# ASYMPTOTIC SHAPE OF THE ERLANG CAPACITY REGION OF A MULTI-SERVICE SHARED RESOURCE

John A. Morrison and Debasis Mitra

*Bell Laboratories, Lucent Technologies, 600 Mountain Avenue*
*Murray Hill, New Jersey 07974*

**Abstract**

We consider a loss model of an unbuffered resource having $C$ channels, which are shared by several different types of service connections. Connections of each type arrive in a Poisson stream and request a number of channels, which depends on the type. An arriving connection is blocked and lost if there are not enough free channels. Otherwise, the channels are held for the duration of the connection, and the holding period is generally distributed. It is assumed that $C$ and the traffic intensities are proportionately large. The admission control problem is considered for specified upper bounds on the blocking probabilities, and the boundary of the admissible set is investigated asymptotically. The results are derived by investigating the local behavior with respect to the tangent hyperplane at a point on the boundary of the admissible set. The lowest order results that hold in the asymptotic limit $C \to \infty$ are given first. Importantly, the boundary is linear for the key critically loaded and also for the overloaded regimes, and weakly convex for the underloaded regime. Next, refined results that hold for $C \gg 1$ are given, which indicate that the boundary is not convex, although only slighty so.

*Key words:* admissible set, asymptotics, Erlang capacity, network design, network economics

## 1 Introduction

We consider an unbuffered resource having $C$ channels, which are shared by $J$ different types of connections. Connections of type $j$ arrive in a Poisson

*Email address:* `jam@research.bell-labs.com` and `mitra@research.bell-labs.com` (John A. Morrison and Debasis Mitra).

stream with mean rate $\lambda_j$, and they require a fixed number $d_j$ of channels. An arriving connection is blocked and lost if there are fewer than $d_j$ free channels. Otherwise, $d_j$ channels are held for the duration of the connection, and the holding period is generally distributed with mean $1/\mu_j$ and is independent of earlier arrival and holding times. The traffic intensity of type $j$ connections is $\rho_j = \lambda_j/\mu_j$, and the product form and the insensitivity property hold [3], [4], [6], i.e., the joint stationary distribution of the number of active connections of each type depends on the distributions only through $\rho_j$. The admissible set in $\mathbb{R}^J$ contains all combinations of $\rho_j$, $j = 1, \ldots, J$, such that the blocking probability for each connection type satisfies specified bounds, i.e., $L_j \leq \ell_j$, $j = 1, \ldots, J$, where $L_j$ is the blocking probability for type $j$, and $\ell_j$ is a parameter. If the boundary of the admissible set is convex, then points on the tangent hyperplane to the boundary lie in the admissible set.

Mitra and Morrison [8] considered this model (as well as the finite-sources version), and they investigated the case when $C$ and the traffic intensities $\rho_j = \alpha_j C$, $j = 1, \ldots, J$, are proportionately large. They derived uniform asymptotic approximations to the blocking probabilities $L_j$, $j = 1, \ldots, J$, for type $j$ connections. The results for Poisson arrivals are obtained from the finite-sources version as a limiting case. They presented numerical results for $J = 2$ and $J = 3$ types, for the finite-sources model. These results constitute a numerical procedure but do not provide a characterization of the admissible set, nor do they resolve specific questions on the linearity and convexity of the boundary.

In this paper we consider the boundary of the admissible set in the case of Poisson arrivals. The uniform asymptotic approximations to the blocking probabilities are specialized [8] to three regimes in which their behavior is markedly different, namely, the overloaded, the critically loaded and the underloaded regimes, corresponding to $\sum_{j=1}^{J} d_j\alpha_j > 1$, $\sum_{j=1}^{J} d_j\alpha_j - 1 = O(1/\sqrt{C})$ and $\sum_{j=1}^{J} d_j\alpha_j < 1$, respectively. The shape of the admissible set is investigated separately for each of the three regimes.

The lowest order results are stated in Section 2. In the asymptotic limit $C \to \infty$, with $\rho_j = \alpha_j C$, $j = 1, \ldots, J$, the boundary of the admissible set is a hyperplane if the resource is critically loaded, which is the regime of greatest interest, or if it is overloaded. If the resource is underloaded, the boundary of the admissible set, in the limit $C \to \infty$, is convex, but not strictly so, except when $J = 2$, and it is nonlinear. For $C \gg 1$, a first order correction to the boundary of the admissible set for a critically loaded resource is also given, and the boundary is slightly nonlinear. The nature of the nonlinear boundary of the admissible set, for both an overloaded resource and a critically loaded one, is summarized in Section 3. Refined results, which pertain to $C \gg 1$, are stated in Section 4. The reader is referred to [10] and [11] for proofs of the results.

In general, the boundary of the admissible set is not convex. If $J = 2$, then the boundary is slightly convex if the resource is critically loaded, but slightly concave if the resource is overloaded. For $J = 2$, the convexity is maintained for an underloaded resource. Unfortunately, for $J \geq 3$, the boundary of the admissible set is *not* convex, although only slightly so, whether the resource is overloaded, critically loaded or underloaded. The reader should note that the above characterizations are interpretations of precise mathematical statements which are given in the paper.

Characterization of the admissible set is useful, not only for connection-level admission control, which is the context in which this topic has typically been considered in the past, but also for higher level objectives, such as network economics, network design and network control. The asymptotic view of the admissible set is particularly appropriate for the higher level objectives, where the fine details are not as important as the qualitative properties of the shape of the set and tractability of the numerical calculations for large systems. For examples of the role of the admissible set in network economics, see [2], [5], [7] and [14], and references therein. See [1] and [9] for applications to network routing and control. For recent work on loss models of optical networking see [12] and [13].

## 2 Lowest order results

Generally, we assume that

$$C \gg 1;\ \rho_j = \alpha_j C, \quad j = 1, \ldots, J, \tag{1}$$

where $C$ is an integer and $\alpha_j > 0$ is $O(1)$ and bounded away from zero. We also assume that $d_j$, $j = 1, \ldots, J$, are distinct positive integers, not large relative to $C$, and that the greatest common divisor of $d_1, \ldots, d_J$ is 1. The admissible set corresponds to

$$L_j\left(\alpha_1, \ldots, \alpha_J;\ C\right) \leq \ell_j, \quad j = 1, \ldots, J, \tag{2}$$

where the function $L_j$ gives the blocking probability for type $j$ connections. It may be shown, in all three regimes, that asymptotically

$$\frac{\partial L_j}{\partial \alpha_k} > 0, \quad j, k = 1, \ldots, J, \tag{3}$$

and the boundary of the admissible set, which depends on $C$, is expressed in the form

$$\alpha_J = \alpha_J(\boldsymbol{\alpha}; C), \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{J-1}) \,. \tag{4}$$

We first give the lowest order asymptotic results, which are obtained from the finite-sources version [8] as a limiting case. We have

**Proposition 2.1** (A). If $\ell_j$, $j = 1, \ldots, J$ is asymptotically bounded away from zero, then on the boundary of the admissible set the resource is overloaded. Let

$$a_j = (1 - \ell_j)^{1/d_j}\,, \quad j = 1, \ldots, J\,, \tag{5}$$

and, by choice of notation,

$$a_J = \max_j \, a_j \,. \tag{6}$$

Then the boundary of the admissible set satisfies

$$\sum_{j=1}^{J} d_j a_J^{d_j} \alpha_j = 1 + O\left(\frac{1}{C}\right)\,, \tag{7}$$

which in the asymptotic limit $C \to \infty$, is a hyperplane.

(B). If

$$\min_j \, \ell_j = O\left(1/\sqrt{C}\right)\,, \tag{8}$$

then on the boundary of the admissible set the resource is critically loaded. The boundary satisfies

$$\sum_{j=1}^{J} d_j \alpha_j = 1 + O\left(1/\sqrt{C}\right)\,, \tag{9}$$

which in the asymptotic limit $C \to \infty$ is a hyperplane. Note, by comparing to (7) that, for a common $C \gg 1$, the admissible set is smaller since $a_J < 1$.

(C). If at least one $\ell_j$ is exponentially small, the resource is underloaded on the boundary of the admissible set. Let

$$\ell_j = \frac{e^{-C\omega}}{\sqrt{2\pi C}}\beta_j, \quad j = 1, \ldots, J; \;\; \min_j \beta_j = 1, \quad \omega > 0\,. \tag{10}$$

Also, let $\tau > 1$ be the unique solution of

$$\sum_{j=1}^{J} \alpha_j \left( d_j \tau^{d_j} \log \tau + 1 - \tau^{d_j} \right) = \omega \,. \tag{11}$$

Then the boundary of the admissible set satisfies

$$\sum_{j=1}^{J} d_j \alpha_j \tau^{d_j} = 1 + O\left(\frac{1}{C}\right) , \tag{12}$$

which is nonlinear in view of (11). $\square$

A refinement [8] of $(B)$ in Proposition 2.1 is given by

**Proposition 2.2** Suppose that (8) holds, so that the resource is critically loaded on the boundary of the admissible set. Let

$$\kappa = \max_j \frac{d_j}{\sqrt{C}\ell_j} , \tag{13}$$

which is $O(1)$ in view of (8), and define

$$\phi(y) = \int_0^\infty e^{2yu} e^{-u^2} du \; . \tag{14}$$

Since $\phi'(y) > 0$, $-\infty < y < \infty$, it follows that $\phi(y)$ has a unique inverse, which we denote by $\phi^{-1}$. Let

$$\sigma^2 = 2 \sum_{j=1}^{J} d_j^2 \alpha_j, \quad \sigma > 0 \; . \tag{15}$$

Then the boundary of the admissible set satisfies

$$\sum_{j=1}^{J} d_j \alpha_j = 1 - \frac{\sigma}{\sqrt{C}} \phi^{-1} \left( \frac{\kappa}{\sigma} \right) + O\left( \frac{1}{C} \right) , \tag{16}$$

which is slightly nonlinear, in view of (15) and the linearity of the lowest order approximation. $\square$

We consider further the nature of the boundary of the admissible set given by (11) and (12), and that given by (16). The results are stated in Section 3.

## 3 The nonlinear boundaries

We first consider the boundary of the admissible set corresponding to (11) and (12), which pertains to an underloaded resource. We have

**Proposition 3.1** Suppose that (10) holds, so that the resource is underloaded. Let $(\boldsymbol{\alpha}^{(0)}, \alpha_J^{(0)})$, where corresponding to (4) $\alpha_J^{(0)} = \alpha_J(\boldsymbol{\alpha}^{(0)}; C)$, be a point on the boundary of the admissible set. Define

$$a_i(\tau) = d_J \tau^{d_J} \left(\tau^{d_i} - 1\right) - d_i \tau^{d_i} \left(\tau^{d_J} - 1\right) , \tag{17}$$

and let $\tau_0 > 1$ be the solution of (11) corresponding to $\alpha_j = \alpha_j^{(0)}$, $j = 1, \ldots, J$. Also, let

$$\xi_J = \left(\tau_0^{d_J} - 1\right) \left[\alpha_J - \alpha_J^{(0)} - \sum_{i=1}^{J-1} \frac{\partial \alpha_J}{\partial \alpha_i} \left(\boldsymbol{\alpha}^{(0)}\right) \left(\alpha_i - \alpha_i^{(0)}\right)\right] , \tag{18}$$

and

$$\xi_{J-1} = \sum_{i=1}^{J-1} a_i(\tau_0) \left(\alpha_i - \alpha_i^{(0)}\right) . \tag{19}$$

The tangent hyperplane to the boundary of the admissible set at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$ is given by $\xi_J = 0$. From (11) and (12), it follows that

$$\left(\tau^{d_J} - 1\right) \frac{\partial \alpha_J}{\partial \alpha_i} = -\left(\tau^{d_i} - 1\right) + O\left(\frac{1}{C}\right), \quad i = 1, \ldots, J-1 , \tag{20}$$

and hence, from (18), that

$$\xi_J = \sum_{j=1}^{J} \left(\tau_0^{d_j} - 1\right) \left(\alpha_j - \alpha_j^{(0)}\right) + O\left(\frac{1}{C}\right) . \tag{21}$$

Let

$$v_0 = \sum_{j=1}^{J} d_j^2 \alpha_j^{(0)} \tau_0^{d_j} . \tag{22}$$

If $\alpha_i - \alpha_i^{(0)} = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, \ldots, J-1$, then

$$\xi_J = \xi_{J-1}^2 \left[ \frac{1}{2v_0 \left(\tau_0^{d_J} - 1\right)^2} + O(\epsilon) \right] + O\left(\frac{\epsilon^2}{C}\right) . \tag{23}$$

Hence in the asymptotic limit $C \to \infty$, the boundary of the admissible set is convex, but not strictly so, except when $J = 2$. □

We remark that, in the limit $C \to \infty$, from (11) and (12), $\tau \equiv \tau_0$ corresponds to the intersection of two hyperplanes of dimension $J-1$, i.e., to a hyperplane of dimension $J-2$. In particular, if $J = 3$, $\tau \equiv \tau_0$ corresponds to a straight line, and the boundary of the admissible set is a ruled surface.

Next, we consider the boundary of the admissible set corresponding to (16). We have

**Proposition 3.2** Suppose that (8) holds, so that on the boundary of the admissible set the resource is critically loaded. The linear transformation of variables

$$\zeta_J = d_J \left[ \alpha_J - \alpha_J^{(0)} - \sum_{i=1}^{J-1} \frac{\partial \alpha_J}{\partial \alpha_i} \left(\boldsymbol{\alpha}^{(0)}\right) \left(\alpha_i - \alpha_i^{(0)}\right) \right] , \tag{24}$$

$$\zeta_{J-1} = \sum_{i=1}^{J-1} d_i (d_i - d_J) \left(\alpha_i - \alpha_i^{(0)}\right) , \tag{25}$$

and

$$\zeta_i = \alpha_i - \alpha_i^{(0)}, \quad i = 1, \ldots, J-2 \ (J \geq 3) , \tag{26}$$

is nonsingular, and $\zeta_J = 0$ corresponds to the tangent hyperplane to the boundary of the admissible set at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$. From (9) and (24) we have

$$\zeta_J = \sum_{j=1}^{J} d_j \left(\alpha_j - \alpha_j^{(0)}\right) + O\left(\frac{1}{\sqrt{C}}\right) . \tag{27}$$

Let

$$\sigma_0^2 = 2 \sum_{j=1}^{J} d_j^2 \alpha_j^{(0)}, \quad \sigma_0 > 0 . \tag{28}$$

If $\alpha_i - \alpha_i^{(0)} = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, \ldots, J-1$, then

$$\zeta_J = \frac{\zeta_{J-1}^2}{2\sqrt{C}} \left[P_2(\sigma_0) + O(\zeta_{J-1})\right] + O\left(\frac{\epsilon^2}{C}\right) , \tag{29}$$

where $P_2(\sigma_0) > 0$. □

If $J = 2$, then the boundary of the admissible set is convex, since the second error term in (29) is $O(\zeta_1^2/C)$. However, if $J \geq 3$, it is necessary to consider a more refined asymptotic approximation to ascertain the nature of the boundary of the admissible set when $\zeta_{J-1}$ is close to zero.

## 4 Refined results

We now consider refined approximations to the boundary of the admissible set. Corresponding to (A) in Proposition 2.1, which is for the overloaded regime, we have

**Proposition 4.1** If $\ell_j$, $j = 1, \ldots, J$ is asymptotically bounded away from zero, we consider the case when

$$C(a_J - a_i) \gg 1, \quad i = 1, \ldots, J-1 , \tag{30}$$

where $a_j$ is given by (5). Then,

(i). If $J = 2$, then $0 > d\alpha_2/d\alpha_1 = O(1)$ and $0 > d^2\alpha_2/d\alpha_1^2 = O(1/C)$, so that the boundary of the admissible set is *concave*, although only slightly.

(ii). If $J \geq 3$, the linear transformation of variables

$$\eta_J = d_J a_J^{d_J} \left[\alpha_J - \alpha_J^{(0)} - \sum_{i=1}^{J-1} \frac{\partial \alpha_J}{\partial \alpha_i}\left(\boldsymbol{\alpha}^{(0)}\right)\left(\alpha_i - \alpha_i^{(0)}\right)\right] , \tag{31}$$

$$\eta_{J-1} = \sum_{i=1}^{J-1} d_i(d_i - d_J) a_J^{d_i} \left(\alpha_i - \alpha_i^{(0)}\right) \tag{32}$$

$$\eta_{J-2} = \sum_{i=1}^{J-1} d_i \left(d_i^2 - d_J^2\right) a_J^{d_i} \left(\alpha_i - \alpha_i^{(0)}\right) , \tag{33}$$

and

$$\eta_i = \alpha_i - \alpha_i^{(0)}, \quad i = 1, \ldots, J-3 \ (J \geq 4) , \tag{34}$$

is nonsingular, and $\eta_J = 0$ corresponds to the tangent hyperplane to the boundary of the admissible set at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$. From (7) and (31) we have

$$\eta_J = \sum_{j=1}^{J} d_j a_J^{d_j} \left(\alpha_j - \alpha_j^{(0)}\right) + O\left(\frac{1}{C}\right) . \tag{35}$$

If $\alpha_i - \alpha_i^{(0)} = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, \ldots, J-1$, then

$$\eta_J = \frac{1}{2C\left(\sum_{j=1}^{J} \alpha_j^{(0)} d_j^2 a_J^{d_j}\right)^2} \left[\frac{\sum_{j=1}^{J} \alpha_j^{(0)} d_j^3 a_J^{d_j}}{\sum_{j=1}^{J} \alpha_j^{(0)} d_j^2 a_J^{d_j}} \eta_{J-2}^2 - \eta_{J-1}\eta_{J-2}\right] \tag{36}$$

$$+ O\left(\frac{\epsilon^2}{C^2}\right) + O\left(\frac{\epsilon^3}{C}\right) .$$

Hence, asymptotically, the boundary of the admissible set is *not* convex, although only slightly so. □

Next, corresponding to Proposition 3.2, we have

**Proposition 4.2** If (8) holds, then we consider the case when

$$\frac{d_J}{\sqrt{C}\,\ell_J} - \frac{d_i}{\sqrt{C}\,\ell_i} \gg \frac{1}{\sqrt{C}}, \quad i = 1, \ldots, J-1 . \tag{37}$$

If $J \geq 3$, the linear transformation of variables given by (24), (25) and

$$\zeta_{J-2} = \sum_{i=1}^{J-1} d_i \left(d_i^2 - d_J^2\right) \left(\alpha_i - \alpha_i^{(0)}\right) , \tag{38}$$

and

$$\zeta_i = \alpha_i - \alpha_i^{(0)}, \quad i = 1, \ldots, J-3 \ (J \geq 4) , \tag{39}$$

is nonsingular and, as before, $\zeta_J = 0$ corresponds to the tangent hyperplane to the boundary of the admissible set at $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)}$. If in (25)

$$\zeta_{J-1} = Z_{J-1}/\sqrt{C} , \tag{40}$$

where $Z_{J-1} = O(1)$, then

$$\zeta_J = \frac{1}{2C^{3/2}} \left[ P_2(\sigma_0) Z_{J-1}^2 + 2R_2(\sigma_0) Z_{J-1} \zeta_{J-2} + T_2(\sigma_0) \zeta_{J-2}^2 \right] + O\left(\frac{1}{C^2}\right) . \quad (41)$$

It is shown [10] that the discriminant of the quadratic form in (41) is negative. Hence, asymptotically, the boundary of the admissible set is *not* convex, although only very slightly so. □

We note that in this result we do not need $\alpha_i - \alpha_i^{(0)} = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, \ldots, J-1$.

Finally, corresponding to Proposition 3.1, we have

**Proposition 4.3** If (10) holds, we let $\tau_0 > 1$ be the solution of (11) corresponding to $\alpha_j = \alpha_j^{(0)}$, $j = 1, \ldots, J$, and define

$$B(\tau) = \min_j \left[ \frac{\beta_j (\tau - 1)}{(\tau^{d_j} - 1)} \right], \quad \tau > 1 . \quad (42)$$

In general, $B(\tau)$ is only piecewise differentiable. We consider the case when $B(\tau)$ is differentiable at $\tau = \tau_0$. For $J \geq 3$, with $a_i(\tau)$ given by (17), the linear transformation of variables (18), (19) and

$$\xi_{J-2} = \sum_{i=1}^{J-1} a_i'(\tau_0) \left( \alpha_i - \alpha_i^{(0)} \right) , \quad (43)$$

where the prime denotes derivative, and

$$\xi_i = \alpha_i - \alpha_i^{(0)}, \quad i = 1, \ldots, J-3 \ (J \geq 4) , \quad (44)$$

is nonsingular. If $\alpha_i - \alpha_i^{(0)} = O(\epsilon)$, $0 < \epsilon \ll 1$, $i = 1, \ldots, J-1$, then

$$\begin{aligned} \xi_J = \xi_{J-1}^2 & \left[ \frac{1}{2v_0 \left( \tau_0^{d_J} - 1 \right)^2} + O(\epsilon) + O\left( \frac{1}{C} \right) \right] \quad (45) \\ & - \frac{\tau_0^2 \xi_{J-2}^2}{4Cv_0^2 \left( \tau_0^{d_J} - 1 \right)^2} + O\left( \frac{\epsilon}{C} \xi_{J-1} \right) + O\left( \frac{\epsilon^2}{C^2} \right) . \end{aligned}$$

Hence, asymptotically, the boundary of the admissible set is *not* convex, although only slightly so. □

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