# Ring Routing and Wavelength Translation

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**Abstract**

Let $G$ be the digraph consisting of two oppositely-directed rings on the same set of $n$ nodes. We provide a polynomial-time algorithm which, given a list of demands—each requiring a path from a specified source node to a specified target node—routes the demands so as to minimize the largest number of paths through any of the $2n$ directed links of $G$. The algorithm makes use of a partial linear relaxation and rounding technique which together, somewhat surprisingly, produce an exact solution.

The problem arises in an optical communications network with wavelength division multiplexing (WDM), configured as a ring. Such a network features a fixed number of wavelengths, each of which (at the optical level) can support a single path of high bandwidth through a given link. If there is no "wavelength translation" available, so that each demand is restricted to a single wavelength, then the combined routing and wavelength assignment problem is NP-complete. Our results imply, however, that the presence of even a single wavelength translator (at any node) guarantees both full capacity and polynomial-time optimizability.

Single-translator sufficiency in the ring is a special case of a simple criterion which, given a set of nodes in an *arbitrary* WDM network, determines whether wavelength translators on those nodes allow the network to run at maximum capacity. Although the problem of minimizing the cardinality of this set is NP-complete (even in the planar case), the high cost of wavelength translators can be expected to make the criterion a useful tool.

## 1 Introduction

Defining multiple channels simultaneously across an optical link by using a different wavelength for each channel is a technique called wavelength-division multiplexing (WDM) [12]. In a WDM network, in order to establish a logical connection from one node to another, a route is chosen (that is, a sequence of links leading from the initial node to the terminal node) and an assignment of a wavelength to the connection from each link in the route is made. The wavelength assignments must be made so that there are no conflicts; that is, no two connections whose routes share a link can be assigned the same wavelength along that link.

Typically, WDM networks have been thought of in two broad categories. In a *wavelength selective* (WS) network the links in the route assigned to a connection must all allocate the same wavelength to that connection [3] whereas in a *wavelength interchanging* (WI) network the links in the route assigned to a connection may allocate different wavelengths to the connection [17]. Clearly the nodes in a WI network require some sort of hardware that takes incoming signals on the different wavelengths and permutes them for the outgoing signals. We call such a device a *wavelength translator*. Since each wavelength translator adds cost to the network, it may seem that a WS network is preferable to a WI network. However a WS network may have lower capacity for a fixed number of wavelengths since there are situations where a connection in the WS network is impossible because no single wavelength is available along any route for the connection even though there are different wavelengths available along a route. Another way to state the difference is that a WS network may require more wavelengths to establish a set of connections than a WI network of the same topology. There has been a good deal of study of the relationship between WS and WI network wavelength requirements such as [1, 3, 5, 15, 16, 18, 20, 23].

We begin by considering such problems for a ring network. Even in a ring, the simplest 2-connected network, the problem of assigning routes and wavelengths in response to connection requests is not trivial. In considering this problem we make the following definitions. For any given routing and any fixed directed link of the ring, the number of paths routed through that link is called its *link load* and the maximum of this value over all the links is the *ring load*.

If each connection path must employ the same wavelength on every link, then what is needed is a routing which minimizes the chromatic number of a corresponding directed circular-arc graph. We consider this case in Section 3 below, showing that the number of wavelengths required could be as much as twice the minimum ring load minus one, and is NP-hard to determine exactly.

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If, on the other hand, the wavelength of a connection path may change from link to link, then there is no coloring problem and the object is merely to route so as to achieve the minimum ring load $L_{\rm OPT}$. We study this case in Section 2. This is a special case of integer multi-commodity flow; luckily, it turns out to be a tractable case. The usefulness of our algorithm is much enhanced by the fact that only one node of the ring needs to be able to change the wavelengths of paths, in order to be able to route with just $L_{\rm OPT}$ wavelengths. The sufficiency of one wavelength translator may be important since wavelength translators currently require opto-electronic conversion and are likely to be expensive even when new technology obviates this requirement.

In a general WDM network, we give a simple characterization of those sets of nodes which, when supplied with wavelength translators, permit valid wavelength assignments for any routing and no more wavelengths are required than if wavelength translators had been placed at every node. In all such cases, a valid assignment can be made in polynomial time. However, finding a minimum cardinality set of such nodes is shown to be NP-complete. Results for general WDM networks are covered in Section 4.

## 2 Ring Routing With Wavelength Translation

In this section we are primarily concerned with determining the minimum ring load $L_{\rm OPT}$, given a list of demands for directed paths. We remark that the undirected case, where demands and link loads are bidirectional, is of particular interest for SONET (Synchronous Optical NETwork) rings. This "ring loading problem" was studied by Cosares and Saniee [4], Schrijver, Seymour and Winkler [21], Khanna [14], and Carpenter, Cosares and Saniee [2].

In the usual formulation SONET demands are equipped with non-negative weights representing amount of traffic. When these weights are $\{0,1\}$, as in our WDM problem, Frank [7] showed using a result of Okamura and Seymour [19] that the undirected problem is solvable in polynomial time. Both the $\{0,1\}$ case (with parity conditions) and the linear relaxation of the SONET ring loading problem enjoy a nice "cut" property which fails in our directed-path setting.

To distinguish the problem considered in this work from that of [21] we add the word "directed" and change "loading" to "routing", reflecting the fact that in a WDM setting we need not consider different-sized demands; or, putting it another way, multiple demands for the same nodes can be routed differently. We wish to find a routing which achieves the minimum ring load $L_{\rm OPT}$.

We begin by noting that only one wavelength translator is necessary. Henceforth $R$ will always be the ring on nodes $\{0,1,\ldots,n-1\}$ modulo $n$, labeled clockwise.

THEOREM 2.1. *Suppose node 0 of the ring $R$ has the capability of permuting the wavelengths of its traversing paths in any desired manner. Then any routing with ring load $L$ can be realized with $L$ wavelengths, and moreover such a realization can be found in time polynomial in $n$ and the number of paths.*

**Proof.** Let us note first that we may assume every node is an endpoint for at least one path; for, otherwise, we may sort the nodes which do occur as endpoints (in time polynomial in $\log n$ and the number of paths) and relabel them by consecutive integers beginning with 0. The resulting collapse of links has no effect on the problem.

Open the ring at node 0 to form a linear network on nodes $0,1,\ldots,n-1,0$; paths through 0 will now be broken into two pieces. Since interval graphs are perfect (see e.g. [6]), the rightward directed paths (paths which were routed clockwise in $R$) can be colored with $L$ colors (wavelengths) in such a way that no two paths which share a link have the same color. In fact, colors can be assigned greedily: the paths are ordered left to right by source node, then each path is assigned the least color not already assigned to an intersecting path. A similar argument applies to the leftward directed paths.

Paths which were cut in two will generally get two different colors, but these can be reconciled by permuting the colors at node 0. ∎

We now proceed to the question of finding a routing which achieves the minimum ring load $L_{\rm OPT}$. It will be useful to deal with intervals (arcs) of nodes around the ring $R$; hence, for nodes $s$ and $t$ we define $[s,t]$ to be $\{u\ :\ s\le u\le t\}$ when $s\le t$ and $[s,n-1]\cup[0,t]$ otherwise. Also, all arithmetic involving nodes will be done implicitly using modulo $n$ operations.

The problem is formally stated as follows.

DIRECTED RING ROUTING

INSTANCE: Positive integers $n$ and $m$, and ordered pairs $(s_1,t_1),\ldots,(s_m,t_m)$ from the set $\{0,\ldots,n-1\}^2$ with $s_i\ne t_i$.

QUESTION: Find values $x_1,\ldots,x_m\in\{0,1\}$ which minimize $L$, where:

$$L=\max\left(\max_k A_k,\ \max_k B_k\right)$$

and

$$A_k=|\{i:\ k\in[s_i,t_i-1]\text{ and }x_i=1\}|$$

and

$$B_k = |\{i: \ k \in [t_i, s_i - 1] \text{ and } x_i = 0\}| \ .$$

To make DIRECTED RING ROUTING a decision problem, as in [10], we append a target value $T$ to the instance and ask whether there are $x_1, \ldots, x_m \in \{0, 1\}$ for which $L \leq T$.

The parameter $n$ is known as the *ring size* and $(s_i, t_i)$ is the $i^{\text{th}}$ *demand*, $d_i$. Setting $x_i = 1$ amounts to routing a path from $s_i$ to $t_i$ clockwise around the ring; $x_i = 0$ indicates the counterclockwise choice. The number of demands routed through the clockwise link $k \to k+1$ is its "link load" $A_k$, and similarly for the counterclockwise link $k+1 \to k$ and $B_k$. The maximum of the link loads is the *ring load* $L$, and the object is to choose a *routing* $x = x_1, \ldots, x_m$ which minimizes $L$.

The decision form of DIRECTED RING ROUTING is clearly in the class NP since the routing provides an economical certificate. In fact, DIRECTED RING ROUTING is an integer multicommodity flow problem (see e.g. Section 8, p. 159 of the survey [8]). General such problems are NP-complete, but we are dealing here with a very special case which turns out to be in the class P.

THEOREM 2.2. DIRECTED RING ROUTING *is solvable in polynomial time.*

**Proof.** We may assume that every node of the ring occurs as $s_i$ or $t_i$ in at least one demand, hence $n \leq 2m$; thus we need not be concerned that $n$ contributes only its logarithm to the instance size.

We begin by considering a "relaxed" version of DIRECTED RING ROUTING, in which demands may be split (that is, sent partly clockwise, partly counterclockwise).

RELAXED DIRECTED RING ROUTING

INSTANCE: Positive integers $n$ and $m$, and ordered pairs $(s_1, t_1), \ldots, (s_m, t_m)$ from the set $\{0, \ldots, n-1\}^2$ with $s_i \neq t_i$.

QUESTION: Find reals $x_1, \ldots, x_m \in [0, 1]$ which minimize $L^*$, where:

$$L^* = \max \left( \max_k A_k^*, \ \max_k B_k^* \right)$$

and

$$A_k^* = \sum_{i: \ k \in [s_i, t_i - 1]} x_i$$

and

$$B_k^* = \sum_{i: \ k \in [t_i, s_i - 1]} (1 - x_i) \ .$$

Since this is now a linear programming problem, it is solvable in polynomial time [13]; and of course its optimal ring load $L^*_{\text{OPT}}$ satisfies $L^*_{\text{OPT}} \leq L_{\text{OPT}}$, where $L_{\text{OPT}}$ is the optimal ring load in the original problem, thus also $\lceil L^*_{\text{OPT}} \rceil \leq L_{\text{OPT}}$.

Our strategy will be to obtain a real routing $x' = (x'_1, \ldots, x'_m)$ with two special properties, but whose ring load $L'$ still satisfies $L' \leq L_{\text{OPT}}$. We will then modify $x'$ to obtain a $\{0,1\}$-routing $x$ whose ring load is strictly less than $L' + 1$. It will then follow that $x$ is an optimal routing for the original problem.

We say that a routing $x'$ is *flush* if its sum $\sum_{i=1}^m x'_i$ is an integer.

PROPOSITION 1. *Given an instance of* RELAXED DIRECTED RING ROUTING, *a flush routing $x'$ with $L' \leq L_{\text{OPT}}$ can be found in polynomial time.*

**Proof.** Such a routing certainly exists since any $\{0,1\}$-routing, in particular an optimal routing for the original problem, is flush. Thus we can consider each possible value $F = 0, 1, \ldots, m$ in turn, adding the equality $\sum_{i=1}^m x_i = F$ to the conditions for RELAXED DIRECTED RING ROUTING and solving the resulting linear programming problem to obtain a minimal ring load $L_F$. The routing among these $m+1$ solutions achieving the least $L_F$ is then taken and so the resulting ring load is no more than $L_{\text{OPT}}$.

In fact, we can do better. Observe that the function $L_f$ (where $f$ is now an arbitrary real in $[0, m]$) is concave down, since if a routing $x$ yields ring load $L$ and $x'$ yields $L'$ then $\lambda x + (1-\lambda)x'$ yields a ring load of at most $\lambda L + (1-\lambda)L'$.

(Note also that $L_f$ is piecewise linear with slopes bounded by 1 in absolute value, since any routing $x'$ with $x'_i \geq x_i$ for each $i$ satisfies $L' \leq L + \sum_{i=1}^m x'_i - \sum_{i=1}^m x_i$. Thus our flush ring load $L'$ cannot be more than $L^*_{\text{OPT}} + \frac{1}{2}$, but conceivably exceeds $\lceil L^*_{\text{OPT}} \rceil$.)

It follows that if $L_f$ is minimal at $f = r$ then one of the values $\lfloor r \rfloor$ and $\lceil r \rceil$ must achieve the minimum of the discrete function $L_F$. Hence, if $r$ is not already an integer, we can just check those two values and take the better one. ∎

It is handy to think of demands geometrically, as directed chords in a circle representing the ring. Two demands $(s_i, t_i)$ and $(s_j, t_j)$ are said to be *parallel* if the intervals $[s_i, t_i]$ and $[t_j, s_j]$, or the intervals $[t_i, s_i]$ and $[s_j, t_j]$, intersect at most at their endpoints. There are essentially four ways that this can happen, illustrated in Fig. 1; it may even be that $s_i = s_j$ and $t_i = t_j$.

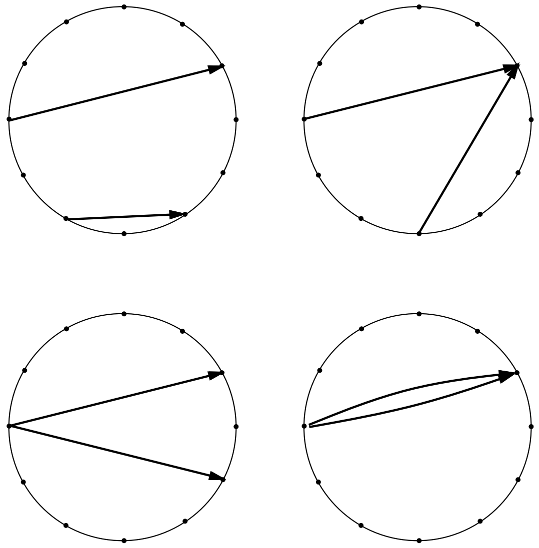

Figure 1: Parallel pairs of demands

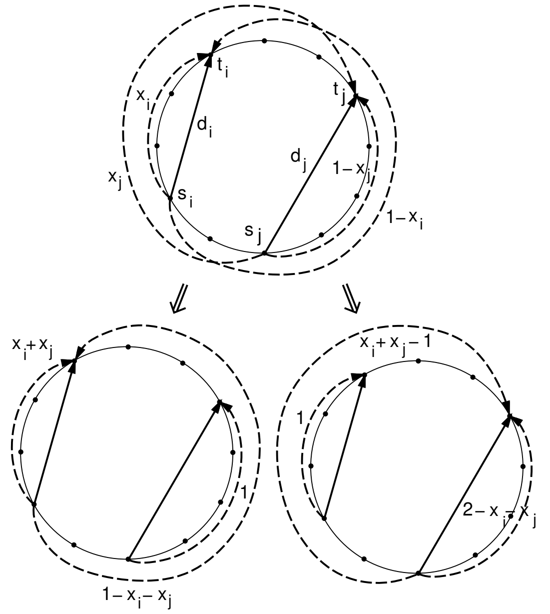


Figure 2: Untangling routings of split parallel demand pairs

Regarding a link also as a chord, we see that that a demand is parallel to a link just when the demand can be routed through that link. Any link partitions the demands into those which are parallel to it, and those which are parallel to its reverse.

A real routing $x'$ is said to *split* the $i^{\text{th}}$ demand if $0 < x'_i < 1$.

PROPOSITION 2. *Given an instance of* RELAXED DIRECTED RING ROUTING *and a flush routing $x$ with ring load $L$, a flush routing $x'$ can be found in polynomial time with ring load $L' \leq L$, having the additional property that no two parallel demands are both split.*

**Proof.** We can immediately ensure that in any set of *identical* demands at most one is split, so let us suppose that there is a pair of unequal, parallel demands $d_i$ and $d_j$ with $0 < x_i < 1$ and $0 < x_j < 1$. We will reroute these demands in such a way that one of them is no longer split, but their collective contribution to every link load is either maintained or reduced, and the routing sum $\sum_{i=1}^{m} x_i$ remains unchanged.

Since the demands are distinct we may assume that the intervals $[s_i, t_i]$ and $[t_j, s_j]$ intersect in at most *one* node and do not cover the ring. Suppose first that $x_i \leq 1 - x_j$; then define a new routing $x'$ by

$$x'_i = x_i + x_j \text{ and } x'_j = 0$$

with $x'_k = x_k$ for $k \notin \{i, j\}$. Then links in $[s_i, t_i]$ and $[t_j, s_j]$ enjoy the same load as before and other links have the same or reduced loads.

If $x_i > 1 - x_j$ then we let

$$x'_i = 1 \text{ and } x'_j = x_i + x_j - 1$$

with a similar effect. These two methods of "untangling" routings of parallel demand pairs are illustrated in Fig. 2, in the case where the demands have distinct endpoints.

Since each untangling reduces by one (or two) the total number of split demands, at most $m$ such procedures will produce the desired routing. ∎

Henceforth we will assume that we have obtained a flush routing $x'$ with link loads $A'_k, B'_k$ and ring load $L' \leq L_{\mathrm{OPT}}$, satisfying the condition that no two parallel demands are both split. Since in particular two non-parallel demands cannot share a source, the number of demands split by $x$ is at most the ring size $n$.

We may assume that the demands are numbered so that $S = \{d_1, \ldots, d_q\}$ is the set of split demands. Moreover, since no two are parallel, we may order them clockwise simultaneously by source $s_i$ and by target $t_i$, as in Fig. 3. Then for any clockwise link $k \to k{+}1$ there is an interval $[i_k, j_k] \subseteq \{1, \ldots, q\}$, interpreted if necessary "around the corner" modulo $q$, which contains exactly the indices of the demands in $S$ which are parallel to the link. For its counterclockwise mate $k{+}1 \to k$, the indices of the parallel links are just those in the complement of $[i_k, j_k]$, namely the interval $[j_k{+}1, i_k{-}1]$.

It is the unsplitting of the demands parallel to a link which affects the load on that link. Suppose $x'$ is a flush routing with link loads $A'_k$ and $B'_k$ and $x$ is the $\{0,1\}$-valued routing with link loads $A_k$ and $B_k$ resulting from

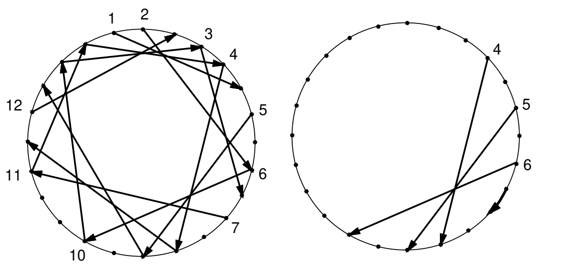


Figure 3: Split demands; split demands parallel to a link

some unsplitting of the split demands of $x'$. Then we have

$$A_k = A'_k + \sum_{i \in [i_k, j_k]} (x_i - x'_i)$$

and

$$B_k = B'_k + \sum_{i \notin [i_k, j_k]} (x'_i - x_i) \ .$$

Let us now define an unsplitting $x$ recursively by putting

$$x_j = \begin{cases} 1 & \text{if } -x'_j + \sum_{i=1}^{j-1}(x_i - x'_i) < -\frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Then every partial sum $\sum_{i=1}^{j}(x_i - x'_i)$ lies in the half-open real interval $[-\frac{1}{2}, \frac{1}{2})$. If $i_k \le j_k$, we have

$$A_k - A'_k = \sum_{i=1}^{j_k}(x_i - x'_i) - \sum_{i=1}^{i_k-1}(x_i - x'_i) < \frac{1}{2} - (-\frac{1}{2}) = 1$$

as desired.

If $i_k > j_k$ then

$$A_k - A'_k = \sum_{i=1}^{q}(x_i - x'_i) + \sum_{i=1}^{j_k}(x_i - x'_i) - \sum_{i=1}^{i_k-1}(x_i - x'_i)$$

which lies in the interval $[-\frac{3}{2}, \frac{3}{2})$, seemingly not good enough. But $x'$ is flush and $x$ is integral, therefore $\sum_{i=1}^{q}(x_i - x'_i)$ is an integer; namely 0, since it lies in $[-\frac{1}{2}, \frac{1}{2})$.

Thus $A_k - A'_k < 1$ as before, and a symmetric argument for the counterclockwise links shows that $B_k - B'_k < 1$ as well. Hence the ring load $L$ induced by $x$ satisfies $L < L' + 1$ where $L'$ is the ring load induced by $x'$.

Since $L$ is integral we conclude that $L \le \lceil L' \rceil \le L_{\mathrm{OPT}}$, and the proof of Theorem 2.2 is complete. ■

We note that the proof of Theorem 2.2 shows also that $L_{\mathrm{OPT}} \le 1 + \lceil L^*_{\mathrm{OPT}} \rceil$ and $L_{\mathrm{OPT}} < L^*_{\mathrm{OPT}} + \frac{3}{2}$. We have no example, however, in which $L_{\mathrm{OPT}} > \lceil L^*_{\mathrm{OPT}} \rceil$.

## 3 Ring Routing Without Wavelength Translation

We now consider the case where all links in a path must be assigned the same wavelength, and no two paths assigned the same wavelength can share a (directed) link. Since we know that we can calculate the number of wavelengths required when translation *is* available, namely the minimum ring load $L_{\mathrm{OPT}}$, we are in a good position to bound the value of wavelength translation.

THEOREM 3.1. *For any positive integer $L$ there is an instance of* DIRECTED RING ROUTING *with minimum ring load $L$, in which routing without wavelength translation requires $2L - 1$ wavelengths.*

**Proof.** The theorem is in effect proved for fixed routings in [22], where it is observed that a set of $2L-1$ arcs can be pairwise overlapping yet no point of the circle is contained in more than $L$ of the arcs. We only need to select demands so as to force such a pattern of paths in each direction.

Assume $L \ge 2$ (there is nothing to prove for $L = 1$). Let $n = 4(2L - 1)$ and for each $i$, $0 \le i < 2L - 1$, we introduce a demand $d_i$ from node $4i$ to node $4i+2(2L-1)-1$ and a demand $e_i$ from $4i-1$ to $4i+2(2L-1)$. If the $d_i$'s are routed clockwise and the $e_i$'s counterclockwise, then the maximum link load is $L$.

However, any routing must contain $2L-1$ demands in one direction (say, clockwise) since there are $2(2L-1)$ demands altogether. It remains only to observe that the clockwise paths for the $d_i$'s are pairwise intersecting, and that for each $i$ the clockwise path for $e_i$ contains the clockwise path of $d_i$. Hence $2L - 1$ wavelengths are required. ■

THEOREM 3.2. *Let $x$ be a routing achieving the minimum ring load $L$ in an instance of* DIRECTED RING ROUTING. *Then there is a polynomial time algorithm which will realize the routing with at most $2L - 1$ wavelengths, without wavelength translation.*

**Proof.** We may assume that at least one demand has node 0 as its source. Then we open the ring as in the proof of Theorem 2.1, cutting paths through 0 in two, and assign wavelengths greedily as before, using at most $L$ distinct wavelengths. Since there are at most $L - 1$ paths cut, there are at most $L-1$ paths that have links of two different wavelengths. Thus we can choose an entirely new wavelength for each of these and still use only $2L - 1$ wavelengths in all. ■

One consequence of Theorem 3.2, already noted in [20] and [18], is that there is a polynomial time algorithm which will solve the routing *and* wavelength assignment problem using no more than twice the optimum number of wavelengths.

Existence of an efficient algorithm for optimal assignment of wavelengths to paths is unlikely in view of the result of Garey, Johnson, Miller and Papadimitriou [11], that it is NP-hard to determine the chromatic number of a circular arc graph. It is not immediate from this result that the *combined* problem of routing and wavelength assignment is NP-hard, but that is indeed the case.

THEOREM 3.3. *The* DIRECTED RING ROUTING *problem, without wavelength translation, is NP-complete.*

**Proof.** It is straightforward to check that the DIRECTED RING ROUTING problem is in NP. In order to show that the problem is NP-hard, it suffices to convert an instance of CHROMATIC NUMBER OF CIRCULAR ARC GRAPH to DIRECTED RING ROUTING in such a way that for the latter to have any chance of realization with the specified number of wavelengths, the routing would have to match the circular arc graph.

Consider an instance of CHROMATIC NUMBER OF CIRCULAR ARC GRAPH consisting of the collection of arcs $(s_1, t_1), \ldots, (s_m, t_m)$ and a positive integer $T < m$ where $s_i$ is the counterclockwise and $t_i$ the clockwise endpoint of the $i^{th}$ arc. Note that the $s_i$'s and $t_i$'s can be assumed to be non-negative integers. Let $n \leq 2m$ be the number of distinct $s_i$'s and $t_i$'s and hence without loss of generality we can further assume that each $s_i$ and $t_i$ is a non-negative integer less than $n$. From such an instance, an instance of DIRECTED RING ROUTING can be defined consisting of ring size $(T+1)n$ and capacity $T$ and the demands $d_1 = ((T+1)s_1, (T+1)t_1), \ldots, d_m = ((T+1)s_m, (T+1)t_m)$ on a ring $R'$ on $(T+1)n$ nodes, to which we add $T$ "short" demands of the form $(j, j-1)$ for every node $j$ of $R'$, $0 \leq j < (T+1)n$.

If the given circular arc graph is $T$-colorable then the long demands can be routed clockwise and colored accordingly, while the short demands can be routed counterclockwise and colored arbitrarily subject to each set of $T$ identical short demands being assigned all $T$ colors.

Suppose, on the other hand, that the circular arc graph is not $T$-colorable, but the DIRECTED RING ROUTING instance can nonetheless be realized with $T$ wavelengths. Then at least one of the long demands, say $(s_1, t_1)$ must be routed counterclockwise; hence for each link $(T+1)s_1 - i \rightarrow (T+1)s_1 - i - 1$, $0 \leq i < T+1$, at least one short demand must be routed the long way (clockwise) around the ring. This adds $T+1$ to the clockwise load of any other link, contradicting the assumption that $T$ wavelengths sufficed for the DIRECTED RING ROUTING instance, thus proving the theorem. ∎

We remark that minor modifications to this proof obviate the need for multiple demands between the same pair of nodes. Independently, Erlebach and Jansen [5] have shown that the problem is NP-hard by a somewhat more complicated transformation.

## 4 Translators in General WDM Networks

One consequence of the work in Section 2 is that in a ring, a single wavelength translator at any node suffices, and the resulting wavelength assignment problem is solvable in polynomial time. Hence, with our polynomial time routing algorithm of Section 2, we can do optimal wavelength assignment in any ring with at least one wavelength translator.

In this section we consider similar questions for general WDM networks. That is, we consider the question of where wavelength translators should be placed in a WDM network so that any collection of demands for directed paths can be satisfied using no more wavelengths than if there were wavelength translators at every node.

Let $G = (V, E)$ be an undirected connected graph and define a network $N_G$ to be the directed graph with vertex set $V$ and where for each edge $e = \{u, v\} \in E$ there correspond two directed links $(u, v)$ and $(v, u)$ of the network. A *routing* $\boldsymbol{R}$ is a collection of directed paths. As before we define the load for $\boldsymbol{R}$ on a link in $G$ to be the number of paths routed over the link and the *load of* $\boldsymbol{R}$, denoted by $L(\boldsymbol{R})$, is the maximum load of any link.

A *wavelength assignment* for $\boldsymbol{R}$ associates a wavelength from $\{1, \ldots, L(\boldsymbol{R})\}$ to each link of each path such that no link gets the same wavelength for two different paths which pass through it.

Let $S \subseteq V$ be the set of nodes equipped with wavelength translators. A wavelength assignment is *valid* with respect to $S$ if for every node $v \notin S$, every path containing links $(u, v)$ and $(v, w)$ has the same color assigned to both links. $S$ is said to be *sufficient* for $G$ if every admissible routing allows a wavelength assignment which is valid with respect to $S$.

The natural questions are: (1) which choices of $S$ are sufficient? and (2) with such an $S$ in place, is the assignment problem tractable?

To answer the former we first need to determine which networks require no wavelength translators at all. A graph is said to be a "spider" if it is a tree with at most one node of degree greater than 2.

THEOREM 4.1. *The empty set is sufficient for $G$ if and only if $G$ is a spider.*

**Proof.** (Sketch) Suppose $G$ is a spider; if it is a path then sufficiency follows from the fact that interval graphs are perfect. Otherwise $G$ consists of $k \geq 3$ "legs" attached to the unique vertex $u$ of degree greater than 2.

We define a bipartite multigraph $H$ on two copies of the set of legs by including an edge from leg $i$ on the left to leg $j$ on the right for each path (through $u$) from leg $i$ to leg $j$; in an admissible routing, $H$ has maximum degree at most $w$.

By Gabow's theorem [9] there is a proper edge-coloring of $H$ with $w$ colors that can be computed in polynomial time, and this provides a valid wavelength assignment for the paths through $u$. The rest of the paths are confined to single legs, and can be assigned colors on a greedy basis from the body on out.

Suppose on the other hand that $G$ is not a spider; if it contains a cycle, let $x$, $y$ and $z$ be three nodes in clockwise order around the cycle and let $\boldsymbol{R}$ consist of clockwise paths from $x$ to $z$, $y$ to $x$ and $z$ to $y$. This is an admissible routing for $w = 2$ wavelengths, but no valid wavelength assignment is possible since any two of the paths have at least one common link.

If $G$ is a tree but contains two nodes of degree at least 3, then it contains a path with two pendant edges attached to each end. An example of an admissible routing in this graph for which there is no valid wavelength assignment is given in [18]. ∎

For a subset $S \subseteq V$, define the graph $G(S)$ as follows. The nodes $V(S)$ of $G(S)$ are the nodes in $V \setminus S$ together with pairs $\langle s, e\rangle$ for each $s \in S$ and each edge $e$ incident to $s$ in $G$. The edges of $G(S)$ consist of the edges $\{u, v\}$ of $G$ where $u, v \notin S$, together with $\{\langle s, e\rangle, v\}$ whenever $e = \{s, v\}$ and $\{\langle s, e\rangle, \langle t, e\rangle\}$ whenever $s$ and $t$ are adjacent nodes of $S$.

We may think of $G(S)$ as the result of "exploding" each node $s$ of $S$ into degree-of-$s$-many copies, each of which becomes a pendant node to one of $s$'s old neighbors. We let $G_1(S), \ldots, G_k(S)$ denote the connected components of $G(S)$. The following is now a straightforward observation:

Theorem 4.2. *$S$ is sufficient for $G$ if and only if every component $G_i(S)$ is a spider.*

One may reasonably ask whether we have made the "sufficiency" condition rather stronger than necessary, by forbidding reroutings. Given a graph $G$, suppose that for every set of connection demands $D$ and routing $\boldsymbol{R}$ of $D$, there is some routing $\boldsymbol{R'}$ of $D$ and assignment of $L(\boldsymbol{R})$ wavelengths for $\boldsymbol{R'}$ which is valid with respect to $S$; then we say that $S$ is *weakly sufficient* for $G$. However,

Theorem 4.3. *If $S$ is weakly sufficient then it is sufficient.*

**Proof.** Omitted. ∎

In the special case where $G$ is a ring, we have that a single wavelength translator at any node is sufficient, justifying our earlier claim that the routing algorithm leads to optimal wavelength assignment in this case.

We now consider the main question of this section. Namely, given a graph $G = (V, E)$ the goal is to find a (weakly) sufficient set $S \subseteq V$ for $G$ of minimum cardinality. By the results above, this means that the goal is to find a minimum cardinality set $S$ such that each $G_i(S)$ is a spider graph. We consider the decision version of this problem: "Given integer $k \geq 0$ and graph $G = (V, E)$ is there a subset $S \subseteq V$ with $|S| = k$ such that each resulting $G_i(S)$ is a spider graph?" This problem will be called the MINIMUM SUFFICIENT SET problem and will be shown to be NP-complete.

Theorem 4.4. *The problem of determining whether, for given G and k, there is a sufficient set of nodes of size k, is NP-complete—even if G is planar.*

**Proof.** For a graph $G = (V, E)$ it is easy to check in polynomial time if $S \subseteq V$ is sufficient for $G$ since all this requires is checking that no $G_i(S)$ contains a cycle or more than one node with degree greater than 2. In order to shown that MINIMUM SUFFICIENT SET is NP-hard, we reduce PLANAR 3SAT [10] to it. That is, given an instance $I$ of PLANAR 3SAT we show how to construct an instance $M_I$ of MINIMUM SUFFICIENT SET that has a solution if and only if $I$ does. The construction of $M_I$ will be polynomial in the size of $I$. As a reminder, $I$ is an instance of PLANAR 3SAT means that the bipartite graph with a node for each variable and a node for each clause with edges between a variable node $x$ and a clause node $K$ exactly when $x$ or $\bar{x}$ is in the clause $K$ is a planar graph.

Let $I$ be an instance of PLANAR 3SAT with $m$ clauses $K_1, K_2, \ldots, K_m$ on $n$ variables $x_1, x_2, \ldots, x_n$. Each clause $K_i$ is a disjunction of three literals $u_{i,1}, u_{i,2}$ and $u_{i,3}$. Each $u_{i,j}$ is some $x_k$ or $\bar{x}_k$. In the definition of a graph that follows we are going to have distinct nodes labeled $u_{i,j}$, $x_k$ and $\bar{x}_k$ even though $u_{i,j} = x_k$ or $u_{i,j} = \bar{x}_k$. We will use the notation $xu_{i,j}$ to mean the node labeled by $x_k$ if $u_{i,j}$ is $x_k$ or the node labeled by $\bar{x}_k$ otherwise.

To define an instance of MINIMUM SUFFICIENT SET we need to define an integer $k$ and a graph $G = (V, E)$. Define $M_I$ as follows. Let $k = 2m + n$. Define $G = (V, E)$ to be the graph with node set

$$V = \left(\bigcup_{i=1}^{n}\{x_i, \bar{x}_i, a_i, b_i, c_i, d_i\}\right) \cup \left(\bigcup_{i=1}^{m}\{u_{i,1}, u_{i,2}, u_{i,3}\}\right)$$

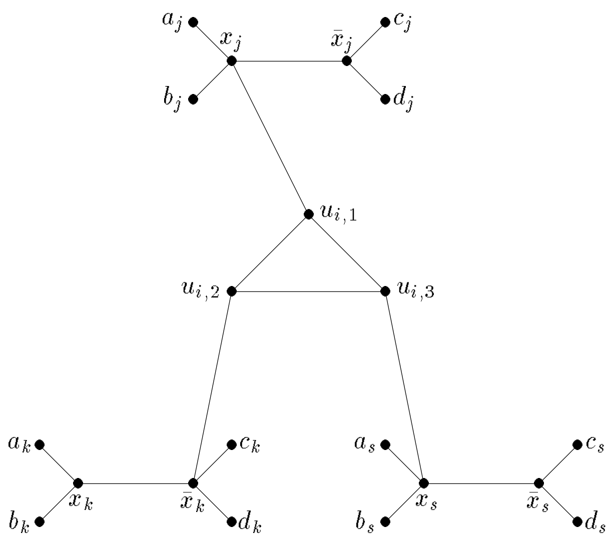


Figure 4: Gadget for clause $K_i = (x_j \vee \bar{x}_k \vee x_s)$.

and edge set

$$
\begin{aligned}
E \;=\;& \left( \bigcup_{i=1}^{n} \{\{x_i, \bar{x}_i\}\} \right) \\
& \cup \left( \bigcup_{i=1}^{n} \{\{x_i, a_i\}, \{x_i, b_i\}, \{\bar{x}_i, c_i\}, \{\bar{x}_i, d_i\}\} \right) \\
& \cup \left( \bigcup_{i=1}^{m} \{\{u_{i,1}, u_{i,2}\}, \{u_{i,2}, u_{i,3}\}, \{u_{i,3}, u_{i,1}\}\} \right) \\
& \cup \left( \bigcup_{i=1}^{m} \{\{u_{i,1}, xu_{i,1}\}, \{u_{i,2}, xu_{i,2}\}, \{u_{i,3}, xu_{i,3}\}\} \right).
\end{aligned}
$$

(See Figure 4.) Thus each clause $K_i$ of $I$ becomes a triangle of nodes $u_{i,1}$, $u_{i,2}$ and $u_{i,3}$ with an edge from each such node to the node labeled by its corresponding variable or negated variable. In addition, each node labeled by a variable $x_i$ has an edge to the node labeled by its negation and to the dummy nodes $a_i$ and $b_i$. Similarly, the node labeled by $\bar{x}_i$ also has edges to dummy nodes $c_i$ and $d_i$. Clearly the size of $G$ is polynomial in the size of $I$. Also, the graph $G$ is obviously planar since the instance $I$ was an instance of PLANAR 3SAT.

We claim that $I$ has a satisfying assignment if and only if $M_I$ has a sufficient set of size $k = 2m + n$. Consider the following four conditions on a subset $S \subseteq V$:

$(i)$ exactly one of $x_i$ or $\bar{x}_i$ is in $S$, $1 \leq i \leq n$

$(ii)$ exactly two of $u_{i,1}, u_{i,2}, u_{i,3}$ are in $S$, $1 \leq i \leq m$

$(iii)$ if $u_{i,j}$ is not in $S$ then $xu_{i,j}$ is in $S$

$(iv)$ $a_i, b_i, c_i, d_i$ are not in $S$, $1 \leq i \leq n$

It will be shown that a set $S$ satisfies conditions (i), (ii), (iii) and (iv) if and only if $S$ is a sufficient set of size $2m + n$.

Suppose $S$ is a sufficient set of size $2m + n$. Note that each node labeled by some $u_{l,j}$, $x_k$ or $\bar{x}_k$ has degree at least 3 in $G$ and so none of them can be in the same $G_i(S)$ since $G_i(S)$ must be a spider graph. Then for each clause $K_i$ at least two of $u_{i,1}, u_{i,2}$ and $u_{i,3}$ must be in $S$. Also at least one of $x_j$ or $\bar{x}_j$ must be in $S$. But since $|S| = 2m + n$ this means that exactly two of $u_{i,1}, u_{i,2}$ and $u_{i,3}$ must be in $S$ and exactly one of $x_j$ and $\bar{x}_j$ must be in $S$ and no other nodes are in $S$. Thus conditions (i), (ii) and (iv) are satisfied. Similarly due to the edges $\{u_{i,j}, xu_{i,j}\}$, if $u_{i,j}$ is not in $S$ then $xu_{i,j}$ must be in $S$ else $u_{i,j}$ and $xu_{i,j}$ are in the same $G_k(S)$ and both have degree at least 3 in $G_k(S)$ contradicting the claim that $S$ is sufficient and hence $G_k(S)$ is a spider graph. Thus condition (iii) is satisfied.

Suppose on the other hand that $S$ satisfies conditions (i), (ii), (iii) and (iv). Clearly then $|S| = 2m + n$ by conditions (i), (ii) and (iv). Consider some $G_j(S)$. Notice that nodes of $G_j(S)$ that result from the exploding of a node in $S$ have degree 1 in $G_j(S)$. Thus if $G_j(S)$ is not a spider graph then it must have at least one node labeled by some $x_i$, $\bar{x}_i$ or $u_{s,t}$ that is not in $S$. Suppose the node labeled $x_i$ is in $G_j(S)$ and not in $S$. (A similar argument holds if the node labeled $\bar{x}_i$ is in $G_j(S)$.) Then the node is adjacent to $a_i$, $b_i$, $\bar{x}_i$ and some number of nodes labeled by some $u_{s,t}$ where $xu_{s,t}$ is $x_i$. By definition $a_i$ and $b_i$ have degree 1. By condition (i), $\bar{x}_i$ must be in $S$ and hence has degree 1 in $G_j(S)$ and by condition (iii) the other nodes adjacent to $x_i$ are also in $S$. Thus $G_j(S)$ must be a spider graph (in fact, it is a star since each arm has length 1) contradicting our assumption. Suppose the node labeled $u_{s,t}$ is in $G_j(S)$ but not in $S$. Then conditions (ii) and (iii) show that it is adjacent in $G_j(S)$ only to nodes with degree 1 in $G_j(S)$ and again this implies that $G_j(S)$ is a spider graph (again it is actually a star). Thus it must be that $S$ is a sufficient set.

We wish now to show that a satisfying assignment $A$ for $I$ can be found if and only if there is a sufficient set of size $2m + n$. Let $A$ be a satisfying assignment for $I$. For $1 \leq i \leq n$, define $t_i$ to be $x_i$ or $\bar{x}_i$ depending on whether $x_i$ is true or not respectively according to the assignment $A$. Define $S_A$ to be the set of nodes of $G$ containing those nodes labeled by each $t_i$ and for each $j$, $1 \leq j \leq m$, two of those nodes labeled by $u_{j,1}, u_{j,2}$ and $u_{j,3}$ so that the one that is not placed in $S_A$ evaluates to true according to $A$. (There is always one such node since $A$ is a satisfying assignment.) Hence $|S_A| = 2m + n$. Clearly $S_A$ satisfies conditions (i), (ii) and (iv). Suppose the node labeled $u_{i,j}$ is not in $S_A$. Then it must be that $u_{i,j}$ is true according to $A$ since it is not in $S_A$. Therefore the node labeled $xu_{i,j}$ is true and hence in $S_A$ thus satisfying condition (iii). Thus $S_A$ satisfies all the conditions and hence is a sufficient set of size $2m + n$.

Suppose $S$ is a sufficient set of size $2m+n$. Then $S$ satisfies conditions (i), (ii), (iii) and (iv). Consider the truth assignment $A_S$ that assigns true to each variable $x_i$ such that the node labeled $x_i$ is in $S$. Consider any clause $K_i$ of $I$. Since $S$ satisfies condition (ii) we know that exactly one of the nodes labeled by the literals $u_{i,1}, u_{i,2}$ and $u_{i,3}$ is not in $S$. Suppose that literal is $u_{i,j}$. Then by condition (iii), the node labeled by $xu_{i,j}$ is in $S$ and hence is assigned true by $S_A$ and so $u_{i,j}$ is true according to $A_S$. Thus in $K_i$ there at least one true literal and so $A_S$ is a satisfying assignment. ∎

## 5 Conclusions

We have given a simple characterization of those WDM networks which have enough wavelength translators to operate at maximum capacity, and an efficient algorithm to assign wavelengths once a routing is fixed. In the case of a ring, one wavelength translator suffices, and in this case the problem of finding an optimal routing is also solvable in polynomial time.

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