "generalized interactions" of block variables are treated the same as "individual" block variables, hence the equivalence of blocking schemes can be defined through the equivalence of the groups $G_{b\otimes t}$.

Definition: *Equivalence of blocking schemes*

For a 2^{n-p} FF design, let $f_1, f_2, \cdots, f_n$ denote the n treatment factors and $G^1_{b\otimes t}$ and $G^2_{b\otimes t}$ denote the cosets of block effects of two blocking schemes. If there exists a permutation map $M(\cdot)$ from $(f_1, f_2, \cdots, f_n)$ to itself $(M(f_1), M(f_2), \cdots, M(f_n))$ (i.e., $(M(f_1), M(f_2), \cdots, M(f_n))$

is a permutation of $(f_1, f_2, \cdots, f_n)$) such that $M(G^1_{b\otimes t})$ is identical to $G^2_{b\otimes t}$, then the two blocking schemes are said to be equivalent. Note that $M(G^1_{b\otimes t})$ is defined as $\{M(v_i w_j); i = 1, \cdots, 2^k-1; j = 0, \ldots, 2^p-1\}$ if $G^1_{b\otimes t} = \{v_i w_j; i = 1, \cdots, 2^k-1; j = 0, \ldots, 2^p-1\}$. Obviously, equivalent blocking schemes have the same W_b and the same values of C1 and C2.

4.2 Construction procedures

Based on these two concepts, we use the following approach to construct optimal blocking schemes. First, construct all the non-equivalent blocking schemes for all the non-equivalent FF designs for a given run size, number of treatment factors, and number of block variables using the FF designs in CSW (1993). Second, eliminate all the inadmissible blocking schemes. This significantly reduces the number of candidate blocking schemes. By following this approach, we were able to compile all the admissible blocking schemes with up to nine treatment factors in 8, 16, 32, 64 and 128 runs. To save space we only give here a much smaller collection of schemes in Tables **??**- **??**, namely those that are ranked highly according to the criteria W_t, W_b, C1 and C2.

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Table 1: Matrices for 16, 32 and 64-run designs.
(For 16-run designs, it consists of the first 4 rows and 15 columns; for 32-run designs, it consists the first 5 rows and 31 columns, and for 64-run designs, it is the whole matrix. Independent columns are numbered 1, 2, 4, 8, 16 and 32. The rows are denoted by a, b, c, d, e, f .)

	1	**2**	3	**4**	5	6	7	**8**	9	10	11	12	13	14	15	**16**	17	18	19	20	21
a	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
b	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
c	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1
d	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
e	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
f	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	22	23	24	25	26	27	28	29	30	31	**32**	33	34	35	36	37	38	39	40	41	42
a	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
b	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1
c	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0
d	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1
e	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1

	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
a	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
b	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
c	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
d	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
e	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
f	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2: Blocking schemes of full factorial designs.
($\times-$ blocking schemes that do not have minimum aberration; $\triangle-$ BH^2 designs.)

Design	Columns (b)	W_b	C1	C2	
3-0.1/B1.1	7	0 1	3	3	$\triangle$
3-0.1/B2.1	3 5	3 0	3	0	$\triangle$
4-0.1/B1.1	15	0 0 1	4	6	$\triangle$
4-0.1/B2.1	13 14	1 2 0	4	5	$\triangle$
4-0.1/B3.1	3 5 15	6 0 1	4	0	$\triangle$
5-0.1/B1.1	31	0 0 0 1	5	10	$\triangle$
5-0.1/B2.1	7 30	0 2 1 0	5	10	$\triangle$
5-0.1/B3.1	15 21 22	2 4 1 0	5	8	$\triangle$
5-0.1/B4.1	3 5 9 17	10 0 5 0	5	0	$\triangle$
6-0.1/B1.1	63	0 0 0 0 1	6	15	$\triangle$
6-0.1/B2.1	15 51	0 0 3 0 0	6	15	$\triangle$
6-0.1/B3.1	7 30 45	0 4 3 0 0	6	15	$\triangle$
6-0.1/B4.1	3 12 21 37	3 8 3 0 1	6	12	
6-0.1/B4.2	3 5 24 41	4 6 3 2 0	6	11	$\times$ $\triangle$
6-0.1/B5.1	3 5 9 17 33	15 0 15 0 1	6	0	$\triangle$
7-0.1/B1.1	127	0 0 0 0 0 1	7	21	$\triangle$
7-0.1/B2.1	115 124	0 0 1 2 0 0	7	21	$\triangle$
7-0.1/B3.1	15 51 85	0 0 7 0 0 0	7	21	
7-0.1/B3.3	30 123 124	0 3 2 1 1 0	7	21	$\times$ $\triangle$
7-0.1/B4.1	7 25 42 75	0 7 7 0 0 1	7	21	$\triangle$
7-0.1/B5.1	3 5 24 41 73	5 12 7 4 3 0	7	16	
7-0.1/B5.2	3 5 24 40 73	6 9 9 6 0 1	7	15	$\times$ $\triangle$
7-0.1/B6.1	3 5 9 17 33 65	21 0 35 0 7 0	7	0	$\triangle$
8-0.1/B1.1	255	0 0 0 0 0 0 1	8	28	
8-0.1/B2.1	31 252	0 0 0 2 1 0 0	8	28	
8-0.1/B3.1	213 218 230	0 0 3 4 0 0 0	8	28	
8-0.1/B4.1	15 51 85 150	0 0 14 0 0 0 1	8	28	
8-0.1/B5.1	3 13 49 84 149	1 10 11 4 3 2 0	8	27	
8-0.1/B6.1	3 5 24 40 73 137	7 18 15 12 9 2 0	8	21	
8-0.1/B7.1	3 5 9 17 33 65 129	28 0 70 0 28 0 1	8	0	

Table 3: Blocking schemes of 8-run FF designs

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2
4-1.1/B1.1	7	0 1	3	2 0 0	4	0
4-1.2/B1.1	3	1 0	5	1 1 0	1	2
4-1.1/B2.1	7	0 1	3 5	6 0 0	4	0
5-2.1/B1.1	3 5	2 1 0	6	2 2 0 0	0	0
6-3.1/B1.1	3 5 6	4 3 0 0	7	3 4 0 0 1	0	0

Table 4: Blocking schemes of 16-run FF designs.

($\times$ — inadmissible blocking schemes; $\triangle$ — BH^2 designs; $\star$ — NBS designs;)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
5-1.1/B1.1	15	0 0 1	3	1 1 0 0	5	9	$\star$
5-1.2/B1.1	7	0 1 0	11	0 2 0 0	5	4	
5-1.3/B1.1	3	1 0 0	13	0 1 1 0	2	7	
5-1.1/B2.1	15	0 0 1	3 5	3 3 0 0	5	7	$\star$
5-1.2/B2.1	7	0 1 0	13 14	2 4 0 0	5	4	
5-1.3/B2.1	3	1 0 0	5 15	2 3 1 0	2	5	
5-1.2/B3.1	7	0 1 0	9 10 12	10 0 4 0	5	0	
6-2.1/B1.1	7 11	0 3 0 0	13	0 4 0 0 0	6	0	$\star$
6-2.2/B1.1	3 13	1 1 1 0	6	1 2 1 0 0	3	5	
6-2.2/B1.2	3 13	1 1 1 0	5	2 1 0 1 0	3	6	
6-2.4/B1.1	3 5	2 1 0 0	14	0 2 2 0 0	1	5	
6-2.1/B2.1	7 11	0 3 0 0	13 14	3 8 0 0 1	6	0	$\star$
6-2.2/B2.1	3 13	1 1 1 0	10 15	4 5 2 1 0	3	4	
6-2.2/B2.2	3 13	1 1 1 0	5 9	6 3 0 3 0	3	6	
6-2.1/B3.1	7 11	0 3 0 0	5 6 9	15 0 12 0 1	6	0	$\star$
7-3.1/B1.1	7 11 13	0 7 0 0 0	14	0 7 0 0 0 1	7	0	
7-3.1/B1.2	7 11 13	0 7 0 0 0	3	3 0 4 0 1 0	7	0	$\times$ $\star$
7-3.2/B1.1	3 5 14	2 3 2 0 0	9	1 4 2 0 1 0	2	1	
7-3.2/B1.2	3 5 14	2 3 2 0 0	10	2 2 2 2 0 0	2	2	
7-3.3/B1.1	3 5 10	3 2 1 1 0	12	1 3 3 1 0 0	0	3	
7-3.3/B1.2	3 5 10	3 2 1 1 0	6	2 3 1 1 1 0	0	4	
7-3.4/B1.1	3 5 9	3 3 0 0 1	14	0 4 4 0 0 0	0	0	
7-3.5/B1.1	3 5 6	4 3 0 0 0	15	0 3 4 0 0 1	1	6	
7-3.1/B2.1	7 11 13	0 7 0 0 0	3 5	9 0 12 0 3 0	7	0	$\star$
7-3.2/B2.1	3 5 14	2 3 2 0 0	9 15	5 10 4 2 3 0	2	0	
7-3.2/B2.2	3 5 14	2 3 2 0 0	10 13	6 7 6 4 0 1	2	2	
7-3.5/B2.1	3 5 6	4 3 0 0 0	9 14	5 8 4 4 3 0	1	4	
7-3.1/B3.1	7 11 13	0 7 0 0 0	3 5 9	21 0 28 0 7 0	7	0	
8-4.1/B1.1	7 11 13 14	0 14 0 0 0 1	3	4 0 8 0 4 0 0	8	0	$\star$
8-4.2/B1.1	3 5 9 14	3 7 4 0 1 0	15	1 7 4 0 3 1 0	1	0	
8-4.2/B1.2	3 5 9 14	3 7 4 0 1 0	6	3 3 4 4 1 1 0	1	1	
8-4.4/B1.1	3 5 6 15	4 6 4 0 0 1	9	2 4 4 4 2 0 0	2	0	
8-4.6/B1.1	3 5 6 7	7 7 0 0 1 0	9	1 3 4 4 3 1 0	1	6	
8-4.1/B2.1	7 11 13 14	0 14 0 0 0 1	3 5	12 0 24 0 12 0 0	8	0	$\star$
8-4.2/B2.1	3 5 9 14	3 7 4 0 1 0	6 10	9 9 12 12 3 3 0	1	1	
8-4.3/B2.1	3 5 10 12	4 5 4 2 0 0	11 13	7 14 10 8 7 2 0	0	0	
8-4.4/B2.1	3 5 6 15	4 6 4 0 0 1	9 14	8 12 8 12 8 0 0	2	0	
8-4.5/B2.1	3 5 6 9	5 5 2 2 1 0	10 13	7 13 10 10 7 1 0	0	2	
8-4.1/B3.1	7 11 13 14	0 14 0 0 0 1	3 5 9	28 0 56 0 28 0 0	8	0	

Table 4: (Continued)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2
9-5.1/B1.1	3 5 9 14 15	4 14 8 0 4 1 0	6	4 4 8 8 4 4 0 0	0	0
9-5.2/B1.1	3 5 10 12 15	6 9 9 6 0 0 1	6	3 7 6 6 7 3 0 0	0	0
9-5.3/B1.1	3 5 6 9 14	6 10 8 4 2 1 0	15	2 8 8 4 6 4 0 0	0	0
9-5.4/B1.1	3 5 6 9 10	7 9 6 6 3 0 0	13	2 7 9 6 4 3 1 0	0	0
9-5.1/B2.1	3 5 9 14 15	4 14 8 0 4 1 0	6 10	12 12 24 24 12 12 0 0	0	0
9-5.2/B2.1	3 5 10 12 15	6 9 9 6 0 0 1	6 11	9 21 18 18 21 9 0 0	0	0

Table 5: Blocking schemes of 32-run FF designs.

($\times$ — inadmissible blocking schemes; $\triangle$ — BH^2 designs; $\star$ — NBS designs;)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
6-1.1/B1.1	31	0 0 0 1	7	0 2 0 0 0	6	15	$\triangle$ $\star$
6-1.2/B1.1	15	0 0 1 0	19	0 1 1 0 0	6	15	
6-1.3/B1.1	7	0 1 0 0	27	0 0 2 0 0	6	9	
6-1.1/B2.1	31	0 0 0 1	13 14	1 4 1 0 0	6	14	$\star$
6-1.2/B2.1	15	0 0 1 0	21 22	1 3 2 0 0	6	14	
6-1.3/B2.1	7	0 1 0 0	11 30	0 4 2 0 0	6	9	
6-1.1/B3.1	31	0 0 0 1	21 22 25	3 8 3 0 0	6	12	$\star$
6-1.3/B3.1	7	0 1 0 0	13 14 27	3 8 2 0 1	6	8	
6-1.4/B3.1	3	1 0 0 0	15 22 28	3 7 3 0 1	3	9	
6-1.1/B4.1	31	0 0 0 1	3 5 9 17	15 0 15 0 0	6	0	
6-1.3/B4.1	7	0 1 0 0	3 5 9 17	15 0 14 0 1	6	0	
7-2.1/B1.1	7 27	0 1 2 0 0	13	0 2 2 0 0 0	7	15	
7-2.1/B1.5	7 27	0 1 2 0 0	3	2 1 0 0 0 1	7	15	$\times$ $\star$
7-2.3/B1.1	7 11	0 3 0 0 0	29	0 0 4 0 0 0	7	6	
7-2.4/B1.1	3 29	1 0 1 1 0	14	0 2 2 0 0 0	4	18	
7-2.6/B1.1	3 13	1 1 1 0 0	22	0 1 2 1 0 0	4	12	
7-2.8/B1.1	3 5	2 1 0 0 0	30	0 0 2 2 0 0	2	11	
7-2.1/B2.1	7 27	0 1 2 0 0	21 30	1 6 4 0 1 0	7	14	$\star$
7-2.1/B2.3	7 27	0 1 2 0 0	13 14	2 5 4 0 0 1	7	15	
7-2.3/B2.1	7 11	0 3 0 0 0	19 30	0 7 4 0 0 1	7	6	
7-2.4/B2.1	3 29	1 0 1 1 0	22 26	1 5 5 1 0 0	4	17	
7-2.5/B2.1	3 28	1 1 0 0 1	13 22	0 6 6 0 0 0	4	12	
7-2.1/B3.1	7 27	0 1 2 0 0	14 22 29	5 12 6 2 3 0	7	12	$\star$
7-2.2/B3.1	7 25	0 2 0 1 0	21 22 28	5 12 5 4 2 0	7	8	
7-2.4/B3.1	3 29	1 0 1 1 0	18 23 27	5 11 7 3 2 0	4	13	
7-2.2/B4.1	7 25	0 2 0 1 0	3 5 9 17	21 0 33 0 6 0	7	0	
7-2.3/B4.1	7 11	0 3 0 0 0	3 5 9 17	21 0 32 0 7 0	7	0	
8-3.1/B1.1	7 11 29	0 3 4 0 0 0	19	0 3 4 0 0 1 0	8	13	
8-3.1/B1.5	7 11 29	0 3 4 0 0 0	5	2 1 2 2 0 1 0	8	13	$\times$ $\star$
8-3.3/B1.1	7 11 19	0 6 0 0 0 1	29	0 0 8 0 0 0 0	8	0	
8-3.4/B1.1	7 11 13	0 7 0 0 0 0	30	0 0 7 0 0 0 1	8	7	
8-3.5/B1.1	3 13 22	1 2 3 1 0 0	25	0 3 3 1 1 0 0	5	13	
8-3.7/B1.1	3 5 30	2 1 2 2 0 0	15	0 3 4 0 0 1 0	3	18	
8-3.8/B1.1	3 12 21	2 1 2 2 0 0	26	0 2 4 2 0 0 0	2	16	
8-3.1/B2.1	7 11 29	0 3 4 0 0 0	19 30	1 10 8 0 3 2 0	8	12	$\star$
8-3.1/B2.8	7 11 29	0 3 4 0 0 0	5 9	6 3 6 6 0 3 0	8	13	
8-3.5/B2.2	3 13 22	1 2 3 1 0 0	25 28	2 8 7 3 3 1 0	5	13	
8-3.7/B2.1	3 5 30	2 1 2 2 0 0	23 25	1 8 10 2 1 2 0	3	17	
8-3.1/B3.1	7 11 29	0 3 4 0 0 0	14 17 18	8 16 11 12 8 0 1	8	8	
8-3.1/B3.2	7 11 29	0 3 4 0 0 0	18 23 30	9 12 16 12 3 4 0	8	10	

Table 5: (Continued)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
8-3.1/B3.4	7 11 29	0 3 4 0 0 0	5 6 9	15 6 12 16 1 6 0	8	13	⋆
8-3.5/B3.1	3 13 22	1 2 3 1 0 0	17 26 31	7 17 13 9 8 2 0	5	10	
8-3.7/B3.1	3 5 30	2 1 2 2 0 0	9 15 18	7 16 14 10 7 2 0	3	13	
8-3.8/B3.2	3 12 21	2 1 2 2 0 0	10 27 30	8 14 13 14 6 0 1	2	14	
8-3.12/B3.1	3 5 24	3 1 0 2 1 0	15 18 20	7 15 14 12 7 1 0	0	10	
8-3.2/B4.1	7 11 21	0 5 0 2 0 0	3 5 9 17	28 0 65 0 26 0 1	8	0	
8-3.3/B4.1	7 11 19	0 6 0 0 0 1	3 5 9 17	28 0 64 0 28 0 0	8	0	
9-4.1/B1.1	7 11 19 29	0 6 8 0 0 1 0	30	0 4 8 0 0 4 0 0	9	8	
9-4.2/B1.1	7 11 13 30	0 7 7 0 0 0 1	17	1 3 4 4 3 1 0 0	9	14	
9-4.2/B1.3	7 11 13 30	0 7 7 0 0 0 1	3	3 1 4 4 1 3 0 0	9	15	⋆
9-4.5/B1.1	7 11 13 14	0 14 0 0 0 1 0	19	0 4 0 8 0 4 0 0	9	8	
9-4.6/B1.1	3 13 21 26	1 5 6 2 1 0 0	28	0 5 5 2 2 1 1 0	6	9	
9-4.7/B1.1	3 13 21 25	1 7 4 0 3 0 0	30	0 3 7 4 0 1 1 0	6	12	
9-4.13/B1.1	3 5 9 30	3 3 4 4 1 0 0	15	0 5 7 0 0 3 1 0	2	15	
9-4.20/B1.1	3 5 6 31	4 3 3 4 0 0 1	9	1 3 4 4 3 1 0 0	3	20	
9-4.20/B1.2	3 5 6 31	4 3 3 4 0 0 1	7	3 5 0 0 5 3 0 0	3	21	
9-4.27/B1.1	3 5 6 24	5 3 0 4 3 0 0	15	0 3 7 4 0 1 1 0	0	18	
9-4.1/B2.1	7 11 19 29	0 6 8 0 0 1 0	5 30	4 8 16 8 4 8 0 0	9	8	
9-4.2/B2.1	7 11 13 30	0 7 7 0 0 0 1	17 31	3 13 8 8 13 3 0 0	9	12	
9-4.2/B2.2	7 11 13 30	0 7 7 0 0 0 1	17 18	5 7 12 12 7 5 0 0	9	13	
9-4.2/B2.3	7 11 13 30	0 7 7 0 0 0 1	3 5	9 3 12 12 3 9 0 0	9	15	⋆
9-4.20/B2.1	3 5 6 31	4 3 3 4 0 0 1	9 17	3 9 12 12 9 3 0 0	3	18	
9-4.20/B2.2	3 5 6 31	4 3 3 4 0 0 1	9 14	5 11 8 8 11 5 0 0	3	19	
9-4.1/B3.1	7 11 19 29	0 6 8 0 0 1 0	5 9 30	12 16 32 24 12 16 0 0	9	8	
9-4.2/B3.1	7 11 13 30	0 7 7 0 0 0 1	17 18 20	13 15 28 28 15 13 0 0	9	11	
9-4.2/B3.2	7 11 13 30	0 7 7 0 0 0 1	3 5 9	21 7 28 28 7 21 0 0	9	15	⋆
9-4.3/B3.1	7 11 21 25	0 9 0 6 0 0 0	13 14 31	9 27 18 27 21 9 0 1	9	0	
9-4.4/B3.1	7 11 13 19	0 10 0 4 0 1 0	25 26 28	10 24 18 32 18 8 2 0	9	2	
9-4.20/B3.1	3 5 6 31	4 3 3 4 0 0 1	9 14 18	9 23 24 24 23 9 0 0	3	15	
9-4.3/B4.1	7 11 21 25	0 9 0 6 0 0 0	3 5 9 17	36 0 117 0 78 0 9 0	9	0	
9-4.4/B4.1	7 11 13 19	0 10 0 4 0 1 0	3 5 9 17	36 0 116 0 80 0 8 0	9	0	

Table 6: Blocking schemes of 64-run FF designs.

($\times$ — inadmissible blocking schemes; $\triangle$ — BH^2 designs; $\star$ — NBS designs;)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
7-1.1/B1.1	63	0 0 0 0 1	7	0 1 1 0 0 0	7	21	$\star$
7-1.2/B1.1	31	0 0 0 1 0	39	0 0 2 0 0 0	7	21	
7-1.3/B1.1	15	0 0 1 0 0	51	0 0 1 1 0 0	7	21	
7-1.4/B1.1	7	0 1 0 0 0	59	0 0 0 2 0 0	7	15	
7-1.1/B2.1	63	0 0 0 0 1	7 25	0 3 3 0 0 0	7	21	$\star$
7-1.2/B2.1	31	0 0 0 1 0	41 46	0 3 2 1 0 0	7	21	
7-1.4/B2.1	7	0 1 0 0 0	27 45	0 0 6 0 0 0	7	15	
7-1.1/B3.1	63	0 0 0 0 1	7 25 42	0 7 7 0 0 0	7	21	$\triangle$ $\star$
7-1.4/B3.1	7	0 1 0 0 0	30 45 56	0 7 6 0 0 1	7	15	
7-1.1/B4.1	63	0 0 0 0 1	3 5 24 40	6 9 9 6 0 0	7	15	$\star$
7-1.2/B4.1	31	0 0 0 1 0	3 12 21 33	5 12 7 4 2 0	7	16	
7-1.3/B4.1	15	0 0 1 0 0	3 5 24 41	5 12 7 3 3 0	7	16	
7-1.2/B5.1	31	0 0 0 1 0	3 5 9 17 33	21 0 35 0 6 0	7	0	
7-1.4/B5.1	7	0 1 0 0 0	3 5 9 17 33	21 0 34 0 7 0	7	0	
8-2.1/B1.1	15 51	0 0 2 1 0 0	21	0 1 2 1 0 0 0	8	28	$\star$
8-2.2/B1.1	7 59	0 1 0 2 0 0	29	0 0 4 0 0 0 0	8	22	
8-2.4/B1.1	7 27	0 1 2 0 0 0	45	0 0 2 2 0 0 0	8	22	
8-2.7/B1.1	7 11	0 3 0 0 0 0	61	0 0 0 4 0 0 0	8	13	
8-2.1/B2.1	15 51	0 0 2 1 0 0	21 42	0 4 5 2 1 0 0	8	28	$\triangle$ $\star$
8-2.2/B2.1	7 59	0 1 0 2 0 0	25 53	0 4 4 4 0 0 0	8	22	
8-2.3/B2.1	7 57	0 1 1 0 1 0	26 44	0 3 6 3 0 0 0	8	22	
8-2.5/B2.1	7 56	0 2 0 0 0 1	27 45	0 0 12 0 0 0 0	8	16	
8-2.1/B3.1	15 51	0 0 2 1 0 0	41 42 61	2 8 10 6 1 0 1	8	26	
8-2.1/B3.2	15 51	0 0 2 1 0 0	22 41 58	2 9 9 4 3 1 0	8	26	$\times$ $\star$
8-2.2/B3.1	7 59	0 1 0 2 0 0	37 46 54	1 10 10 4 1 2 0	8	21	
8-2.1/B4.1	15 51	0 0 2 1 0 0	5 9 18 34	7 18 15 10 8 2 0	8	21	$\star$
8-2.2/B4.1	7 59	0 1 0 2 0 0	5 9 18 35	7 18 14 12 7 2 0	8	17	
8-2.3/B4.1	7 57	0 1 1 0 1 0	3 12 21 37	7 18 14 11 9 1 0	8	17	
8-2.2/B5.1	7 59	0 1 0 2 0 0	3 5 9 17 33	28 0 69 0 26 0 1	8	0	
8-2.5/B5.1	7 56	0 2 0 0 0 1	3 5 9 17 33	28 0 68 0 28 0 0	8	0	
9-3.1/B1.1	7 27 45	0 1 4 2 0 0 0	51	0 1 4 2 0 1 0 0	9	30	
9-3.1/B1.11	7 27 45	0 1 4 2 0 0 0	6	2 0 1 4 0 0 1 0	9	30	$\times$ $\star$
9-3.2/B1.1	7 25 43	0 2 3 1 1 0 0	52	0 1 3 3 1 0 0 0	9	24	
9-3.3/B1.1	7 27 43	0 2 4 0 0 1 0	53	0 0 4 4 0 0 0 0	9	24	
9-3.12/B1.1	7 11 13	0 7 0 0 0 0 0	62	0 0 0 7 0 0 0 1	9	15	
9-3.1/B2.1	7 27 45	0 1 4 2 0 0 0	49 63	0 6 8 5 4 0 0 1	9	30	
9-3.1/B2.2	7 27 45	0 1 4 2 0 0 0	21 51	0 6 9 4 2 2 1 0	9	30	$\times$ $\star$
9-3.2/B2.1	7 25 43	0 2 3 1 1 0 0	49 60	0 6 8 5 3 1 1 0	9	24	
9-3.3/B2.1	7 27 43	0 2 4 0 0 1 0	13 62	0 4 12 4 0 4 0 0	9	24	

Table 6: (Continued)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
9-3.6/B2.1	7 11 53	0 3 2 0 2 0 0	45 59	0 4 11 6 0 2 1 0	9	21	
9-3.1/B3.1	7 27 45	0 1 4 2 0 0 0	35 53 62	2 14 17 8 8 6 1 0	9	28	
9-3.1/B3.3	7 27 45	0 1 4 2 0 0 0	49 50 60	3 13 14 11 11 3 0 1	9	29	
9-3.1/B3.15	7 27 45	0 1 4 2 0 0 0	49 50 52	6 10 9 16 12 2 1 0	9	30	$\star$
9-3.2/B3.1	7 25 43	0 2 3 1 1 0 0	37 46 49	2 14 16 9 9 5 1 0	9	22	
9-3.3/B3.1	7 27 43	0 2 4 0 0 1 0	49 50 60	2 14 16 8 10 6 0 0	9	24	
9-3.1/B4.1	7 27 45	0 1 4 2 0 0 0	3 13 17 37	9 27 26 23 25 9 0 1	9	23	
9-3.1/B4.4	7 27 45	0 1 4 2 0 0 0	3 5 9 48	12 20 25 36 18 4 5 0	9	24	$\star$
9-3.4/B5.1	7 11 61	0 3 0 4 0 0 0	3 5 9 17 33	36 0 123 0 80 0 9 0	9	0	
9-3.5/B5.1	7 25 42	0 3 0 4 0 0 0	3 5 9 17 33	36 0 123 0 80 0 9 0	9	0	
9-3.9/B5.1	7 11 49	0 4 0 2 0 1 0	3 5 9 17 33	36 0 122 0 82 0 8 0	9	0	

Table 7: Blocking schemes of 128-run FF designs.

($\times$ — inadmissible blocking schemes; $\triangle$ — BH^2 designs; $\star$ — NBS designs;)

Design	Columns (t)	W_t	Columns (b)	W_b	C1	C2	
8-1.1/B1.1	127	0 0 0 0 0 1	15	0 0 2 0 0 0 0	8	28	$\star$
8-1.2/B1.1	63	0 0 0 0 1 0	71	0 0 1 1 0 0 0	8	28	
8-1.3/B1.1	31	0 0 0 1 0 0	103	0 0 0 2 0 0 0	8	28	
8-1.5/B1.1	7	0 1 0 0 0 0	123	0 0 0 0 2 0 0	8	22	
8-1.1/B2.1	127	0 0 0 0 0 1	15 51	0 0 6 0 0 0 0	8	28	$\star$
8-1.3/B2.1	31	0 0 0 1 0 0	39 108	0 0 5 0 1 0 0	8	28	
8-1.4/B2.1	15	0 0 1 0 0 0	51 85	0 0 3 3 0 0 0	8	28	
8-1.1/B3.1	127	0 0 0 0 0 1	15 51 85	0 0 14 0 0 0 0	8	28	$\star$
8-1.5/B3.1	7	0 1 0 0 0 0	27 45 120	0 0 13 0 0 0 1	8	22	
8-1.1/B4.1	127	0 0 0 0 0 1	3 12 49 84	2 8 10 8 2 0 0	8	26	$\star$
8-1.2/B4.1	63	0 0 0 0 1 0	7 25 42 65	1 10 11 4 3 1 0	8	27	
8-1.3/B4.1	31	0 0 0 1 0 0	3 13 52 85	1 10 11 4 2 2 0	8	27	
8-1.4/B4.1	15	0 0 1 0 0 0	3 21 41 77	1 10 11 3 3 2 0	8	27	
8-1.1/B5.1	127	0 0 0 0 0 1	3 5 9 48 81	8 16 14 16 8 0 0	8	20	$\star$
8-1.2/B5.1	63	0 0 0 0 1 0	3 5 24 40 73	7 18 15 12 9 1 0	8	21	
8-1.3/B5.1	31	0 0 0 1 0 0	3 12 21 33 68	7 18 15 12 8 2 0	8	21	
8-1.4/B5.1	15	0 0 1 0 0 0	3 5 24 40 73	7 18 15 11 9 2 0	8	21	
8-1.1/B6.1	127	0 0 0 0 0 1	3 5 9 17 33 65	28 0 70 0 28 0 0	8	0	
8-1.3/B6.1	31	0 0 0 1 0 0	3 5 9 17 33 65	28 0 70 0 27 0 1	8	0	
9-2.1/B1.1	31 103	0 0 0 3 0 0 0	43	0 0 3 0 1 0 0 0	9	36	$\star$
9-2.2/B1.1	15 115	0 0 1 1 1 0 0	53	0 0 2 2 0 0 0 0	9	36	
9-2.4/B1.1	15 51	0 0 2 1 0 0 0	85	0 0 1 2 1 0 0 0	9	36	
9-2.10/B1.1	7 27	0 1 2 0 0 0 0	109	0 0 0 2 2 0 0 0	9	30	
9-2.13/B1.1	7 11	0 3 0 0 0 0 0	125	0 0 0 0 4 0 0 0	9	21	
9-2.1/B2.1	31 103	0 0 0 3 0 0 0	43 85	0 0 9 0 3 0 0 0	9	36	$\star$
9-2.3/B2.1	15 113	0 0 2 0 0 1 0	54 90	0 0 6 6 0 0 0 0	9	36	
9-2.10/B2.1	7 27	0 1 2 0 0 0 0	45 120	0 0 5 6 0 0 1 0	9	30	
9-2.1/B3.1	31 103	0 0 0 3 0 0 0	41 46 85	0 6 9 9 3 0 0 1	9	36	
9-2.1/B3.2	31 103	0 0 0 3 0 0 0	46 81 90	0 7 9 6 3 3 0 0	9	36	$\times$ $\triangle$ $\star$
9-2.3/B3.1	15 113	0 0 2 0 0 1 0	19 54 90	0 4 14 6 0 4 0 0	9	36	
9-2.1/B4.1	31 103	0 0 0 3 0 0 0	3 13 37 84	2 14 18 12 7 6 1 0	9	34	$\star$
9-2.2/B4.1	15 115	0 0 1 1 1 0 0	5 18 35 73	2 14 18 11 9 5 1 0	9	34	
9-2.3/B4.1	15 113	0 0 2 0 0 1 0	6 19 35 74	2 14 18 10 10 6 0 0	9	34	
9-2.4/B4.1	15 51	0 0 2 1 0 0 0	3 20 41 69	2 14 18 10 9 6 1 0	9	34	
9-2.1/B5.1	31 103	0 0 0 3 0 0 0	7 9 18 33 66	9 27 27 27 24 9 0 1	9	27	
9-2.2/B5.1	15 115	0 0 1 1 1 0 0	3 5 24 40 73	9 27 27 26 26 8 0 1	9	27	
9-2.4/B5.1	15 51	0 0 2 1 0 0 0	5 9 18 34 67	9 27 27 25 26 9 0 1	9	27	
9-2.1/B6.1	31 103	0 0 0 3 0 0 0	3 5 9 17 33 65	36 0 126 0 81 0 9 0	9	0	
9-2.6/B6.1	7 121	0 1 0 1 0 1 0	3 5 9 17 33 65	36 0 125 0 83 0 8 0	9	0	