A Model of Coverage Probability under Shadow Fading

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Abstract

We give a simple analytic model of coverage probability for CDMA cellular phone systems under lognormally distributed shadow fading. Prior analyses have generally considered the coverage probability of a single antenna; here we consider the probability of coverage by an ensemble of antennas, using some independence assumptions, but also modeling a limited form of dependency among the antenna fades. We use the Fenton-Wilkinson approach of approximating the external interference I_0 as lognormally distributed. We show that our model gives a coverage probability that is generally within a few percent of Monte Carlo estimates, over a wide regime of antenna strengths and other relevant parameters.

1 Introduction

In modeling a spread-spectrum cellular phone system, we are interested in the conditions under which the quality of the radio link between the mobile (phone) and the base station antenna is adequate. An important measure of that quality is the E_c/I_0 of the pilot signal, since important decisions in starting a call are based on it. Here E_c/I_0 for a given mobile m and antenna a is the ratio of the signal strength E_c received by m from a to the interference I_0 received by m from all other sources; such interference is due to external noise, and to the power received from all antennas. (As measured, the interference includes all the power received from a itself, but this only approximates the fact that some power received from a is interference for this mobile.)

If E_c/I_0 is too low, then the call may not be carried by a, or only carried with poor quality. If the E_c/I_0 from a at a particular location is above a quality threshold, then we say that the location is "covered" by a, and in a given cellular market, it is important to know what the probability that locations are covered.

The situation is complicated by the phenomenon of *fading*, where motion of the mobile results in variation of the received signal strength. We will ignore here *fast fading*, the rapid variation due to constructive and destructive interference of signals arriving via different paths to the phone, and concentrate on *shadow fading*, a slower variation due to obstructions. It is common to model shadow fading as a lognormally distributed random variable[Gud91].

Such a model would imply that at a given location, we are interested in the ratio of a lognormally distributed random variable E_c to an interference term I_0 that is the sum of such random variables, together with some noise. The *coverage probability* is the probability that such a ratio is above a given threshold. In particular, we are interested in the probability that there is *some* antenna above threshold, which provides a certain "gain": if fading increases the E_c of some antenna, that not only reduces the chance that other antennas are above threshold, by increasing I_0, but it also, of course, increases the chance that the given antenna is above threshold. We will derive an expression for coverage probability that conservatively accounts for such gain.

Our analysis reduces the ensemble-coverage problem to the problem of estimating the probability that a given antenna is above threshold. Here there is a substantial related literature, mostly concerned with approximating the probability distribution of I_0, the sum of lognormally-distributed random variables. See, for example, the paper of Abu-Dayya and Beaulieu for references in the wireless literature[ADB94], the paper of Datey, Gauthier and Simonato for references in the computational finance literature[DGS03], and the paper of Rasmusson for further references and an application in network design.[Ras02] The techniques applied to this problem include cumulant matching[JR82, Sch77], approximation using the Inverse Gamma distribution[MS98], upper bounds[Sli01], and characteristic function or moment-generating function techniques[ATB01, Zha99]. (Note that the lognormal distribution, alone, has no moment-generating function, so the latter techniques are applied to fading models where the lognormal is compounded with some other distribution.)

Here we will use the approximation due to Fenton[Fen60] and to Wilkinson[SY82], where the sum is approximated as a single lognormal distribution, whose parameters are such that its mean and variance match those of the original sum.

We compare our overall coverage probability estimate to the results of Monte Carlo experiments. By exploring the space of relevant parameters for such a comparison, we show that our estimate is generally accurate within a few percent absolute error. Therefore, a Monte Carlo coverage probability estimate can be replaced with our analytic expression. This has the advantage of a very large speedup in time needed for evaluation, and also that the resulting function of the parameters is much smoother than a Monte Carlo estimate would be.

2 The model

First, we will define some notation, and give some simplifying assumptions. We have signals E_j from antenna j to a location, for $j = 1 \ldots m$, and additional external interference term η. We will use the following assumptions:

1. The values $\ln E_j$ are normally distributed with mean μ_j;
2. The values $\ln E_j$ all have the same variance σ^2;
3. The random variables E_j and η are all independent.

4. We can regard the value $\ln \eta$ as normally distributed, with mean μ_η and variance σ_η;

As noted above, assumption (1) is common in the literature. It is based on experimental evidence, and is suggested by the Central Limit Theorem, as applied to the sequence of semi-independent obstructions and terrain variations between the location and the antenna.

Assumptions (2) and (3) are due to ignorance: there may be some correlations among the signals, and each signal will have a different variance, but often we will not have such data.

Assumption (4) is non-physical, but simply reflects per-location, "correlated" fading: it is equivalent to such fading since we are interested in E_c/I_0 ratios $E_k/(\eta + \sum_j E_j)$. Such correlations are treated with greater generality by some authors, using a general covariance matrix A. Note, however, that a model often tested is one where the off-diagonal entries of A have a single common value, and the diagonal entries of A have another common value. (For example, the distributions tested by Abu-Dayya and Beaulieu all have this property[ADB94]) Our model satisfies those conditions.

The means μ_j are due to the path loss from the antenna to the location, and also the antenna pattern and the antenna power level.

3 Estimating the coverage probability

We are interested in the probability that a location is uncovered, so that

$$\frac{I}{E_k} > t_k \text{ for all } k,$$

where $I \equiv \eta + \sum_j E_j$. (We write I_0 as just I here.) To simplify the discussion we will assume that all $t_k = t$ for some t, but it is easy to remove this assumption. The desired probability is equal to

$$\prod_k \operatorname{Prob}\left\{ I > tE_k \mid I \geq t \max_{j<k} E_j \right\}. \tag{1}$$

Let

$$I_k \equiv \eta + \sum_{j>k} E_j.$$

The conditions for given k imply that

$$tI = t\sum_{j<k} E_j + tE_k + tI_k \leq (k-1)I + tE_k + tI_k,$$

so that

$$I \leq \frac{t}{t-k+1}(E_k + I_k).$$

We will use the estimate

$$\begin{aligned}
&\text{Prob}\left\{I > tE_k \mid I \geq t\max_{j<k} E_j\right\} \\
\leq\ & \text{Prob}\left\{\frac{t}{t-k+1}(E_k + I_k) > tE_k \mid I \geq t\max_{j<k} E_j\right\} \\
\approx\ & \text{Prob}\left\{\frac{t}{t-k+1}(E_k + I_k) > tE_k\right\} \\
=\ & \text{Prob}\left\{\frac{I_k}{E_k} > t-k\right\}.
\end{aligned}$$

Here we have approximated in two ways: the upper bound on non-coverage in one step, and the more questionable approximation in the next step, where we assume the condition $I \geq t\max_{j<k} E_j$ does not affect our revised condition too much. We will use Monte Carlo simulation to check our severe these approximations were.

It seems to be better, based on our Monte Carlo experiments, as discussed below, to use $t - t_d k$ in place of $t - k$ in the above, where the best value of t_d, found experimentally, is 0.4.

We can estimate the probabilities $\text{Prob}\{I_k/E_k > t - t_d k\}$, under the assumption that each I_k is lognormal. Let $\hat{\mu}_k$ and $\hat{\sigma}_k^2$ denote the mean and variance of $\ln I_k$; these values can be readily determined.[ADB94] The mean of $\ln(\frac{I_k}{E_k})$ is then $\hat{\mu}_k - \mu_k$, while the variance of $\ln(\frac{I_k}{E_k})$ is $\hat{\sigma}_k^2 + \sigma^2$, since I_k and E_k are independent. We use these quantities, and the error function, to estimate the coverage probability.

3.1 Handling σ_η

This method of estimating the coverage probability heuristically and experimentally accurate when $\sigma_\eta = 0$. It is not accurate when σ_η is large, but it can be extended for $\sigma_\eta \neq 0$ by using numerical integration: take a weighted combination of probability estimates for trial values μ_η^t and trial assumption $\sigma_\eta = 0$, for values of $\mu_\eta^t = \mu_\eta - m\sigma_\eta/2, \ldots \mu_\eta + m\sigma_\eta/2$, where m is ten or so. Plainly this integration can be refined and extended to be as accurate as desired, up to the accuracy of the underlying estimates.

4 Experimental results

While the derivation of the coverage probability estimate was rigorous "most of the time," it used several approximations, beyond the assumptions mentioned in Section 2. We can, however, check its accuracy by means of comparison to Monte Carlo computations. Here we do many such computations, over a broad range of values of the relevant parameters: μ_j, σ, μ_η, and the threshold t. Note that, for the purpose of checking the usability of our estimate, that these are the relevant input values. In all the experiments the noise variation

σ_η	0	dB
μ_η	-1	dB
M	1000	
t	2, 7	dB
t_d	0.4	
σ	3, 5, 7	dB
$\Delta\mu_0$	0 to 12 step 1	dB
$\Delta\mu_1$	0 to 12 step 1	dB
$\Delta\mu_2$	0, 3, 6, 9	dB
$\Delta\mu_3$	0, 3, 6, 9	dB
$\Delta\mu_4$	0, 5, 10	dB
$\Delta\mu_5$	0, 5, 10	dB
$\Delta\mu_6$	0, 8, 16	dB
$\Delta\mu_7$	0, 8, 16	dB

Figure 1: Range of experimental parameters, Study 1

$\sigma_\eta = 0$ because, as noted in S 3.1, a non-zero σ_η can be handled using a single numerical integration.

Our first results show the range of errors in using our estimate. In Figure 2, we show a histogram of the differences between Monte Carlo calculations and our estimates, for all the combinations of values shown in Table 1.

Here for given values of the $\Delta\mu_i$, we have μ_i set to $\mu_{i-1} - \Delta\mu_i$, for $i > 0$. We also restrict the evaluations to values of μ_i that are not too small: if some μ_j is less than 20 dB below μ_0, we only consider $\mu_{j'} = \mu_j$ for $j' \geq j$.

In Figure 3, we show the range of probabilities associated with the combinations of values in Table 1. We want to make sure that we are not considering combinations of conditions for which the coverage probability is "easily" zero or one, and indeed, while the probabilities are skewed a bit toward the high end, a broad range of probabilities is found.

Table 4 shows the combinations of conditions for a second round of comparisons. Here we are trying to more closely monitor the effect of variations in antennas that are closer together in power levels. The histograms in Figures 5 and 6 show the general pattern of results.

In Table 7 are the combinations of conditions for a set of experiments intended to help find the best value of t_d, the amount by which the threshold is reduced in the uncoverage calculation, as discussed in §3. Figure 8 shows the distribution of errors for different values of t_d, and shows that $t_d = 0.4$ seems, by a narrow margin, to be the best.

The conditions explored in experiment 4 are the same as for experiment 1, but only $\sigma = 0.5$ is considered. Here the errors are typically larger, and the limits of the applicability of our estimates may be visible. The parameters considered are shown in FigureTable 9, the errors in Figure 10, and the range of probabilities in Figure 11.

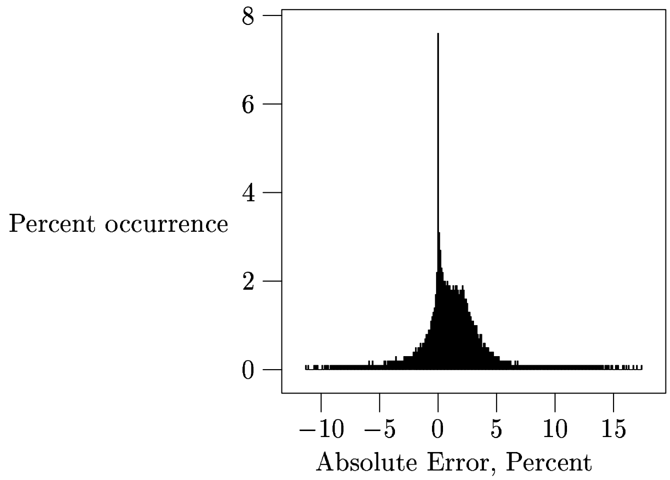

Figure 2: Error of our analytic estimate vs. Monte Carlo, Study 1

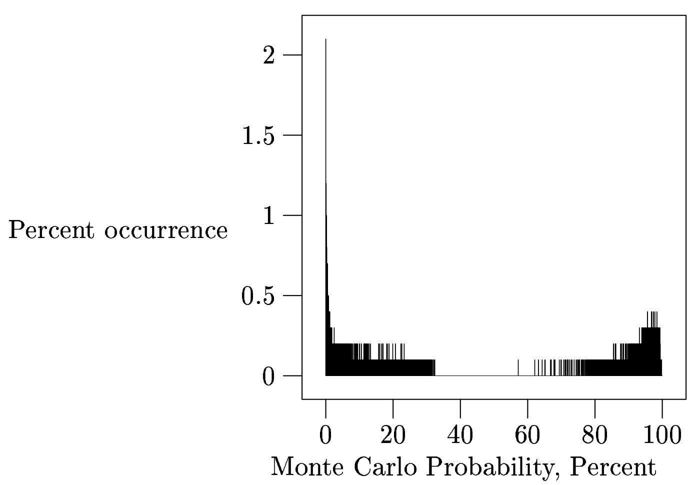

Figure 3: Distribution of Monte Carlo Probabilities, Study 1

σ_η	0	dB
μ_η	-1	dB
M	1000	
t	2, 7	dB
t_d	0.4	
σ	2, 4	dB
$\Delta\mu_0$	0 to 4 step 1	dB
$\Delta\mu_1$	0 to 4 step 1	dB
$\Delta\mu_2$	0 to 4 step 1	dB
$\Delta\mu_3$	0 to 4 step 1	dB
$\Delta\mu_4$	0 to 4 step 1	dB
$\Delta\mu_5$	0 to 4 step 1	dB
$\Delta\mu_6$	0	dB
$\Delta\mu_7$	0	dB

Figure 4: Range of experimental parameters, Study 2

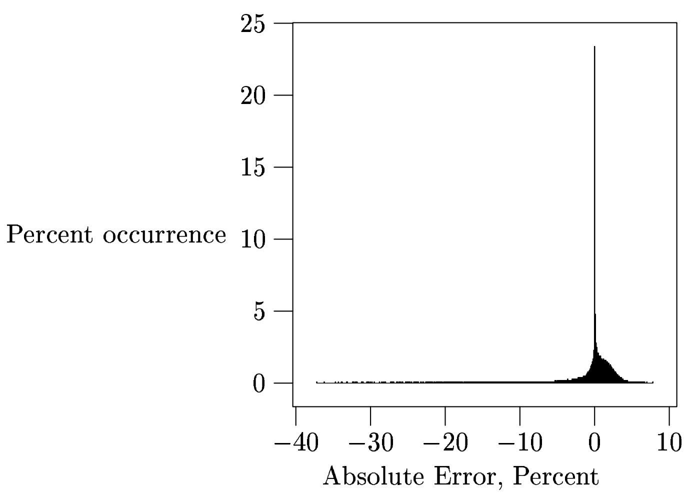

Figure 5: Error of our analytic estimate vs. Monte Carlo, Study 2

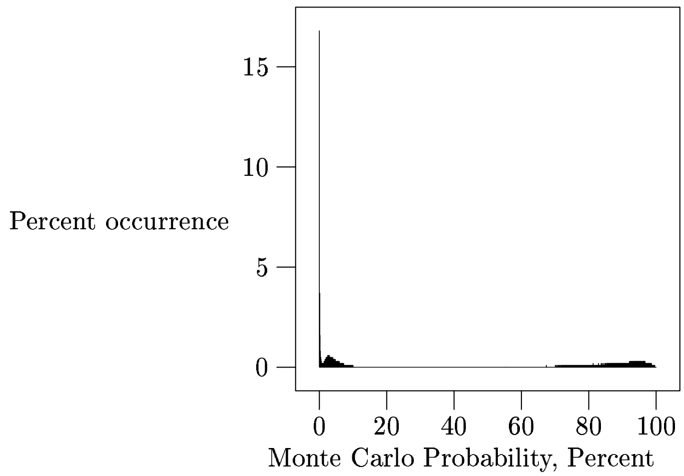

Figure 6: Distribution of Monte Carlo Probabilities, Study 2

σ_η	0	dB
μ_η	-1	dB
M	1000	
t	12	dB
t_d	0.2 to 1.2 step 0.2	
σ	4	dB
$\Delta\mu_0$	0 to 12 step 1	dB
$\Delta\mu_1$	0 to 12 step 1	dB
$\Delta\mu_2$	0, 3, 6, 9	dB
$\Delta\mu_3$	0, 3, 6, 9	dB
$\Delta\mu_4$	0, 5, 10	dB
$\Delta\mu_5$	0, 5, 10	dB
$\Delta\mu_6$	0, 8, 16	dB
$\Delta\mu_7$	0, 8, 16	dB

Figure 7: Range of experimental parameters, Study 3

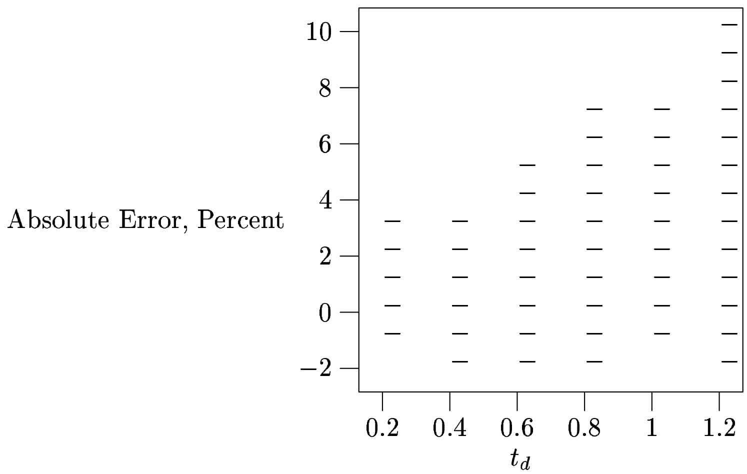

Figure 8: Distribution of probability errors vs. t_d, Study 3

σ_η	0	dB
μ_η	-1	dB
M	1000	
t	2, 7	dB
t_d	0.4	
σ	0.5	dB
$\Delta\mu_0$	0 to 12 step 1	dB
$\Delta\mu_1$	0 to 12 step 1	dB
$\Delta\mu_2$	0, 3, 6, 9	dB
$\Delta\mu_3$	0, 3, 6, 9	dB
$\Delta\mu_4$	0, 5, 10	dB
$\Delta\mu_5$	0, 5, 10	dB
$\Delta\mu_6$	0, 8, 16	dB
$\Delta\mu_7$	0, 8, 16	dB

Figure 9: Range of experimental parameters, Study 4

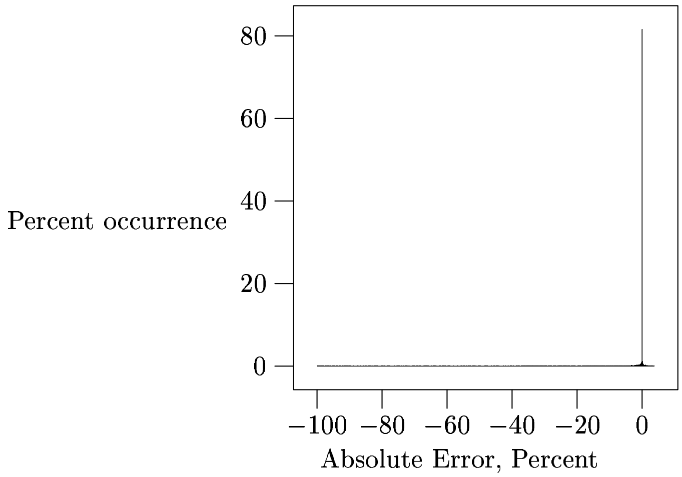

Figure 10: Error of our analytic estimate vs. Monte Carlo, Study 4

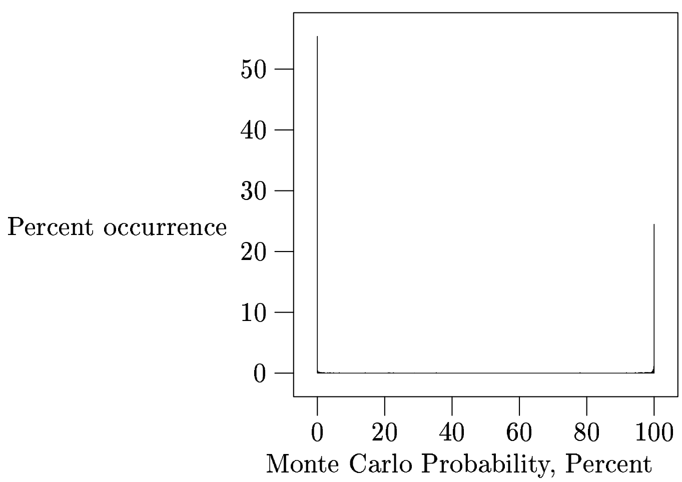

Figure 11: Distribution of Monte Carlo Probabilities, Study 4

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A The lognormal distribution

If X is lognormal, so that $\ln X$ is normal with mean μ_x and variance σ_x^2, then X^p is lognormal and $\ln X^p$ is normal with mean $p\mu_x$ and variance $p^2\sigma_x^2$. The product of two independent lognormal variates X and Y is lognormal, and $\ln(XY)$ has mean $\mu_x + \mu_y$ and variance $\sigma_x^2 + \sigma_y^2$. The expected value $\mathbf{E}X$ of lognormal X is $\exp(\mu_x + \sigma_x^2/2)$.

If X is lognormal,

$$\begin{aligned} \mathbf{Var}\, X &= \mathbf{E}X^2 - [\mathbf{E}X]^2 \\ &= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2) \\ &= \exp(2\mu)\exp(\sigma^2)(\exp(\sigma^2) - 1). \end{aligned} \tag{2}$$

Finally, if X is lognormal,

$$\begin{aligned} \frac{\mathbf{E}X^2}{[\mathbf{E}X]^2} &= \frac{\exp(2\mu + 2\sigma^2)}{\exp(2\mu + \sigma^2)} \\ &= \exp(\sigma^2), \end{aligned} \tag{3}$$

and similarly,

$$\frac{\mathbf{Var}\, X}{[\mathbf{E}X]^2} = \exp(\sigma^2) - 1 \tag{4}$$

B Mean and variance of I

We are interested in interference ratios I/E_k, where

$$I \equiv \eta + \sum_j \alpha_j E_j,$$

for some noise level η and fixed coefficients α_j. A mobile is *covered* when there is some antenna k with

$$I/E_k \leq t_k,$$

for some given threshold t_k.

We will immediately change notation to assume all $\alpha_j = 1$; since $E'_j \equiv \alpha_j E_j$ is lognormal if and only if E_j is, it's easy to translate from the $\alpha_j = 1$ case to the more general case.

In the remainder of this section, we find the mean $\mathbf{E}I$ and variance $\mathbf{Var}\, I$ of I.

We have

$$\begin{aligned} \mathbf{E}I &= E\eta + \mathbf{E}\sum_j E_j \\ &= \exp(\mu_\eta + \sigma_\eta^2/2) + \sum_j \mathbf{E}E_j \end{aligned}$$

$$\begin{aligned} &= \exp(\mu_\eta + \sigma_\eta^2/2) + \sum_j \exp(\mu_j + \sigma^2/2) \\ &= \exp(\mu_\eta + \sigma_\eta^2/2) + \exp(\sigma^2/2) \sum_j \exp(\mu_j) \\ &= \exp(\mu_\eta + \sigma_\eta^2/2) + \exp(\sigma^2/2)\mathtt{Io}, \end{aligned} \tag{5}$$

where

$$\mathtt{Io} \equiv \sum_j \exp(\mu_j).$$

Since $I \equiv \eta + \sum_j E_j$, and the E_j and η are independent, we have

$$\begin{aligned} \mathbf{Var}\, I &= \mathbf{Var}\, \eta + \sum_j \mathbf{Var}\, E_j \\ &= (\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\sum_j \exp(2\mu_j) \\ &= (\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\mathtt{Io}_2, \end{aligned} \tag{6}$$

where

$$\mathtt{Io}_2 \equiv \sum_{j>1} \exp(2\mu_j).$$

Since $\mathbf{Var}\, I = \mathbf{E}I^2 - [\mathbf{E}I]^2$, we have

$$\begin{aligned} \mathbf{E}I^2 &= [\mathbf{E}I]^2 + \mathbf{Var}\, I \\ &= [\mathbf{E}I]^2 + (\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\mathtt{Io}_2 \end{aligned} \tag{7}$$

C Assuming I is lognormal

We have exact expressions for the mean (5) and variance (6) of I, assuming that the E_j and η are lognormally distributed, so $\ln E_j$ is normal with mean μ_j and variance σ^2.

We now assume that I is lognormally distributed, which seems to be a common simplifying assumption in the literature. Under this assumption, we can solve for the mean $\hat{\mu}$ and variance $\hat{\sigma}$ of $\ln I$. Having obtained these values, it is easy to find the mean and variance of $\ln(I/E_1)$, or more general ratios, as in the next section.

From (3), (5) and (7), we have

$$\begin{aligned} \exp(\hat{\sigma}^2) &= \frac{\mathbf{E}I^2}{[\mathbf{E}I]^2} \\ &= \frac{[\mathbf{E}I]^2 + (\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\mathtt{Io}_2}{[\mathbf{E}I]^2} \\ &= 1 + \frac{(\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\mathtt{Io}_2}{[\exp(\mu_\eta + \sigma_\eta^2/2) + \exp(\sigma^2/2)\mathtt{Io}]^2} \end{aligned}$$

$$
\begin{aligned}
&= 1 + \frac{(\exp(\sigma_\eta^2) - 1)\exp(\sigma_\eta^2)\exp(2\mu_\eta) + (\exp(\sigma^2) - 1)\exp(\sigma^2)\mathtt{Io}_2}{\exp(\sigma^2)\mathtt{Io}^2[1 + \exp(\mu_\eta + \sigma_\eta^2/2 - \sigma^2/2)/\mathtt{Io}]^2} \\
&= 1 + \frac{(\exp(\sigma_\eta^2) - 1)\exp(2\mu_\eta + \sigma_\eta^2 - \sigma^2) + (\exp(\sigma^2) - 1)\mathtt{Io}_2}{\mathtt{Io}^2[1 + \exp(\mu_\eta + \sigma_\eta^2/2 - \sigma^2/2)/\mathtt{Io}]^2}. \qquad (8)
\end{aligned}
$$

Note that when all E_j are equal, so that significant averaging occurs, this quantity is approximately $1 + \exp(\sigma^2)/n$ for larger values of σ and neglecting η.

So,

$$
\begin{aligned}
\exp(\hat{\mu}) &= \mathbf{E}[I]\exp(-\hat{\sigma}^2/2) \\
&= (\exp(\mu_\eta + \sigma_\eta^2/2) + \exp(\sigma^2/2)\mathtt{Io})\exp(-\hat{\sigma}^2/2) \\
&= \exp(\sigma^2/2 - \hat{\sigma}^2/2)\mathtt{Io}(1 + \exp(\mu_\eta + \sigma_\eta^2/2 - \sigma^2/2)/\mathtt{Io}), \qquad (9)
\end{aligned}
$$

where $\exp(\hat{\sigma}^2)$ is estimated above.

While this estimate of $\exp(\hat{\mu})$ is adequate for computation, it may be worthing noting that **fix**

$$
\begin{aligned}
(\eta + \exp(\sigma^2/2)\mathtt{Io})\exp(-\hat{\sigma}^2/2) &\approx (\eta + \exp(\sigma^2/2)\mathtt{Io})\frac{1}{\sqrt{1 + \frac{\mathtt{Io}_2}{\mathtt{Io}^2}\frac{\exp(\sigma^2)}{[1+\eta\exp(-\sigma^2/2)/\mathtt{Io}]^2}}} \\
&\approx \mathtt{Io}\frac{\mathtt{Io}}{\sqrt{\mathtt{Io}_2}}\frac{(1 + \eta\exp(-\sigma^2/2)/\mathtt{Io})^2}{\sqrt{1 + \frac{\mathtt{Io}^2}{\mathtt{Io}_2}(1 + \eta\exp(-\sigma^2/2)/\mathtt{Io})^2\exp(-\sigma^2)}}.
\end{aligned}
$$