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# Combined Network Design and Multiperiod Pricing: Modeling, Solution Techniques, and Computation

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In this paper we describe an efficient algorithm for solving novel optimization models arising in the context of multiperiod capacity expansion of optical networks. We assume that the network operator must make investment decisions over a multiperiod planning horizon while facing rapid changes in transmission technology, as evidenced by a steadily decreasing per-unit cost of capacity. We deviate from traditional and monopolistic models in which demands are given as input parameters, and the objective is to minimize capacity deployment costs. Instead, we assume that the carrier sets end-to-end prices of bandwidth at each period of the planning horizon. These prices determine the demands that are to be met, using a plausible and explicit price-demand relationship; the resulting demands must then be routed, requiring an investment in capacity. The objective of the optimization is now to simultaneously select end-to-end prices of bandwidth and network capacities at each period of the planning horizon, so as to maximize the overall net present value of expanding and operating the network. In the case of typical large-scale optical networks with protection requirements, the resulting optimization problems pose significant challenges to standard optimization techniques. The complexity of the model, its nonlinear nature, and the large size of realistic problem instances motivates the development of efficient and scalable solution techniques. We show that while general-purpose nonlinear solvers are typically not adequate for the task, a specialized decomposition scheme is able to handle large-scale instances of this problem in reasonable time, producing solutions whose net present value is within a small tolerance of the optimum.

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## 1. Introduction

In this paper, we describe a novel model and develop efficient algorithms for addressing capacity expansion and allocation combined with bandwidth pricing, in the context of designing resilient optical transport networks.

A network is modeled as a set of nodes, representing cities and/or metropolitan areas, which are connected by links representing physical routes owned by the network operator; transmission systems deployed on links are used to carry traffic. The carrier, or network operator, makes investments for deploying those systems and incurs periodic operating expenses, while collecting revenue from carrying demand for customers.

We consider network planning over a time horizon that spans many years. During such a period several generations of transmission technologies will typically emerge. Although a newer system may have a higher deployment and operating cost, the magnitude of capacity improvement can be expected to far outpace that of cost increase. As a result, a new technology results in a lower cost per unit of capacity than previous ones, thus making it an attractive candidate for new deployment.

In addition, after a new technology becomes available, its deployment cost decreases over time due to a learning effect. As a result, there may be an economic incentive to delay the deployment of a new technology to exploit savings from future cost reductions.

Such a dynamic technology environment immediately poses two interesting problems. First, timing: When should the operator start to deploy the new systems and phase out old technologies? Second, sizing: How much capacity should be deployed on each link at each time period?

Traditional network-planning problems have been discussed extensively in the literature, and it would be impractical to provide a thorough review here. See Alevras et al. (1998), Balakrishnan et al. (1995), Bienstock et al.

(1998), Bienstock and Günlük (1996), Bienstock and Saniee (2001), Günlük (1999), Magnanti et al. (1995), Stoer and Dahl (1994), and the references therein. Typically, the critical decisions involve adding capacity, at minimum cost, to existing capacity levels, so that one can route known demands (fully or partly, with a penalty for shortfalls). Lisser et al. (1999) and Sen et al. (1994) consider models in which demands are imperfectly known.

While such models rationalize decision making from an engineering perspective, they reflect a monopolistic approach to decision making under price regulation. Consequently, more comprehensive approaches have been proposed in recent studies to integrate capacity planning into the overall business optimization strategy, in particular in the context of optical networks (Lanning et al. 2000).

In Lanning et al. (2000), instead of assuming fixed forecast demands between pairs of nodes, an explicit nonlinear price-demand relationship is taken into consideration, and demands are treated as flexible quantities that are determined by prices. Prices, which must be chosen for every pair of nodes, are critical decision variables playing two roles: They determine the revenue for each unit of service rendered, and through the price-demand relationship they affect the amount of deployed capacity, and thus, incurred deployment cost. Consequently, capacity planning is driven not by the need to meet a fixed-demand target, but by the desire to generate higher profits. In summary, the objective of the optimization model is to simultaneously choose network capacities and prices of network services, so as to maximize the overall net present value of operating the network; i.e., we want to maximize the discounted total revenue minus total cost over the planning horizon. An important qualitative observation in Lanning et al. (2000) is that, as the price elasticity of demand grows larger than 1, the observed (optimal) capacities become large.

The study in Lanning et al. (2000), while innovative, incorporated several limitations: The networks that were considered were of small size and were simple, the number of time periods in the model was small, and the underlying network design problem was simplified. We expand on these limitations next.

First, in Lanning et al. (2000) the planning model was limited to a five-node ring network over five time periods. The small size of the network and small number of time periods obviated the need for truly effective optimization algorithms. We note that in the case of large-scale optical networks (e.g., national networks, quite likely entailing hundreds of nodes), the optimization problem poses significant computational challenges. In comparison with standard minimum-cost capacity-planning models, the new approach introduces a price variable and a demand variable for each period for each node pair; thus, the total number of price (and demand variables) is on the order of $N^2T$. Here, $T$ is the number of planning periods, while $N$ is the number of nodes in the network. In addition, each pricing variable appears in a nonlinear term in the objective function. As a result, when planning a node network with possibly hundreds of nodes, typical for national or international optical transport networks, over a 10-period horizon, we obtain an optimization model that could have even millions of variables that explicitly appear in the nonlinear component of the objective.

Another modeling component of current interest, not considered in Lanning et al. (2000), is that of survivability. In the context of traditional network design problems, survivability was considered in Alevras et al. (1997), Bienstock and Saniee (2001), Bienstock and Muratore (2000), Stoer and Dahl (1994), and others. Survivability modeling seeks to ensure reliable communications in the presence of possible failures. When considering optical transport networks, one way to achieve this end is to have multiple nonoverlapping paths between every node pair, with enough capacity allocated so that recovery from a link failure is guaranteed. Several specific routing and protection schemes that need different formulations will be considered in the following sections. In terms of our optimization model, the survivability feature calls for additional variables used to specify the rerouting of traffic when a network failure occurs. Formulations of some capacity-efficient survivability schemes, such as shared protection, require the introduction of new flow variables whose number is again on the order of $N^2T$.

A third feature not addressed in Lanning et al. (2000), and implicit in the prior paragraph, concerns routing schemes—in the case of a ring network, routing is trivial. Typically, standard network design models have assumed that any path may be used to route traffic (for exceptions, see Bley et al. 2000, Günlük et al. 1996). In realistic settings, however, one cannot route using arbitrary paths—for example, allowable paths are typically limited in length, or they may be constrained by an underlying routing protocol. In general, the decision as to which paths can be used may be defined somewhat idiosyncratically, and for optimum flexibility in the design of an algorithm, it should be assumed that an allowable path family is given as an input to the problem. The restriction to a (potentially large) given path family will almost certainly increase the difficulty of the model.

Clearly, a decision-support tool that could handle large, complex networks over many time periods, while incorporating survivability and routing features, and, of course, the price/demand model, would be of great strategic importance—it would serve as a means to forecast future demand trends, and to ascertain the interplay between different price/demand models and estimations of future per-unit capacity costs.

In this paper, we present the development of such a tool. First, we extend the model in Lanning et al. (2000) so as to incorporate all the features we described above. Second, we develop and test an optimization algorithm geared for large instances. Our algorithm is able to handle very large problems endowed with all of the above modeling features in reasonable time, while providing feasible solutions

whose net present value is guaranteed to be within a small tolerance of the optimum. Here, we stress once more that the optimization problems we tackle are truly difficult. As we will show, general-purpose commercial solvers prove unequal to the task. Finally, our solution techniques are *robust* in the sense that they can handle a broad range of models without modification.

The rest of this paper is organized as following. In §2, we present a detailed description of our overall problem. In §3, we present a mathematical formulation of the optimization model. Our solution techniques are presented in §4, and §5 presents computational experiments. Section 6 contains concluding remarks.

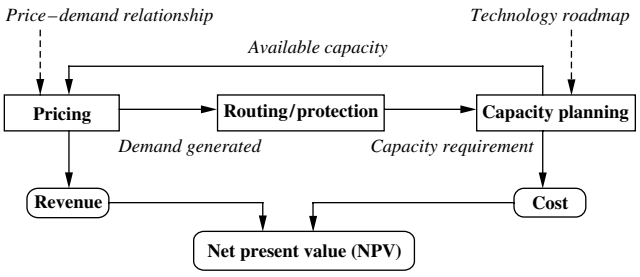
## 2. Modeling

In this section, we discuss several key concepts and assumptions of the pricing and capacity-planning model presented in this paper. The scope of our study is outlined in Figure 1 and is explained as follows. First, the model we use differs fundamentally from the norm in that demand is not given in advance. Rather, based on a given price-demand relationship, we make pricing decisions so as to generate corresponding amounts of demands and, as a result, revenues. Next, routing and protection decisions relate generated demands to link capacity requirements. These requirements are satisfied by the deployment of transmission systems, whose availability, cost, and capacity are prespecified as a technology "roadmap," the principal ingredient of which is that it models (per-unit) capacity costs that decrease over time.

Thus, pricing decisions and capacity planning are interrelated, as installed capacity must be sufficient to route the generated demands; and the issues of pricing, routing/protection, and capacity planning are embedded in the overall optimization framework, whose objective is to maximize the net present value of the total cash flow of all planning periods.

We note that the general topic of combining resource allocation with revenue management has been taken up in other problem settings, notably in supply chain management; our price-demand relationship described below (and close variants) has been studied in that context. See Federgruen and Heching (1999), Chen and Simchi-Levi (2002), and references therein for recent results.

**Figure 1.** Model framework.




In the rest of this section, we discuss the precise details of our models.

Throughout this paper, we will model the network by a graph $G = (N, L)$, where $N$ is the set of nodes and $L$ is the set of physical links (rights-of-way). Pairs of nodes in the network are denoted by $\sigma \in N \times N$. We also denote the length of the planning horizon by $T$, and index each period by $t = 1, \ldots, T$.

We will first describe the individual components of our model; a complete formulation that puts together these components is given in §3.

### 2.1. Price-Demand Relationship

We first introduce the notion of the price elasticity of demand.

Let $d^t_\sigma \geqslant 0$ be the demand between node pair $\sigma$ in period $t$, which relies on price per unit of demand $p^t_\sigma \geqslant 0$. The price elasticity of demand is defined as the negative ratio of the percentage change in demand to the percentage change in price, i.e.,

$$\epsilon^t_\sigma = -\frac{\partial d^t_\sigma}{\partial p^t_\sigma}\frac{p^t_\sigma}{d^t_\sigma}.$$

Following Lanning et al. (2000), we assume that $\epsilon^t_\sigma$ is constant and solve the above differential equation to arrive at the following price-demand relationship:

$$d^t_\sigma = A^t_\sigma (p^t_\sigma)^{-\epsilon^t_\sigma}. \tag{1}$$

The revenue $R^t_\sigma$ collected from node pair $\sigma$ in period $t$ is then

$$R^t_\sigma = p^t_\sigma d^t_\sigma = (A^t_\sigma)^{1/\epsilon^t_\sigma}(d^t_\sigma)^{1-1/\epsilon^t_\sigma}. \tag{2}$$

In this paper, we assume that $\epsilon^t_\sigma > 1$ for all $\sigma$ and $t$. Therefore, $R^t_\sigma$ is an increasing and concave function of $d^t_\sigma$, i.e., the carrier always receives more revenue by carrying more demand, but the slope of revenue growth declines as demand volume increases.

In our model, $d^t_\sigma$ will be a decision variable, with price implicitly determined through the demand function (1).

### 2.2. Routing and Protection

In fiber transport networks, node-pair demands give rise to required link capacities through routing and protection design. The latter is of vital importance: It is mandatory to allocate capacity to protect carried traffic so as to be able to weather a failure, such as a fiber cut or equipment outage. Typically, all traffic affected by a failure must be rerouted. There are many possible schemes for ensuring survivability, each characterized by a particular trade-off between efficiency of capacity use and complexity of the implementation. See, e.g., Alevras et al. (1997).

In this paper, we consider several routing/survivability schemes under the common assumption that for each node

pair, there is a preselected set of disjoint paths to carry the demand; we must choose which paths will be used and how much to route on each path. We will assume that at most one failure can occur, and that in the event of a failure we must be able to reroute all affected traffic.

There is an added modeling benefit that ensues from regarding the set of allowable paths as an input to the optimization. The reason for this is that in a realistic application, the question of which paths can be used to route traffic may be outside the scope of the optimization process, e.g., it may be determined by some qualitative service requirements. It is worth discussing this point with a little more detail. In the traditional network design literature, in particular in work with an optimization focus (especially integer programming aspects of network design), it is frequently assumed that the optimization algorithm has complete control over routing decisions. That is to say, the optimization engine produces the paths used to route demands. From a networking standpoint, this is usually not a realistic assumption, as the optimization may route demands using an excessive amount of "splitting" over many paths, or may produce overly long and circuitous paths, in particular when there is a survivability requirement. A more satisfactory framework is one where a limited path family is input to the optimization engine: Then, the problem becomes that of choosing paths (and corresponding traffic amounts) from among this family. For example, the path family may embody simple structural restrictions (such as limited path length), or may reflect idiosyncratic business objectives of the network operator, or may describe constraints that are imposed on the network operator, such as being forced to use an existing protocol to route traffic (see Bley et al. 2000).

In fact, all of the above restrictions may well arise at the same time. This makes their incorporation into a mathematical programming model problematic, because, typically, it is impossible or extremely costly to describe the allowable set of paths through the use of, e.g., linear inequalities. Thus, in this paper we assume that an explicit path family is given—the precise nature of the family is irrelevant to the design of our solution procedures.

Let $\mathcal{P}(\sigma)$ be the set of candidate paths for demand pair $\sigma$—recall that $d^t_\sigma$ measures the amount of demand for $\sigma$ at time $t$. These paths, as discussed above, are assumed to be pairwise disjoint. Also recall that $d^t_\sigma$ is a *variable*: It will be determined by the optimization. Denote $f^t_r \geqslant 0$ as the amount of flow carried on path $r$ in the normal (nonfault) condition, and $\phi^t_{r'}(r) \geqslant 0$ as the amount of flow diverted from route $r$ to $r'$ when $r$ is disrupted by failure ($r, r' \in \mathcal{P}(\sigma)$, and $\phi^t_{r'}(r) = 0$ if $r' = r$). It follows that to carry all demand under the normal condition,

$$\sum_{r\in\mathcal{P}(\sigma)} f^t_r = d^t_\sigma, \tag{3}$$

and to restore all traffic when failure occurs,

$$\sum_{r'\in\mathcal{P}(\sigma)} \phi^t_{r'}(r) = f^t_r \quad \forall\, r \in \mathcal{P}(\sigma). \tag{4}$$

There are two alternate models that can be built on these equations. In the first, and simplest, we assume that there are given constants $\theta^t_r$ and $\eta^t_{r'}(r)$ such that

$$f^t_r = \theta^t_r d^t_\sigma, \quad \text{where} \sum_{r\in\mathcal{P}(\sigma)} \theta^t_r = 1 \text{ and } \theta^t_r \geqslant 0,$$

$$\phi^t_{r'}(r) = \eta^t_{r'}(r) * f^t_r, \quad \text{where}$$

$$\sum_{r'\in\mathcal{P}(\sigma)} \eta^t_{r'}(r) = 1 \text{ and } \eta^t_{r'}(r) \geqslant 0.$$

In the second model, $f^t_r$ and $\phi^t_{r'}(r)$ are decision variables to be optimized by the model, subject to (3) and (4). The second approach will, of course, typically produce better solutions, at the cost of increased problem size.

Regardless of which approach we take, on link $l$ at time $t$ the total bandwidth that is consumed can be expressed as the sum of two terms:

$$\left(\sum_\sigma \sum_{r\in\mathcal{P}(\sigma): l\in r} f^t_r\right) + z^t_l. \tag{5}$$

In this expression, the summation in parentheses is the amount of bandwidth used for carrying traffic on primary routes, while $z^t_l \geqslant 0$ is the amount of ("redundant") bandwidth used for protecting traffic on other routes when a failure occurs.

There are different ways for planning backup capacity, resulting in different formulas for $z^t_l$. One approach is to set aside dedicated protection capacity for each demand, i.e.,

$$z^t_l = \sum_\sigma \bar{z}^t_l(\sigma), \quad \text{where } \bar{z}^t_l(\sigma) \geqslant \phi^t_{r'}(r)$$
$$\forall\, l \in r',\ \forall\, r,\ r' \in \mathcal{P}(\sigma):\ r \neq r'. \tag{6}$$

This ensures that for each demand $d^t_\sigma$, when any route $r$ fails, every link $l$ on a protection route $r'$ has enough capacity to carry $\phi^t_{r'}(r)$, which is the amount of traffic rerouted to $r'$.

A more efficient but complicated approach is to allow protection bandwidth to be shared by different demands. In this case, we first define for each pair $l$, $l'$ of distinct links,

$$\tilde{z}^t_l(l') = \sum_\sigma \sum_{r\in\mathcal{P}(\sigma): l'\in r} \sum_{r'\in\mathcal{P}(\sigma): l\in r'} \phi^t_{r'}(r),$$

as the amount of redundant bandwidth consumed on link $l$ to protect traffic against the failure of $l'$. The equation shows that when $l'$ fails, all paths $r$ that contain the link are disconnected, and flows on them are rerouted to some backup paths $r$. The redundant bandwidth needed on $l$ is then the sum of rerouted traffic, $\phi^t_{r'}(r)$, for all $r$ that contains link $l$. To satisfy rerouting requirements in the presence of any link failure, the redundant capacity on link $l$ should be the maximum of protection bandwidth set aside for each failure scenario, i.e.,

$$z^t_l \geqslant \tilde{z}^t_l(l')$$

for each $l' \neq l$.

To summarize, we formulate flow values in two ways: We either make them prespecified fractions of the corresponding demand variables (in which case the flows do not appear in the formulation as explicit variables), or include them in the optimization model as decision variables. There are also two ways to derive required link capacity from flows; one allows the sharing of protection bandwidth by multiple demands and the other does not. The combination of these options generates four routing and protection schemes. The two that do not allow sharing of link protection capacity are called single-demand protection with fixed percentage (SDP), and single-demand protection with no fixed percentage (SDN). The other two that allow sharing are called multiple-demand protection with fixed percentage (MDP), and multiple-demand protection with no fixed percentage (MDN).

## 2.3. Capacity Planning

Our model deploys capacity on the links of the network—at each time period, there must be enough capacity to cover the required bandwidth on each link, as described in Equation (5). Of course, the combined pricing/routing/capacity optimization model that we will describe makes all decisions simultaneously, to maximize net present value of profit, but in this section, we focus on the capacity component. Our model uses continuous variables to model capacity decisions. We will comment on the rationale for this approach at the end of this section. First, we will provide a broad description of our model.

Our primary modeling goal is to incorporate the empirical fact that per-unit capacity costs are expected to decrease over time. This is due to improving economies of scale, a result of improving technologies. Our numerical data reflect these effects through a simple "technology roadmap" that posits a progressive decrease of per-unit capacity costs as time goes by. Thus, our experiments reflect the interplay between our price/demand relationship and the technology roadmap.

For additional discussion, please refer to Raskina (2003). Also see Dixit and Pindyck (1994, especially Chapter 11) for related material from an economic viewpoint. We will now describe our detailed capacity model.

For a link $l$ and time period $t$, we denote by $c_l^t$ the cost of deploying one unit of capacity on link $l$ at time $t$. This cost is paid at time $t$. Having deployed capacity at time $t$, this capacity can be kept in the link for use during future time periods, or can be retired (at zero cost) at future time periods under a retirement schedule of our choice. On the other hand, each unit of capacity that is not retired incurs a maintenance/operating cost during each future time period that it remains in operation. For time periods $1 \leqslant s < t \leqslant T$, the cost incurred by each unit of capacity deployed in link $l$ at time $s$ and still in operation at time $t$ is denoted by $c_l^{s,t}$ (which is paid at time $t$).

We will use two types of variables:

(1) For each link $l$ and time period $1 \leqslant t \leqslant T$, set $y_l^t =$ total capacity deployed on link $l$ at time $t$, and

(2) For each link $l$ and time periods $1 \leqslant s < t \leqslant T$, set $y_l^{s,t} =$ amount of capacity installed on $l$ at time $s$, which we still keep in the network at time $t$.

Using these variables, the cost that we incur for at time $t$ deploying and maintaining this capacity configuration on link $l$ is

$$c_l^t y_l^t + \sum_{s<t} c_l^{s,t} y_l^{s,t}.$$

Further, we have to have enough capacity in link $l$ at time $t$ to handle the load on this link. Using Equation (5), this is expressed as

$$y_l^t + \sum_{s<t} y_l^{s,t} - \left( \sum_{\sigma} \sum_{r \in \mathcal{P}(\sigma): l \in r} f_r^t \right) - z_l^t \geqslant 0,$$

where the first two terms indicate the total capacity we have available on $l$ at time $t$. Finally, we need constraints to describe the possibility of capacity retirement. Put in other words, we want to mandate that the amount of capacity that can be carried over from one period to the next should not exceed the amount that is currently in use. Thus, we have

$$y_l^{s,s+1} \leqslant y_l^s$$

for each $l$ and $1 \leqslant s < T$, and

$$y_l^{s,t} \leqslant y_l^{s,t-1}$$

for each $l$ and $1 \leqslant s < t-1 < T$.

Finally, we impose $y_l^{s,t} \geqslant 0$ for all $s$, $t$, and $l$.

**2.3.1. On the Use of Continuous Variables.** In a tactical network design setting, "capacity" is deployed by means of discrete transmission "systems." Typically there are a number of available different system types (e.g., technologies) with different cost/performance profiles. In addition, deployed capacity systems must be "configured"—this entails making decisions about configuration of subcomponents of the systems (ports, interfaces, cables, etc.). The configuration of each deployed system affects the actual capacity of that system and its cost, and also constrains the configuration of other transmission systems. In network design models, the discrete nature of the transmission systems, as well as their configuration, is handled by relying on mixed-integer programming formulations. These formulations draw heavily on precise technical knowledge of the existing transmission technologies. For example, suppose that there are two available transmission systems, one with throughput 48, and another with throughput 96. Then, we would describe the capacity constraint on link $e$ using an inequality of the form

$$r_e - 48x_e^1 - 96x_e^2 \leqslant 0. \tag{7}$$

Here, $r_e$ is a continuous variable used to describe the total load on link $e$, while $x_e^1$ and $x_e^2$ are integer-valued variables

used to describe the *number* of systems of throughput 48 and 96 that we purchase for link $e$.

In addition, tactical models assume fairly accurate knowledge of demand levels, which are not decision variables.

In this paper, however, we do not consider a tactical network design model. On the contrary, ours is a strategic, multiyear model devised to extract long-range information. Namely, we would like to predict demand levels, as a function of our projected price/demand relationship. The usefulness of this type of model is that it can be used to predict future profit levels, future cash flows, resource needs, and so on. An important point is that when using, say, a 10-year model, it is neither feasible nor desirable to use a precise description of the transmission technologies—in particular, we cannot even know what technologies will be available during the later stages of the planning horizon (historically, such technologies have changed rapidly, with steeply decreasing per-unit costs). As a result, a detailed mixed-integer programming model becomes an unnecessary hindrance—restricting the model to the set of technologies that are known at the start of the planning horizon would limit the scope of the model.

## 3. The Formulation

In this section, we provide the complete nonlinear programming formulation of our model, summarizing what we discussed above. Below we refer to this formulation as MULTIREV.

Our formulation describes decisions over $T$ periods. For $1 \leqslant t \leqslant T$, let $h_t$ denote the discount factor used to weigh cash flows at time $t$. The objective function is formulated as

$$\max F(d, y) \equiv \sum_{t=1}^{T} h_t \left[ \sum_{\sigma} (A_\sigma^t)^{1/\epsilon_\sigma^t} (d_\sigma^t)^{1-1/\epsilon_\sigma^t} - \sum_{l} \left( c_l^t y_l^t + \sum_{s<t} c_l^{s,t} y_l^{s,t} \right) \right]. \tag{8}$$

In the above equation, items inside the square brackets reflect the cash flow in period $t$, where the first summation refers to the revenue (see §2.1), aggregated over all node pairs, and the second summation refers to the cost (see §2.3), aggregated over all links. Thus, the objective, which we seek to maximize, is the sum of discounted cash flows, or net present value.

Our formulation has two broad sets of constraints. The first set models the fact that capacity purchased in a given period can be reused in the next period (and in future periods), while at the same time allowing the possibility of "retiring" capacity at any time.

$$y_l^{s,s+1} - y_l^s \leqslant 0 \tag{9}$$

for each $l$ and $1 \leqslant s < T$, and

$$y_l^{s,t} - y_l^{s,t-1} \leqslant 0 \tag{10}$$

for each $l$ and $1 \leqslant s < t-1 < T$.

The second set of constraints requires that in any period there should be enough capacity on each link to carry and protect all traffic.

$$y_l^t + \sum_{s<t} y_l^{s,t} - \left( \sum_{\sigma} \sum_{r \in \mathcal{P}(\sigma): l \in r} f_r^t \right) - z_l^t \geqslant 0. \tag{11}$$

Flow variables $f_r^t$ and protection bandwidth $z_l^t$ are related to demand variables $d_\sigma^t$ in different ways, depending on routing and protection schemes.

*Case* 1. In the case of SDP, for simplicity of demonstration, the formulation of $z_l^t$ below, as well as that for Case 3, is based on the assumption that paths $r \in \mathcal{P}(\sigma)$ are disjoint from each other. This restriction can be easily removed by grouping nondisjoint paths and defining new flow variables associated with groups. This extension will not affect our main point, which is the linear structure of constraints.

$$f_r^t = \theta_r^t d_\sigma^t, \quad \text{and} \quad \forall\, t, l, \quad z_l^t \geqslant \sum_{\sigma} \max_{r, r' \in \mathcal{P}(\sigma): l \in r'} [\eta_{r'}^t(r) f_r^t], \tag{12}$$

where $\theta_r^t$, $\eta_{r'}^t$ are constants that are an input to the algorithm. Here, the $f_r^t$ are *not* variables—rather, we substitute the first equation in (12) into, e.g., (11). The $z_l^t$ are decision variables.

*Case* 2. In the case of MDP, the quantities $\theta_r^t$, $\eta_{r'}^t$, and $f_r^t$ are as in the previous case, but instead we impose

$$z_l^t \geqslant \sum_{\sigma} d_\sigma^t \left[ \sum_{r \in \mathcal{P}(\sigma): l' \in r} \left( \sum_{r' \in \mathcal{P}(\sigma): l \in r'} \eta_{r'}^t(r) \right) \right] \quad \forall\, l' \neq l. \tag{13}$$

*Case* 3. In the case of SDN, $f_r^t$ are decision variables, and we impose the constraint

$$\sum_{r \in \mathcal{P}(\sigma)} f_r^t = d_\sigma^t. \tag{14}$$

To specify $z_l^t$, we define another set of decision variables, $\phi_{r'}^t(r)$, representing the amount of traffic rerouted to route $r'$ if the primary path $r$ fails. We impose

$$\sum_{r' \in \mathcal{P}(\sigma)} \phi_{r'}^t(r) = f_r^t.$$

Let $\tilde{\phi}_{r'}^t(\sigma) = \max_{r \in \mathcal{P}(\sigma): r' \neq r} \phi_{r'}^t(r)$. Then, we also impose

$$z_l^t = \sum_{\sigma} \sum_{r' \in \mathcal{P}(\sigma): l \in r'} \tilde{\phi}_{r'}^t(\sigma). \tag{15}$$

*Case* 4. In the case of MDN, $f_r^t$ are decision variables constrained by Equation (14), and $z_l^t$, the constraint we impose, is

$$z_l^t \geqslant \sum_{\sigma} \sum_{r \in \mathcal{P}(\sigma): l' \in r} \sum_{r' \neq r: l \in r'} \phi_{r'}^t(r), \tag{16}$$

where $\phi_{r'}^t(r)$ is defined as in the previous case.

Finally, all variables are assumed to be nonnegative.

We emphasize that the decision variables appearing in our model can be divided into two categories. Those variables in the first category, $d = \{d_\sigma^t\}$, are nonlinear in the objective function but linear in all constraints; while those in the second category, $w = \{f_r^t, y_l^t, y_l^{s,t}, z_l^t, \phi_{r'}^t(r)\}$, are linear in both the objective and constraints. We will exploit this problem structure in developing our algorithms.

## 4. The Algorithm

In this section, we describe a computationally efficient algorithm to solve MULTIREV to a practicable tolerance. One of our goals in the design of this algorithm is that it should be *scalable*: Its running-time performance should deteriorate gracefully as the problem size increases. Motivated by the nature of MULTIREV (in particular, its multiyear nature and the inherent inaccuracy of the input data), we view this scalability goal as more significant than that of obtaining extremely accurate solutions, and this view has shaped our overall design of the algorithm. We will return to this topic later.

Note that an instance of MULTIREV can be very large even when the underlying network is of moderate size. For example, a network with 50 nodes and 70 links considered over 14 time periods can result in a formulation with approximately 60,000 variables for the MDN model. The formulation resulting from a larger network with 150 nodes and 800 links over the same time horizon will already have over one million variables for the same model. The large number of variables is primarily due to the complex routing component of our model—every pair of nodes, in every time period, constitutes a commodity, which must be routed using the given path families.

The nonlinear objective function in MULTIREV strongly suggests the use of an approach that relies, at the core, on the use of Newton's method or a similar second-order method (see Luenberger 1989, Gill et al. 1991). The attractiveness of such an approach is, however, tempered by the risk that the complex structure of the constraints, and the large size of the models, will conspire to turn the necessary matrix computations into an insurmountable obstacle. This, in fact, is precisely what we found when testing general-purpose solvers, which proved unusable, as will be detailed in §5.

Thus, the challenge at hand is to develop a second-order method that intelligently leverages the structure of the model so that the critical computational matrix algebra is only carried out on "small," sparse matrices.

In this section, we provide a high-level description of an algorithm that successfully relies on this approach, with enough details for the purposes of this paper. For a complete description of the algorithm and related details, see Raskina (2003).

To motivate our algorithm, note that the objective function to be maximized in MULTIREV, $F(d, y)$ (see Equation (8)), can be written as

$$F(d, y) = R(d) - c^T y, \tag{17}$$

where $R$ is the sum of present values of all revenues (and is thus nonlinear) and $c^T y$ is the sum of present values of all capacity purchase and maintenance costs, a linear function.

The key insight that we will use is that, for a given demand vector $d$, the capacity vector $y$ is chosen so that the demands $d$ can be routed at minimum cost. More precisely, for a fixed-demand vector $d = \hat{d}$, $y$ is determined from $\hat{d}$ by the solution of the *linear program* LP($\hat{d}$) given by

$$C(\hat{d}) = \min\ c^T y$$
$$\text{s.t. constraints (9)–(16)}, \tag{18}$$

where in (18) we are fixing demands at $\hat{d}$. This statement of the linear program is perhaps misleadingly simple, as we have other decision variables besides capacities (for example, percentages in the cases where these are variables). Nevertheless, what we have is that, intuitively, having fixed demands we should then choose capacities as cheaply as possible. The following is a consequence of standard results in linear programming theory (see Schrijver 1999 for background).

THEOREM 1. *Consider the function $C(d)$, which is the value of the linear program* LP($d$).

(a) *$C(d)$ is a piecewise-linear convex function of $d$.*

(b) *The space of all nonnegative demand variables is the union of a finite collection of polyhedral regions $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_K$, such that for any such region $\mathcal{F}_j$ there is a vector $\kappa_j$ with the following property: For any $d \in \mathcal{F}_j$, we have that $C(d) = \kappa_j^T d$.*

TECHNICAL NOTE. The regions in (b) can be chosen to have disjoint *relative interiors* but, for example, may overlap on lower-dimensional faces.

According to the theorem, inside each region the cost function is *linear* in the demands. In view of this result, we can recast our overall optimization problem as

$$\max_{1 \leqslant j \leqslant K} \max_{d \in \mathcal{F}_j} \{R(d) - \kappa_j^T d\}. \tag{19}$$

Based on this observation, we can now state a "prototype" algorithm.

*Step* 1. Find an initial feasible solution, contained in region $\mathcal{F}_j$ for some $j$.

*Step* 2. Use a Newton-like method, and optimize $R(d) - \kappa_j^T d$ over $\mathcal{F}_j$. Let $\hat{d}$ be the optimal solution.

*Step* 3. Use a first-order step to find a better solution than $\hat{d}$ in a different region $\mathcal{F}_k$. If no such improvement is found, Stop: $\hat{d}$ is optimal. Otherwise, reset $\mathcal{F}_j \leftarrow \mathcal{F}_k$, and go to Step 2.

END.

The correctness of this ideal algorithm follows from the fact that, as we will see below, the objective function is a concave function of the demands. The essential simplification embodied by this algorithm is that the cost function is now linear in the demands, while, at the same time, the dimensionality of the problem has been substantially decreased. Nevertheless, Step 2 is difficult to implement: It entails constrained nonlinear optimization. Another way to put this is that, while running (say) Newton's method, a step may take us from a feasible solution in region $\mathcal{F}_j$

to outside of this region. In spite of this difficulty, one key property enjoyed by this algorithm is that, when we are in the relative interior of the given region $\mathcal{F}_j$, we can cast our problem as an unconstrained optimization problem in the appropriate space of variables. Our actual algorithm, given below in §4.2, uses this point of view. Another potential difficulty lies in the fact that, quite likely, the total number of regions could be very large—but we do not need to enumerate the regions in advance; they are simply "discovered" in the course of the algorithm. We will return to this point later.

To develop a complete algorithm, we also need a starting-point heuristic to carry out Step 1, and we also need a termination criterion that, unlike that in Step 3, allows us to stop the algorithm early when it has achieved sufficient accuracy. The heuristic is outlined next, while the termination criterion is described in §4.3.

### 4.1. Outline of the Heuristic

The starting-point heuristic is quite simple and we will only outline it here (for full details, see Raskina 2003). To understand the heuristic, note that in problem MULTIREV there is a separate demand for each pair of nodes—the amount of the demand is controlled by the price of that demand, as in Equation (1). One could, in principle, study a version of the problem where demands are defined for only a certain restricted subset of all pairs of nodes. In other words, we would be given, as an input to the problem, a subset $\Sigma \subseteq N \times N$ of all node pairs, and Equations (1) and (2) would only be defined for node pairs $\sigma \in \Sigma$. All other constraints would be adjusted to reflect this restricted demand set. It would be straightforward to adapt our algorithm to consider such a restricted version of the problem, but in this paper we are concerned with the case that $\Sigma = N \times N$, and all our computational tests are for instances of this type. Nevertheless, the restriction is useful to develop heuristics. Specifically, in our heuristic we repeatedly consider the version of the problem where there is a demand for precisely one node pair $\sigma$ (i.e., $|\Sigma| = 1$).

The heuristic consists of performing the following two steps.

(H.a) Solve the optimization problem obtained by restricting MULTIREV to *one demand at a time*. That is to say, we focus on a single demand (a single node/destination pair), and then solve the optimization problem: This involves choosing prices and amounts for this demand (and this demand only) over the entire planning horizon, and also simultaneously choosing capacities so as to feasibly route the demand during each of the time periods; all of this done so as to maximize the present value of profit. We stress that during this step all price/demand variables for *other* demands (i.e., other node/destination pairs) are *removed* from the model—the reader may view this as restricting the model we have to a particular demand.

(H.b) The solutions obtained in (H.a) are simply put together. This means that the prices, demands, flows (and, when appropriate, percentages) obtained in (H.a) are used verbatim, and we simply add (for each time period and for each link) all capacity variables computed in (H.a) (this includes $z$ and $y$ variables) so as to obtain a feasible solution.

This heuristic is admittedly myopic—it ignores potential savings that would result from intelligent capacity decisions and how this would impact the pricing process. However, on the other hand, the heuristic already does incorporate some basic understanding of the nonlinear price/demand relationship. As may be expected, on small, simple problem instances the heuristic performs fairly well; less so on larger and more complex instances (see Raskina 2003 for details). In any case, all we require from the heuristic is that it produce a feasible starting point, and it is the job of the core optimization algorithm to refine this initial solution.

One point that should be clear is that each of the optimization tasks in (H.a) is a significantly simpler problem than MULTIREV—while the latter has on the order of $N^2T$ variables, each of the single-demand problems solved in (H.a) has $O(N + T^2)$ variables (in fact, this can be reduced to $O(T^2)$) and is in addition far simpler. In the case of a large network, this has a major impact on problem difficulty, and as a result many solution approaches are likely to be effective; we used a first-order method (see Luenberger 1989) to carry out (H.a) to reasonable accuracy with small computational overhead. Briefly, first-order methods are algorithms that at each iteration compute a step direction that maximizes the inner product with the gradient, while maintaining feasibility. Once the step direction has been computed, a line search is conducted so as to compute the next iterate (which maximizes the objective along the step direction). First-order methods are notorious for slow convergence to an optimal solution in the case of nontrivially nonlinear problems (such as ours)—but the risk of slow convergence is mitigated by the relatively much smaller size of the problem that we consider, and by the fact that we do not really need to converge to a very precise solution.

### 4.2. Core of the Algorithm

Now we return to the critical part of the algorithm, which primarily corresponds to Step 2 in the above prototype. To make this approach effective, we need to change our definition of "region" a bit. Rather than viewing regions as defining subsets of the demand space only, the regions we will use involve all variables. However, the (critical) component that in each given region the cost function is a linear function of the demands will be maintained. From a standard nonlinear programming point of view, our method is (essentially) an active-set method that operates in a reduced space of coordinates. It differs from a standard active-set method in that it treats the demand variables in a special way, and that it only focuses on the capacity constraints. In this last regard, the method also bears some resemblance to Lagrangian relaxation schemes, although it

always maintains a primal feasible solution. Details can be found in Raskina (2003); a similar algorithm was employed in Bienstock (1996). See Luenberger (1989) and Fiacco and McCormick (1968) for general background. Additional comments concerning the design of our method are given later.

The constraints of our optimization problem, including nonnegativity, can be written as

$$-Pw + Qd \leqslant 0, \tag{20}$$

where $w$ is the vector containing all variables other than demand variables. Hence, $w$ includes not only the capacity variables, but also the flow variables, and all other variables as needed in the various versions of our model. Consequently, with a slight abuse of notation, we denote the cost incurred by $w$ as $c^T w$. This is an abuse of notation in the sense that in Equation (17) the vector $c$ was restricted to the capacity variables, but we would rather not introduce new notation.

Definition 1. A working set consists of a triple $(I, J, K)$, where $I$ is a set of rows of the matrix $[P\ \ Q]$, $J$ is a set of $w$-columns, $K$ is a set of $d$-columns, and such that the submatrix of $P$ indexed by $I$ and $J$ has full column rank.

Definition 2. Let $(I, J, K)$ be a working set. We say that a vector $(\hat{w}, \hat{d})$ is consistent with $(I, J, K)$ if:

(i) $-P\hat{w} + Q\hat{d} \leqslant 0$ with equality precisely for those rows $i \in I$,

(ii) $J$ is the support of $\hat{w}$, i.e., $\hat{w}_j > 0$ if and only if $j \in J$, and

(iii) $\hat{d}_k > 0$ if and only if $k \in K$.

Now we have the following (straightforward) results (we refer the reader to Schrijver 1999 for basic linear programming facts):

Lemma 2. *Suppose that there is an optimal solution to problem* MULTIREV. *Then, there is an optimal solution* $(w^*, d^*)$ *that is consistent with some working set* $(I^*, J^*, K^*)$.

Proof. Pick an optimal solution $(w^*, d^*)$ such that, in addition, the support of $w^*$ has minimum cardinality among all optimal solutions. Because MULTIREV has an objective that is linear in the variables $w$, standard linear programming theory implies that the set of columns of $P$ corresponding to the support of $w^*$ is linearly independent. The rest of the conditions follow similarly. □

Lemma 3. *Let* $(I, J, K)$ *be a working set. Then, there is a vector* $\kappa = \kappa_{I,J,K} \in R^{|K|}$ *with the property that, for any vector* $(\hat{w}, \hat{d})$ *consistent with* $(I, J, K)$,

$$c^T \hat{w} = \kappa^T \hat{d}_K. \tag{21}$$

Proof. Denote by $P_{I,J}$ the submatrix of $P$ indexed by $I$ and $J$. Because $P_{I,J}$ has full column rank, it contains a $|J| \times |J|$ invertible submatrix $Z$. Let $\bar{Q}$ denote the submatrix of $Q$ indexed by the row set of $Z$. By definition of working set and consistency,

$$\hat{w}_J = Z^{-1}\bar{Q}\hat{d},$$

where $\hat{w}_J$ is the subvector of $\hat{w}$ indexed by the column set $J$. As a result,

$$c^T\hat{w} = c_J^T\hat{w}_J = c_J^T Z^{-1}\bar{Q}\hat{d}, \tag{22}$$

and the result follows because $d_k = 0$ for $k \notin K$. □

As a consequence of Lemmas 2 and 3 we can now state the critical result:

Lemma 4. *Let* $(I, J, K)$ *be a working set. Then, if* $(w, d)$ *is consistent with* $(I, J, K)$, *its objective value in the overall optimization problem is*

$$R(d_K) - \kappa_{I,J,K}^T d_K,$$

*where* $d_K$ *is the subvector of* $d$ *indexed by the columns* $K$, *and, with a slight abuse of notation,* $R(d_K)$ *is the revenue associated with* $d$.

Using the above definitions and results, we state our core algorithmic step.

### Routine $\Omega(\hat{w}, \hat{d}, \hat{I}, \hat{J}, \hat{K})$

*Initialization.* We are given a working set $(\hat{I}, \hat{J}, \hat{K})$ and a vector $(\hat{w}, \hat{d})$ that is consistent with $(\hat{I}, \hat{J}, \hat{K})$.

*Step* A. Using Newton's method, solve the *unconstrained* optimization problem

$$u = \max R(d_{\hat{K}}) - \kappa_{\hat{I},\hat{J},\hat{K}}^T d_{\hat{K}},$$

with solution $\bar{d}_K$.

*Step* B. Let $\bar{w}_{\hat{J}}$ be the (unique) solution to

$$-P_{\hat{I},\hat{J}}\bar{w}_{\hat{J}} + Q_{\hat{I},\hat{K}}\bar{d}_{\hat{K}} = 0,$$

where $P_{\hat{I},\hat{J}}$ and $Q_{\hat{I},\hat{K}}$ are, respectively, the submatrices of $P$ and $Q$ indexed by row set $\hat{I}$ and column sets $\hat{J}$ and $\hat{K}$.

*Step* C. Define $\bar{d}$ to be a vector of demands that equals $d_{\hat{K}}$ on the column set $\hat{K}$ and is 0 otherwise; and similarly define the $w$-vector $\bar{w}$ so that it agrees with $w_{\hat{J}}$ on the column set $\hat{J}$ and is 0 otherwise.

*Step* D. If $(\bar{w}, \bar{d})$ is consistent with $(\hat{I}, \hat{J}, \hat{K})$, then $(\bar{w}, \bar{d})$ is feasible, and is optimal over all solutions consistent with $(\hat{I}, \hat{J}, \hat{K})$. Routine exits and outputs $(\bar{w}, \bar{d})$.

*Step* E. Otherwise, $(\bar{w}, \bar{d})$ violates a constraint (possibly nonnegativity); hence, there is a maximum value $0 < \lambda < 1$ such that

$$(\breve{w}, \breve{d}) \doteq (1 - \lambda)(\hat{w}, \hat{d}) + \lambda(\bar{w}, \bar{d})$$

is feasible. Because $\lambda < 1$, $(\check{w}, \check{d})$ satisfies with equality some inequalities that are slack for $(\hat{w}, \hat{d})$. Then, without loss of generality, $(\check{w}, \check{d})$ is consistent with a working set $(\check{I}, \check{J}, \check{K})$.
We reset $(\hat{w}, \hat{d}) \leftarrow (\check{w}, \check{d})$ and $(\hat{I}, \hat{J}, \hat{K}) \leftarrow (\check{I}, \check{J}, \check{K})$, and go to A.
END.

COMMENT. Essentially, this method can be viewed as a Newton method that operates on reduced coordinates, but the choice of which coordinates to eliminate is specific to problem MULTIREV. From an intuitive standpoint, our strategy in devising this algorithm went as follows: Step A, at the core of routine $\Omega$, is an unconstrained problem that only uses the demand variables. Thus, during Step A, the capacity variables and constraints have been effectively removed (although of course they reappear in other steps). This is a "good" idea in that, by removing the capacity inequalities, the problem naturally tends to decompose on separate problems that reflect the demand structure (this is a feature shared with Lagrangian relaxation schemes for solving, e.g., multicommodity flow problems). A complete decomposition is not always possible because of the multiperiod structure of the problem, but nevertheless a substantial simplification of the problem is attained. We have:

LEMMA 5. *Routine* $\Omega$ *is finite and always exits as in Step* D.

PROOF SKETCH. Any time that the routine executes Step E, one more tight inequality is found, and henceforth the routine operates in a lower-dimensional face. Thus, the routine must terminate in Step D. □

### 4.3. Termination Criterion

The primary termination criterion we use relies on a simple, yet experimentally effective, upper bound on the optimum discounted profit. This upper bound, furthermore, is already obtained when running procedure $\Omega$, and hence entails no additional computational burden.

The key insight to understand the upper-bounding procedure lies in Theorem 1. To restate it, recall that the profit function associated with a particular demand vector $d$ is of the form $R(d) - C(d)$, where $R$ is the (discounted) revenue function and $C$ is the (discounted) cost of routing $d$. Theorem 1 states that $C(d)$ is convex, and that the space of all demands is the union of a collection of polyhedral regions $\mathcal{F}_j$, such that $C$ is linear when restricted to any one such region. As a result, we have:

THEOREM 6. *Let* $V^*$ *denote the optimal value of* MULTIREV. *Suppose that* $\mathcal{F}_j$ *is one of the regions, and let* $\kappa_j$ *be a vector such that* $C(d) = \kappa_j^T d$ *for all* $d \in \mathcal{F}_j$. *Write*

$$V_j^* = \max_d \{R(d) - \kappa_j^T d\}. \tag{23}$$

*Then,*

$$V_j^* \geqslant V^*. \tag{24}$$

PROOF. First, we stress that in (23) the maximum is taken over *all* $d$, not just all $d \in \mathcal{F}_j$.

Further, as a technical point, both maxima *are* attained, without loss of generality, if all capacity costs are positive. In this case, $R(d) - C(d)$ goes to $-\infty$ as $|d| \to +\infty$.

Now consider some other region $\hat{d} \in \mathcal{F}_k$. Then, once more $C(d)$ is linear over all $d$ in this region, i.e., if $d \in \mathcal{F}_k$, then $C(d) = \kappa_k^T d$ for some vector $\kappa_k$. In summary,

$$C(d) = \kappa_j^T d \quad \forall\, d \in \mathcal{F}_j, \tag{25}$$

$$C(d) = \kappa_k^T d \quad \forall\, d \in \mathcal{F}_k. \tag{26}$$

Because $C$ is in addition convex, we have

$$\kappa_k^T d \geqslant \kappa_j^T d \quad \forall\, d \in \mathcal{F}_k \tag{27}$$

(to see this, it may help to think of the line segment joining a point in $\mathcal{F}_j$ and a point in $\mathcal{F}_k$). Thus, for any $d \in \mathcal{F}_k$,

$$R(d) - C(d) = R(d) - \kappa_k^T d \leqslant R(d) - \kappa_j^T d, \tag{28}$$

and now (24) follows by definition of $V_j^*$ in (23). □

Suppose that we temporarily think of a working set $(I, J, K)$ as defining a region $\mathcal{F}_j$. Then, each time we execute Step A in procedure $\Omega$ we are computing a new upper bound on the value of MULTIREV, and we can use this step to keep track of the *best* upper bound found so far.

Now note that, as shown in Lemma 4, the cost function $C(d)$ is linear over the points consistent with any given working set; consequently, if somewhat informally, we can say that each polyhedral region $\mathcal{F}_j$ is a union of working sets. More precisely: It is the projection to the demand space of those points that are consistent with a finite family of working sets, all of which define the same linear function $\kappa_{I,J,K}^T d$. Hence, Theorem 6 applies, and the upper-bounding procedure described in the previous paragraph can be used.

### 4.4. The Complete Algorithm

Using §§4.2 and 4.3, we can now systematically lay out a formal algorithm to solve MULTIREV derived from our prototype algorithm given above. As the procedure iterates, it will keep track of two values: $v^F$, the *maximum* objective value attained by any feasible solution found by the algorithm so far, and $v^U$, an upper bound on the value of the problem. The algorithm uses two parameters, $\omega$ and $\tau$, to handle termination. For full details, see Raskina (2003).

*Step* I. Use the starting-point heuristic to find an initial feasible solution $(\hat{w}, \hat{d})$. Without loss of generality, $(\hat{w}, \hat{d})$ is consistent with some working set $(\hat{I}, \hat{J}, \hat{K})$. Let $v^F$ be the objective value associated with this solution, and set $v^U \leftarrow +\infty$.

*Step* II. Run routine $\Omega(\hat{w}, \hat{d}, \hat{I}, \hat{J}, \hat{K})$, which outputs a vector $(\bar{w}, \bar{d})$. Let $\bar{u}$ be the smallest value $u$ computed in any execution of Step A during this run of $\Omega$. Reset $v^U = \min\{v^U, \bar{u}\}$.

*Step* III. Reset $v^F$ to be the objective value of the solution $(\bar{w}, \bar{d})$. If $v^F$ has not improved by at least $\omega\%$ during the last $\tau$ iterations, exit.

*Step* IV. Perform a first-order step from $(\bar{w}, \bar{d})$, to find a better solution, $(\hat{w}, \hat{d})$. Go to Step II.
END.

In our implementation, we used $\omega = 5$ (i.e., 5%) and $\tau = 10$. How effective would we expect this algorithm to be? In the worst case, it could display the classical behavior of a first-order algorithm, i.e., severe tailing-off reflected by too many iterations of Step II. Another potential difficulty is that, if the optimum is highly degenerate, the algorithm might also use too many iterations. On the positive side, however, we have that:

(a) Our method greatly reduces the effective number of variables used at any time—this could make the difference between being able to run a problem or not,

(b) Within a region our method is "careful:" It implements a second-order algorithm, and

(c) The anecdotal experience with first-order methods is that while they can tail-off, they also can provide substantial improvements during the first few iterations.

Point (c) is important: As we have stated before, the nature of the application demands an algorithm that scales well with problem size, with less emphasis on being able to obtain solutions of very high accuracy.

## 5. Computational Experiments

We tested our algorithms using real network data. This data involved international long-distance networks. The data included the (geometrical) length of links.

First, we used five medium-sized networks; by varying some of the numerical parameters we generated a total of 7,750 problem instances of MULTIREV on which we ran our implementation.

In addition, to stress-test our algorithm, we used five large networks to generate a set of 50 problem instances. The medium-sized problems were solved on a 336 MHz UltraSPARC machine with 1.7 GB of RAM, while the tests using larger networks were conducted on a 1.89 GHz Xeon with 3 GB of RAM.

We will next describe how we constructed the problem instances.

### 5.1. Inputs

**5.1.1. Cost Data.** We generated capacity investment and maintenance data that are consistent with our modeling goals, as described in §2.3. To do so, for each link $l$ we generated the per-unit capacity investment at time $t$, $c_l^t$, and the per-unit maintenance costs, $c_l^{t,s}$ (see §2) for $1 \leqslant t < s \leqslant T$. The values $c_l^t$ should be nonincreasing (e.g., $c_l^{t+1} \leqslant c_l^t$) so as to model the emergence of improved technologies. Finally, to make the data realistic, we wanted to incorporate into the costs a dependence on the geometrical length of the links.

To achieve these ends, we first set

$$c_l^t = \gamma^{t-1}\Delta_l. \tag{29}$$

Here, $\Delta_l$ is the geometrical length of link $l$, and $0 < \gamma < 1$ is a constant chosen in a manner described below. The point of Equation (29) is that in any given time period, the unit capacity investment costs are proportional to link lengths, and that they become progressively cheaper as time goes by. Note that ideally one might want some proportionality constant in (29)—but the objective of MULTIREV (see Equation (8)) can be scaled without changing the nature of the problem.

Second, we generated maintenance costs of the form

$$c_l^{t,s} = c_l^t \mu \alpha^{s-t}$$

for any link $l$, and each pair of time periods $t < s$, where $\mu < 1$ and $\alpha > 1$. This models the case where (per-unit) maintenance costs are a fixed fraction of (per-unit) investment costs, but become more expensive with age. For our experiments, we set $\mu = 0.05$ and $\alpha = 1.05$.

**5.1.2. Price-Demand Data.** As discussed in §2.1, we used a price-demand relationship of the form $d_\sigma^t = A_\sigma^t (p_\sigma^t)^{-\epsilon_\sigma^t}$ for any given service $\sigma$. Denoting by $N$ the number of demands (e.g., the number of node pairs), we set $A_\sigma^t = A/N$ for all $t$ and $\sigma$, where the parameter $A$ was randomly chosen using a procedure described below. This procedure was also used to create random values to the parameters $\epsilon_\sigma^t$.

**5.1.3. Network Topology.** In Table 2 we describe basic properties of networks and describe resulting instances of MULTIREV. Columns 2 and 3 show the numbers of nodes and links in each network. The next three columns show the number of node pairs that have maximum of two, three, and four disjoint paths, respectively. No node pair had more than four disjoint paths. This is due to the sparsity of the underlying graph, typical in real transport networks. Thus, the total number of paths that can be used in optimization is limited to four in all five networks. (In large instances of the problem, we used up to 12 disjoint paths per node pair, see Table 11.)

The paths were generated using the *SPIDER* code (Davis et al. 2001), a tool that generates paths so as to achieve various survivability criteria.

The columns headed "Vars," "Const," and "Nonzero" describe, respectively, the number of variables, constraints, and nonzeros in the optimization problem. These values are determined by the network size and the length of time horizon, which is set to 14 in all cases.

**Table 1.** Data parameters.

| $\hat{A}$ | $\hat{\gamma}$ | $\hat{\epsilon}$ |
|---|---|---|
| 50,000 | 0.85 | 1.3 |
| 500,000 | 0.9 | 1.4 |
| 5,000,000 | 0.95 | 1.5 |

**5.1.4. Numerical Parameters.** In summary, to completely specify a problem instance, we had to assign values to three types of parameters: $\gamma$ (in Equation (29)), $A$, and $\epsilon_\sigma^t$.

To do so, we used the following procedure.

1. First, choose a triple $(\hat{A}, \hat{\gamma}, \hat{\epsilon})$ from among the values in Table 1.

2. Second, having chosen $(\hat{A}, \hat{\gamma}, \hat{\epsilon})$, we then choose $A$ randomly from a uniform distribution in the interval $(4\hat{A}/5, 6\hat{A}/5)$. Similarly, $\gamma$ is chosen from a uniform distribution in $(\hat{\gamma}-0.05, \hat{\gamma}+0.05)$. Finally, each $\epsilon_\sigma^t$ is chosen from a uniform distribution in $(1, 2\hat{\epsilon}-1)$.

Note that, using this procedure, the mean of $A$ is $\hat{A}$, the mean of $\gamma$ is $\hat{\gamma}$, and the mean of $\epsilon_\sigma^t$ (for each $\sigma$ and $t$) is $\hat{\epsilon}$. All possible triples $(\hat{A}, \hat{\gamma}, \hat{\epsilon})$ in Table 1 were used (a total of 27 combinations), and for each triple, 50 problem instances were generated as indicated for a grand total of 1,350 data sets.

Finally, we used the discount factor $h_t = 0.86^{t-1}$ for every time period $t$ except for period $T$ ($T = 14$), where we let $h_T$ take a larger value to account for the terminal value of installed network capacity. In our example, $h_T = 2.0$, which is calculated based on the assumption of a 7% continuous cash flow growth after the planning horizon.

## 5.2. Test Results

We can now describe the results of our computational tests. Table 3 shows the average, maximum, and minimum percentage error ("GAP") yielded by our algorithm on different parameter groups, while Table 4 presents running-time information, both for runs using the SDP model. (Here the percentage error is the relative gap between the upper and lower bounds computed by the algorithm.)

Table 3 shows that the problem becomes easier to solve as the elasticity $\epsilon$ increases and the yearly reduction rate $\gamma$ decreases. On the other hand, the scaling constant $A$ does not appear to affect the performance. In addition, using a smaller value of $\gamma$ induces a large decrease in cost from one year to another and helps to reduce the problem degeneracy, thus allowing a faster convergence.

Tables 5 and 6 give summary performance data for the algorithm, on the same data sets as above, for the SD

**Table 2.** Medium-sized problems.

| | | | Node pairs with $X$ disjoint paths | | | | | |
|---|---|---|---|---|---|---|---|---|
| Net | Nodes | Links | $X=2$ | $X=3$ | $X=4$ | #Vars | Const | Nonzero |
| snet1 | 14 | 22 | 36 | 49 | 6 | 3,584 | 2,310 | 17,472 |
| snet2 | 38 | 48 | 670 | 30 | 3 | 12,782 | 2,940 | 96,222 |
| snet3 | 47 | 55 | 1,043 | 38 | 0 | 20,909 | 5,775 | 196,623 |
| snet4 | 49 | 64 | 1,125 | 48 | 3 | 23,870 | 6,720 | 223,272 |
| snet5 | 70 | 94 | 2,006 | 400 | 9 | 43,680 | 9,870 | 431,382 |

**Table 3.** Performance summary—SDP model.

| | | $\hat{A}$ | | | $\hat{\gamma}$ | | | $\hat{\epsilon}$ | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Net | Gap (%) | 50,000 | 500,000 | 5,000,000 | 0.85 | 0.9 | 0.95 | 1.3 | 1.4 | 1.5 |
| snet1 | Ave | 0.00081 | 0.00077 | 0.00082 | 0.00062 | 0.00078 | 0.001 | 0.00098 | 0.00075 | 0.00065 |
| | Min | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Max | 0.0085 | 0.0082 | 0.0088 | 0.0068 | 0.0081 | 0.0098 | 0.0096 | 0.0082 | 0.0071 |
| snet2 | Ave | 0.006 | 0.0061 | 0.0061 | 0.0041 | 0.006 | 0.008 | 0.0086 | 0.0062 | 0.004 |
| | Min | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Max | 0.051 | 0.052 | 0.051 | 0.039 | 0.051 | 0.063 | 0.071 | 0.053 | 0.04 |
| snet3 | Ave | 0.019 | 0.018 | 0.016 | 0.009 | 0.017 | 0.044 | 0.039 | 0.018 | 0.01 |
| | Min | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Max | 0.078 | 0.076 | 0.075 | 0.064 | 0.076 | 0.088 | 0.089 | 0.075 | 0.065 |
| snet4 | Ave | 0.019 | 0.023 | 0.021 | 0.011 | 0.021 | 0.031 | 0.033 | 0.021 | 0.012 |
| | Min | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| | Max | 0.11 | 0.125 | 0.14 | 0.1 | 0.13 | 0.25 | 0.21 | 0.12 | 0.096 |
| snet5 | Ave | 0.021 | 0.024 | 0.025 | 0.012 | 0.023 | 0.041 | 0.033 | 0.022 | 0.013 |
| | Min | 0.011 | 0.014 | 0.009 | 0.007 | 0.013 | 0.018 | 0.019 | 0.014 | 0.007 |
| | Max | 0.193 | 0.198 | 0.21 | 0.176 | 0.2 | 0.29 | 0.28 | 0.196 | 0.18 |

**Table 4.** Average running time (sec)—SDP model.

| | $\hat{A}$ | | | $\hat{\gamma}$ | | | $\hat{\epsilon}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Net | 50,000 | 500,000 | 5,000,000 | 0.85 | 0.9 | 0.95 | 1.3 | 1.4 | 1.5 |
| snet1 | 15 | 15 | 14 | 15 | 15 | 16 | 16 | 16 | 14 |
| snet2 | 45 | 47 | 45 | 44 | 46 | 47 | 46 | 46 | 44 |
| snet3 | 100 | 100 | 99 | 99 | 99 | 102 | 100 | 100 | 98 |
| snet4 | 150 | 152 | 151 | 148 | 151 | 153 | 153 | 152 | 148 |
| snet5 | 213 | 212 | 214 | 213 | 213 | 215 | 214 | 214 | 212 |

**Table 5.** Performance summary—SDN, MDP, and MDN models.

| Model | Gap (%) | snet1 | snet2 | snet3 | snet4 | snet5 |
|---|---|---|---|---|---|---|
| SDP | Ave | 0.0008 | 0.0061 | 0.021 | 0.021 | 0.024 |
| | Min | 0 | 0 | 0 | 0 | 0.007 |
| | Max | 0.0098 | 0.071 | 0.089 | 0.25 | 0.29 |
| SDN | Ave | 0.32 | 0.35 | 0.48 | 0.47 | 0.65 |
| | Min | 0.22 | 0.17 | 0.2 | 0.21 | 0.4 |
| | Max | 0.44 | 0.61 | 0.88 | 0.85 | 1.13 |
| MDP | Ave | 0.27 | 0.29 | 0.33 | 0.33 | 0.41 |
| | Min | 0.16 | 0.18 | 0.16 | 0.18 | 0.2 |
| | Max | 0.39 | 0.43 | 0.52 | 0.59 | 0.64 |
| MDN | Ave | 1.19 | 1.77 | 2.01 | 2.2 | 2.3 |
| | Min | 0.82 | 1.16 | 1.29 | 1.56 | 1.45 |
| | Max | 1.56 | 2.47 | 2.89 | 3.08 | 3.89 |

model without percentages (SDN), and the MD model with (MDP) and without (MDN) percentages. Table 5 shows the average and boundary optimality gaps, while Table 6 shows the average running time.

An issue of interest is the relative effectiveness of our starting-point heuristic. To obtain data on the performance of the heuristic, we ran 27 additional models of the SDP type. The results are presented in Table 7. The first three columns concern the optimality gap attained by the heuristic, while the last three columns are for the overall algorithm. Note that while the heuristic is generally effective, in some cases it is less so, and is substantially improved upon by the algorithm.

Another relevant issue is the experimental convergence rate of the algorithm, and the number of iterations it performs. Recall that the algorithm iterates through Steps II–IV as detailed in §4.4. We will refer to these as "major" iterations to distinguish them from the iterations produced by the call to routine $\Omega$ in Step II, which we will call "minor" iterations. Figure 2 presents an overlay of five different runs of the algorithm on network snet5, plotting the gap computed by the algorithm as a function of the major iteration count. Thus, two of the runs terminated in six iterations, and the remainder in 11 iterations, always achieving a final gap below 1%.

A major iteration count on the order of 10 was typical for the *snet*1–5 instances. On these instances, the typical minor iteration count was on the order of 50. In the much larger problem instances that we will discuss in §5.4, the minor iteration count was on the order of 1,000.

**Table 6.** Running-time (sec) summary—SDN, MDP, and MDN models.

| Model | snet1 | snet2 | snet3 | snet4 | snet5 |
|---|---|---|---|---|---|
| SDP | 15.1 | 45.5 | 99.6 | 150.8 | 213.3 |
| SDN | 10 | 79 | 119 | 108 | 284 |
| MDP | 11 | 80 | 126 | 130 | 304 |
| MDN | 13 | 93 | 142 | 158 | 332 |

**Table 7.** Effectiveness of the heuristic and overall algorithm (%).

| | Heuristic (%) | | | Algorithm (%) | | |
|---|---|---|---|---|---|---|
| | Ave | Min | Max | Ave | Min | Max |
| snet1 | 2.12 | 0.61 | 4.06 | 0.32 | 0.22 | 0.44 |
| snet2 | 4.33 | 1.81 | 6.81 | 0.35 | 0.17 | 0.61 |
| snet3 | 5.14 | 1.71 | 10.09 | 0.48 | 0.20 | 0.88 |
| snet4 | 5.51 | 2.36 | 8.32 | 0.47 | 0.21 | 0.85 |
| snet5 | 3.24 | 0.79 | 8.60 | 0.79 | 0.28 | 1.13 |

**5.2.1. Economic Interpretation of Results.** Here we will compare the behavior of the different models from the perspective of the practical model that we discuss in this paper.

One of the goals of our study was to analyze the impact of survivability on profit. In particular, we would like to study the effect on profit of using fixed percentage routing schemes, and to compare the SD and MD models, as they were designed to provide different degrees of survivability, at potentially higher cost. Table 8 shows the (percent) decrease in the optimal objective value when fixed percentages are used, and Table 9 demonstrates the average increase in total profit when the MD model is used instead of the SD model.

The average improvement is about 2% of the overall profit for all cases considered. Thus, a weaker survivability model offers a considerable increase in profit. On the

**Figure 2.** Convergence plots, showing the major iteration counts in five different runs of the algorithm in §4.4 on snet5.

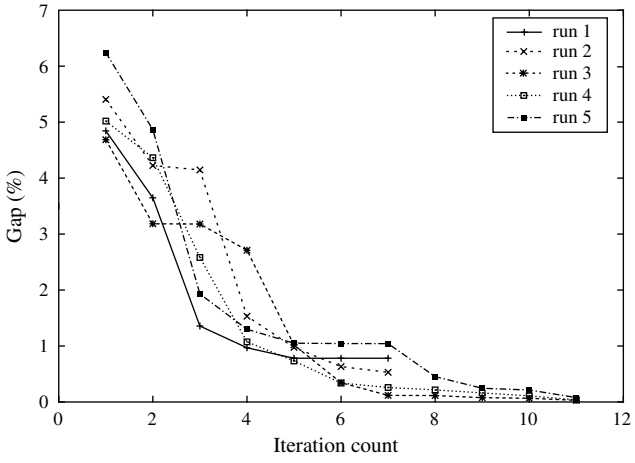


**Table 8.** Decrease in profit (%) when fixing percentages.

| Model | snet1 | snet2 | snet3 | snet4 | snet5 |
|---|---|---|---|---|---|
| SDN/SDP | 1.5 | 1.7 | 1.6 | 1.8 | 1.85 |
| MDN/MDP | 1.6 | 1.9 | 1.86 | 1.91 | 1.9 |

**Table 9.** Change in profit (%)—SD vs. MD comparison.

| Model | snet1 | snet2 | snet3 | snet4 | snet5 |
|---|---|---|---|---|---|
| SDP/MDP | 1.92 | 1.79 | 1.98 | 2.01 | 2.02 |
| SDN/MDN | 2.4 | 2.34 | 2.22 | 2.3 | 2.19 |

other hand, the decrease in profit resulting from using fixed percentages is also about 2% on average. However, it is evident from Tables 5 and 6 that reducing the size and complexity of the problem by using fixed percentages does not provide a considerable decrease in the running time or noticeable improvement of the algorithm performance. This in turn implies that it is most likely not worthwhile to use fixed percentages.

### 5.3. Comparison to Other Solvers

To evaluate our algorithm, we compared it against LOQO (Vanderbei 1997) and SNOPT (Gill et al. 1997), two well-known and highly regarded general-purpose nonlinear solvers. Recall that Table 2 describes statistics on the optimization problems.

Before we describe our comparison, we would like to expand on the philosophy and purpose of the comparison. Inherently, a direct comparison between our solver and a generic solver is both unfair and complex. Our solver was designed to scale well for the specific instance of the optimization problem MULTIREV with many degrees of freedom, e.g., 1M–18M variables for problem instances *bnet*1–5 (see Table 11), possibly at the cost of obtaining less accurate solutions. In contrast, generic solvers are typically designed for much tighter convergence requirements, possibly at the cost of scalability to large problem sizes. For example, generic solvers may generate dense matrices involving many variables and as a result will not scale well beyond a few thousand variables. The critical point we investigate in our comparisons is whether our algorithm can solve realistic instances of problem MULTIREV with modest accuracy, when generic solvers could not provide a solution with high predefined accuracy. Is it possible to lower the accuracy of generic solvers so that a fair comparison can be conducted with our scheme?

As we will see, the answer to this question appears to be no—in fact, scalability proves to be an obstacle for generic solvers regardless of desired solution accuracy. In some sense, this is hardly surprising. For example, SNOPT is not designed to handle many degrees of freedom; so-called "superbasic" variables, which our formulation will produce in large numbers, are stored in dense format.

In Table 10, we present a performance comparison between our algorithm and the two solvers on the Table 2 networks, using the SDP models that, as we have described, produce comparatively easier problem instances. The results in Table 10 describe, for each network, the average behavior on 20 random instances. AMPL was used to input the problems to both LOQO and SNOPT. We tested several settings for the control parameters for both solvers, without noticeable improvement, and the runs below reflect default settings. The LOQO runs used our heuristic solution as the starting point (which proved beneficial). The SNOPT runs used its default starting point.

In each cell of Table 10, we show the average optimality gap (in percents) followed by the average running time, in seconds. A "—" in a cell indicates that the solver failed to solve *any* of the instances corresponding to that cell. Failure occurred in one of three ways:

(a) Failure to converge in 30 minutes (run aborted).

(b) Solver could not run due to excessive memory requirements.

(c) Solver crashed (core dump).

We stress that in the case of LOQO or SNOPT, each solver had a consistent behavior with regards to failures—either all runs corresponding to a network converged, or they all failed. Also, in failures of type (a), at termination the attained solution quality was typically poor (optimality error of the order of 40%). Finally, in the case of network snet5 most of the failed runs were of type (c).

It seems clear that the general-purpose solvers cannot directly handle the larger medium-sized models. What is more, in those cases where the solvers ran successfully, our algorithm ran in significantly faster time while still providing a solution with negligible optimality error.

The primary difficulty experienced by LOQO and SNOPT clearly appears to be caused by large-scale linear algebra issues, which already arise in the Table 2 instances. Conceivably, a general-purpose solver other than LOQO or

**Table 10.** Performance comparison on real problems—average optimality gap (%) and time (sec).

| Solver | snet1 | snet2 | snet3 | snet4 | snet5 |
|---|---|---|---|---|---|
| LOQO | 0.000001/30 | 0.000001/600 | 0.000001/720 | 0.000001/900 | — |
| SNOPT | 0.000001/90 | — | — | — | — |
| Our code | 0.0008/15 | 0.006/46 | 0.018/100 | 0.021/152 | 0.023/213 |

**Table 11.** Larger problem instances.

| Net | Nodes | Links | Paths (min) | Paths (max) | Vars | Const | Nonzero |
|---|---|---|---|---|---|---|---|
| bnet1 | 100 | 500 | 3 | 6 | 745,500 | 406,000 | 5,299,700 |
| bnet2 | 150 | 900 | 3 | 7 | 1,659,000 | 889,350 | 15,340,500 |
| bnet3 | 200 | 1,500 | 3 | 9 | 4,336,500 | 1,850,100 | 40,267,500 |
| bnet4 | 250 | 2,000 | 3 | 10 | 6,746,250 | 2,852,500 | 69,618,500 |
| bnet5 | 300 | 4,500 | 4 | 12 | 18,053,700 | 5,558,700 | 311,100,300 |

SNOPT might do better. In the next section, we will address this issue from the point of view of our primary area of interest: scalability to large-network size.

### 5.4. Larger Problems

To study the performance of our algorithm in a more comprehensive fashion, we also carried out tests involving 50 data sets arising from large, dense networks. Table 11 describes the five larger-problem instances that we generated.

The numerical parameters for these runs were chosen using the same recipe as for the medium-sized problems; but because these larger problem instances were likely to be far more demanding problem instances, we only used the triple $\hat{A} = 50{,}000$, $\hat{\epsilon} = 1.5$, and $\hat{\gamma} = 0.95$—recall that these values generated the hardest instances on the medium-sized data sets. For the same reason, the performance was tested on the largest model (MDN). The triple $(\hat{A}, \hat{\gamma}, \hat{\epsilon})$ was used to generate 10 random triples $(A, \gamma, \epsilon)$ as explained above for each network. Thus, we considered a total of 50 large-problem instances.

**5.4.1. Comparing to Other Solvers on the Large Models.** Based on the comparisons given above, it is reasonable to expect that LOQO and SNOPT will be unable to directly handle the larger problems. In fact, our tests showed that the underlying computational linear algebra (for example, Cholesky factorizations) was a critical stumbling block: All runs using either solver either produced no output (even after a very long time had elapsed) or were aborted, by the solver. In some cases this resulted from a "crash," or core dump, and it is difficult to say exactly why this happened, but it appears likely that this was due to excessive memory requirements, despite having 3 GB of physical memory available.

However, hypothetically, a nonlinear solver endowed with an extremely efficient Cholesky factorization *might* be able to mount a challenge against our method. There are large numbers of nonlinear solvers (both academic and commercial codes) that are available; we would like to tackle our hypothetical question without engaging in wholesale testing of every code. Fortunately, a simple test turned out to provide a negative answer to our question.

CPLEX (1999) is known to have a very fast Cholesky factorization scheme that makes effective use of sparsity, and is thus a natural algorithm to test on the larger problems. However, CPLEX does *not* handle general nonlinear optimization problems, making a direct comparison impossible. On the other hand, CPLEX *does* solve (convex) quadratic minimization problems. To accommodate this issue, we *replaced* the nonlinear objective in MULTIREV with its second-order Taylor series approximation, evaluated at the solution computed by our heuristic, yielding, after changing the sign of the objective, a single convex quadratic program. While this negates a direct comparison with our algorithm, it makes it possible to provide a clear (negative) answer to our hypothetical question, as we will see next. Table 12 shows a running-time comparison of our algorithm (run on the nonlinear problem) with CPLEX (run on the quadratic approximation of the problem).

As we can see from this table, CPLEX's efficiency deteriorates rapidly as the network size and density increase, while our algorithm shows stable and scalable performance. From this test, we conclude that it is rather unlikely that a general-purpose second-order method will dramatically outspeed our implementation. Of course, it is conceivable that a first-order method might "work"—but we are skeptical, given the large size, nonlinear objective, and complex constraints of MULTIREV.

Table 13 illustrates the optimality gap and the running time of our algorithm on the large-problem instances.

On these problems, the number of outer iterations of our algorithm (calls to routine $\Omega$) was typically of the order of 10, while the number of inner iterations (within

**Table 12.** Average running times on large models (sec).

| Solver | bnet1 | bnet2 | bnet3 | bnet4 | bnet5 |
|---|---|---|---|---|---|
| CPLEX (QP approx) | 2,341 | 7,982 | 14,324 | 18,943 | * |
| Our code | 1,943 | 2,457 | 2,691 | 2,893 | 3,103 |

**Table 13.** Performance summary—optimality gap and running time on the large models.

| Gap (%) | bnet1 | bnet2 | bnet3 | bnet4 | bnet5 |
|---|---|---|---|---|---|
| Ave gap (%) | 2.91 | 2.99 | 3.15 | 3.26 | 3.51 |
| Min gap (%) | 1.71 | 1.82 | 1.83 | 1.93 | 1.96 |
| Max gap (%) | 4.56 | 4.87 | 4.91 | 5.1 | 5.08 |
| Ave time (sec) | 1,943 | 2,457 | 2,691 | 2,893 | 3,103 |

routine $\Omega$) was much larger than on the smaller problems—on the order of several hundred.

## 6. Conclusion

Our numerical results indicate that our approach leads to a viable algorithm that provides a high-quality solution within a reasonable time frame, and scales well with problem size. Note that the specific form of the nonlinear component of the objective function is not essential for the development of the algorithm. The idea of projecting out the capacity variables and working only with the demand or flow variables can thus be extended to different pricing models as long as the resulting objective can be formulated as a separable concave function. We expect that this approach will prove efficient for other network design problems with linear or concave objective functions.

Further, we have demonstrated that the integration of pricing and design can be efficiently accomplished, and that it should be possible to handle more detailed design models (for example, using more sophisticated protection requirements).

An interesting extension would be to incorporate stochastic components into the price/demand relationship (as in Federgruen and Heching 1999, Chen and Simchi-Levi 2002). Another would be to create a hybrid tactical-strategic network layout model that serves to capacitate networks in a medium-term setting while incorporating both economic considerations and a more accurate model of capacity systems. Some work in this direction is described in Raskina (2003).

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