Models of network access using feedback fluid queues

Michel Mandjes [*,†,**] — michel@cwi.nl
Debasis Mitra [*] — mitra@lucent.com
Werner Scheinhardt [†,**] — werner@math.utwente.nl

[*] Bell Laboratories/ Lucent Technologies, P.O. Box 636, Murray Hill, NJ 07974, United States

[†] Dept. of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, the Netherlands

[**] CWI, Kruislaan 413, P.O. Box 94079, 1090 GB Amsterdam, the Netherlands

Abstract

At the access to networks, in contrast to the core, distances and feedback delays, as well as link capacities are small, which has network engineering implications that are investigated in this paper. We consider a single point in the access network which multiplexes several bursty users. The users adapt their sending rates based on feedback from the access multiplexer. Important parameters are the user's peak transmission rate p, which is the access line speed, the user's guaranteed minimum rate r, and the bound ϵ on the fraction of lost data.

Two feedback schemes are proposed. In both schemes the users are allowed to send at rate p if the system is relatively lightly loaded, at rate r during periods of congestion, and at a rate between r and p, in an intermediate region. For both feedback schemes we present an exact analysis, under the assumption that the users' file sizes and think times have exponential distributions. We use our techniques to design the schemes jointly with admission control, i.e., the selection of the number of admissible users, to maximize throughput for given p, r, and ϵ. Next we consider the case in which the number of users is large. Under a specific scaling, we derive explicit large deviations asymptotics for both models. We discuss the extension to general distributions of user data and think times.

Key words—access network, feedback regulation, fluid models, spectral expansion, many-sources scaling, large deviations.

AMS classification—Primary 60K25; secondary 68M20

1 Introduction

In today's communication networks, design and control of the network core and access are different, primarily because of the differences in scale in bandwidth and distance. Quite often the bottleneck is the access, rather than the core. This may happen because the access network is characterized by relatively low line speeds and the limited ability of users to buffer and shape traffic (think of the extreme case of a user with a wireless handset). Access control, supported by the use of feedback, is an important mechanism for dealing with this problem. Since distances between users/clients and network access points are relatively short, feedback delay due to propagation is negligible, which contributes to the efficacy of feedback control. In this paper we investigate the problems of access control by introducing some models and techniques for their evaluation, design and performance optimization.

We make three main contributions. First, we present two simple models of network access. The models provide a framework for the joint design of the adaptation of the source rates and admission control in the feedback-based schemes. Second, we show how to compute the stationary behavior of the aforementioned feedback queues. We illustrate the use of these techniques to solve the design problem. Finally, we show how to use the theory of large deviations to obtain explicit results when the system and the number of sources are large.

In our model each user alternates between 'on' periods of transmission, and 'off' periods or 'think times'. The user model here differs from the familiar on-off source models, e.g. [2], in that *file sizes* (where a file size is the amount of data the source transmits during an active period) are independent, identically distributed (iid) random variables, but the *on periods* are not specified *a priori.* The on-periods are determined by the combination of the file sizes and the transmission rates allocated by the access control scheme, to be described below, which depend on the interaction of the multiplexer with the collective behavior of users. In contrast, the think times are iid random variables. The lengths of on periods and the throughputs of the individual users are key performance quantities to be obtained from an analysis of the model.

The access line speed (p for each of the users) is typically small compared to the output rate of the access multiplexer, and therefore constitutes an important constraint.

Another model feature is the minimum throughput rate (r) that is guaranteed to the users. In a number of applications clients derive zero utility if the throughput is below a threshold. This point has been made by Massoulié and Roberts [22] for the case of TCP traffic, in which performance collapse may ensue. As soon as the notion of a minimum guaranteed rate is introduced, admission control needs to be considered, together with the calculation of the capacity of the network.

We present two schemes for feedback-based adaptation of the users' rates. In both schemes users are allowed to transmit at rate p if the system is relatively lightly loaded, at rate r during periods of congestion, and at a rate between r and p, which is determined by the processor sharing discipline, in an intermediate region. In our model the feedback signal from the buffer to the sources is assumed to arrive with negligible delay, which is reasonable when the round trip distances are small. In the first scheme the feedback is based on the number of active users and whether the queue is empty or not. The second scheme utilizes a threshold B^* on the buffer content. Depending on whether the buffer content is less than, greater than, or equal to B^*, the user rate is p, r, or determined by the processor sharing discipline, respectively. The analysis of the first queue is simpler than that of the second queue and our results for it are in closed form. However, as discussed below, the combined use of a buffer threshold with a limit on the number of users in the second model provides a substantially greater facility for controlling important performance trade-offs.

We note a feature of the behavior of our models which may initially appear to be counterintuitive: the effect of feedback is to *increase* the mean length of the period that the buffer is empty, while simultaneously *increasing* the throughputs of the users. The explanation is that the rates allocated to the users are *higher* during the periods that the buffer is empty (namely p, or between p and the lower rate r, whereas in the system without feedback all sources are allowed to send at rate r). Hence feedback has the effect of reducing the on periods and consequently the cycle time of each user, and thereby it increases individual users' throughput and the system throughput, which is defined as the sum of the individual users' throughputs.

Next consider the role of the feedback parameter B^*, the threshold level in the buffer content in the second model. By reasoning as above, we infer that increasing B^* has the effect of increasing the throughput of the users. However, this is at the cost of increasing the probability of buffer overflow. Similarly, with all other parameters held fixed, increasing the number of users N has the effect of increasing both the total system throughput as well as the probability of buffer overflow. Hence, this model allows the study of interesting trade-offs between several important quantities, including the individual users' throughput, the system throughput, the loss probability and the number of users. The trade-offs may be regulated by properly selecting the parameters B* and N. We note parenthetically that by making the trade-off such that the loss rate is low, the error we make by not taking into account the delay due to losses (e.g. retransmissions) is typically small.

There are several possible frameworks in which the trade-offs may be studied and quantified. For instance, we may require that the fraction of source data that is lost does not exceed a given QoS parameter ϵ. In the first design problem N and ϵ are given, and the objective is to obtain the value of the buffer threshold B^* that maximizes the throughput. In the second, more general, problem the objective is the joint design of the feedback and admission controls, i.e., the selection of the parameters B^* and N such that the system throughput is maximized. The numerical procedure that is developed in this paper allows such design questions to be addressed. Indeed one of the highlights of the numerical results that we present later in the paper is the computed solution for an instance of the above design problem.

The two proposed feedback schemes both fall into the category of feedback fluid queues, which were introduced in [28] as generalizations of the well-known Markov-modulated fluid models in e.g. [2], [17]. In the latter, a fluid buffer receives or depletes fluid at rate r_i (positive or negative) at times when a background continuous-time Markov chain is in state i. Typically, stochastic fluid models are characterized by the generator (Q) of the background process and a diagonal rate matrix (R) which contains all fluid rates r_i. In feedback fluid queues most of the above remains true, except that the behavior of the background process (i.e., the matrix Q), and possibly the matrix R as well, depends on the current buffer content. As a result, the background process is no longer an autonomous Markov process. In this paper we confine ourselves to feedback fluid queues in which Q and R are piecewise constant functions of the buffer content level, see also [29]. Notice a crucial difference from [11] in which there are thresholds, as in this paper, but where the thresholds affect R, but not Q. The analysis of our feedback fluid queues is one of the key contributions of this paper. The analysis is based on spectral expansions, which have been used previously in other contexts [2], [10], [11], [17].

An intrinsic drawback of the analytical approach described so far is that it is computationally intensive. This provides motivation for simpler, asymptotic approaches to deal with the access models. In both access models, we study the large-deviations asymptotics for the scaling introduced by Weiss [33], i.e. the regime in which the number of users grows large and resources (buffer and bandwidth) scale proportionally. We derive 'exponential approximations', comparable to those in [3], [4], [9] for the ordinary FIFO discipline. Exponential approximations of the first feedback model, i.e., without the threshold, were obtained earlier by Ramanan and Weiss [25] for exponential file sizes and think times. Our major contribution is that these results are explicit and the computations are simple. Some of these results have nice insensitivity properties, i.e., depend on the distributions of the think times and file sizes only through their means.

The role of feedback in packet networks has a long history, see [30, Ch. 7] for a review. Recently there has been a resurgence of interest, driven in part by work on explicit congestion notification (ECN) marking schemes by Gibbens and Kelly [12], [13], [16], and others. In [24] a feedback model with feedback delays is considered based on the marking scheme of [10]. However, the feedback considered there regulates the marking of packets to be dropped later on, whereas in our model the feedback regulates the actual source rate. The processor sharing discipline has been highlighted in recent work on QoS delivery by Roberts *et al.* [26], [27], albeit in the bufferless framework.

The organization of the paper is as follows. Section 2 deals with the first access model. Section 3 considers the more advanced feedback scheme that utilizes a threshold. Section 4 shows how the mathematical analysis can be extended to the more general feedback fluid queue with multiple thresholds and a (possibly) finite buffer. The large deviations asymptotics are described in Section 5. Finally Section 6 reports on several numerical examples.

2 Feedback model without threshold

2.1 Model

We model a single aggregation or multiplexing point in the network. The output trunk speed is C, and there are N access links. The peak (or line) rate of each access link is p. The minimum rate given to each access user is r. We make the simplifying modeling assumption that traffic can be considered to be continuous fluid. For the benefit of the reader, we note that in fluid models there may be considerable transfer of fluid, i.e., high throughput, even during periods when the buffer is empty. This is because in fluid models, once the buffer is empty, it remains empty for as long as the total input rate does not exceed the output capacity.

In this section we describe the scheme without threshold. When the multiplexer buffer is empty, the access rate will be determined by dividing equally the trunk rate between the (small) number of active users, truncated to p. At the other extreme, when the number of active users exceeds a critical number, N'' in this work, the fair share of the trunk rate for an active user would drop below the guaranteed rate of r. In this case each user retains an access rate r and consequently the buffer starts filling up. As long as the buffer content is positive, each source is assigned the rate of just r. Clearly, N'' is the largest integer not exceeding C/r.

While the buffer is empty and the number of active users is small, say below a critical number N', then each active user transmits at peak access rate p. It is easily seen that N' is the largest integer not exceeding C/p. When the number of active users is between N' and N'', which is the middle range, and the buffer is empty, then the trunk speed is shared equally, i.e., the processor sharing regime holds. Hence, when the buffer is empty there are two regimes characterized respectively by the access line speed and the processor sharing rate. In contrast, when the buffer is not empty, there is a unique transmission rate, namely r, the guaranteed minimum.

Table 1 summarizes the feedback protocol. Let $Y(t)$ denote the number of active sources at time t, and $W(t)$ the buffer content at time t. In the table the allocated rate, as well as the sign of the 'drift' of the buffer content are given, as functions of $Y(t)$ and $W(t)$.

Number of active sources	$W(t) = 0$	$W(t) > 0$
$0 \leq Y(t) \leq N'$	p (0)	r (−)
$N' < Y(t) \leq N''$	$C/Y(t)$ (0)	r (−)
$N'' < Y(t) \leq N$	NA	r (+)

Table 1: *Allowed user transmission rates (and sign of the buffer drift), as functions of number of active sources and buffer content.*

An important aspect of our model is the behavior of the homogeneous sources. Each source alternates between activity ('on') and inactivity ('off'). The inactivity periods are independent, exponentially distributed random variables with mean λ^{-1}. Each source transmits a file during its activity period, whose size (in bits) is independent of everything else and exponentially distributed with mean μ^{-1}. The length of the induced activity period is not given *a priori*, since this depends on the rate(s) at which the file is transmitted. An important parameter for the Quality of Service is ϵ, which the buffer overflow probability is required not to exceed.

We follow a conventional approach in inferring finite-buffer performance from an infinite-buffer model. Since the long run fraction of time a source is on, given that the buffer level is large, is $\lambda/(\lambda + \mu r)$ (note that $(\mu r)^{-1}$ is the maximum time to transmit a file of size μ^{-1}), the stability condition of this infinite-buffer model is given by

$$N \frac{\lambda r}{\lambda + \mu r} < C. \tag{1}$$

In practice there is a need to prevent N from being too large. One of the objectives of the analysis is to calculate the capacity of our system, which is the largest value of N such that the overflow probability of the queue does not exceed ϵ. In particular, the number $N_r(\epsilon)$ of connections admissible as a function of ϵ and the guaranteed rate r, is an important design parameter. We return to the calculation of this quantity in Section 6.3.

2.2 Preliminary results

To analyze the model described in Section 2.1, we first review some results from the model *without* feedback. Anick, Mitra, and Sondhi [2] consider the model in which the sources transmit at a constant rate, say r, while in the on-state, i.e., the allowable transmission rate does not depend on current occupancy of the system. Detailed results on this model are available [2, 10, 17].

Buffer content distribution. For this case without feedback we denote the state of the background process (i.e. the number of sources that are transmitting) and the buffer content process at time t as $X(t)$ and $V(t)$ respectively. In [2] the stationary distribution of the content process is given. It is computed as follows. Clearly $X(\cdot)$ constitutes a continuous-time Markov chain on the state space $\{0,\ldots,N\}$. The (i,j)th element $(i \neq j)$ of its generator Q is given by

$$Q(i,j) := \begin{cases} (N-i)\lambda & \text{if } j = i+1, \\ ir\mu & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal elements $(i = j)$ are such that the rowsums are zero. Element (i,j) represents the probability flux of the continuous-time Markov chain from state i to state j. Define by R the diagonal matrix $\mathrm{diag}\{r_0,\ldots,r_N\}$ with r_i the net input rate if there are i sources in the on state, i.e., $r_i = ir - C$.
To find the stationary buffer content distribution, we first define

$$F_i(x) := \mathbb{P}(X = i, V \leq x).$$

It is not hard to show that the vector $\mathbf{F}(\cdot) := (F_0(\cdot),\ldots,F_N(\cdot))$ satisfies $\mathbf{F}'(x)R = \mathbf{F}(x)Q$. The spectral expansion of the solution is given by

$$\mathbf{F}(x) = \sum_{j=0}^{N} a_j \mathbf{v}_j \exp\left[z_j x\right], \tag{2}$$

where the a_j are coefficients determined later, and $(z_j, \mathbf{v}_j)$ is an eigenvalue-eigenvector pair, i.e., obtained from $z_j \mathbf{v}_j R = \mathbf{v}_j Q$.
The coefficients a_j are calculated as follows. Define D, the set of states with downward drifts, by all states i such that $ir < C$, and U, the set of states with upward drifts, by all other states. Assume that there is no state i such that $ir = C$. Let $Q_{DD}, Q_{DU}, Q_{UD}, Q_{UU}$ be the submatrices that are obtained by partitioning Q. The vector $\mathbf{F}_D(x)$ consists of the $F_i(x)$ with $i \in D$; $\mathbf{F}_U(x)$ is defined analogously. It is easily seen that $a_j = 0$ if $\mathrm{Re}(z_j) > 0$, as the distribution should range between 0 and 1. The remaining a_j follow from $\mathbf{F}_U(0) = \mathbf{0}$. It turns out that there are just as many unknowns as equations. Clearly, $\mathbb{P}(V \leq x) = \sum_i F_i(x)$.

Idle and busy periods. Elwalid and Mitra [10] give explicit expressions for a number of quantities that are related to the busy and idle periods of the queue. A busy period is defined as a period in which the buffer content is positive, whereas an idle period is a period in which the buffer is empty. It is easily seen that at the beginning of a busy period the number of sources in the on-state is equal to $N''+1$; at the end of the busy period the number of sources in the on state is in D. The lengths of consecutive busy and idle periods are independent.

Denote by $\mathbf{P}$ the distribution of $Y(t)$ at the end of the busy period. Then it is not hard to prove that

$$\mathbf{P} = \frac{1}{\langle \mathbf{F}_D(0)Q_{DD}, \mathbf{1}\rangle}\mathbf{F}_D(0)Q_{DD}, \tag{3}$$

see Equation 5.9 of [10]; $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product of two vectors. The mean idle period $\mathbb{E}I$ is given by

$$\mathbb{E}I = -\frac{\sum_{i\in D} F_i(0)}{\langle \mathbf{F}_D(0)Q_{DD}, \mathbf{1}\rangle}.$$

Finally, the mean busy period $\mathbb{E}B$ can be calculated using $\sum_{i\in D} F_i(0) = \mathbb{E}I/(\mathbb{E}I+\mathbb{E}B)$:

$$\mathbb{E}B = \mathbb{E}I \cdot \frac{1-\sum_{i\in D} F_i(0)}{\sum_{i\in D} F_i(0)}. \tag{4}$$

2.3 Analysis

The model analyzed in this section has been described in Section 2.1. Recall that $Y(t) \in \{0,\dots,N\}$ denotes the number of sources that are transmitting at time t in the feedback model of Section 2.1. Notice that this does *not* constitute a Markov chain, unlike $X(t)$ in Section 2.2. This is because the sojourn times and transition probabilities depend on the amount of fluid stored in the buffer. However, as long as the buffer is empty, $Y(t)$ behaves as a continuous-time Markov chain.

Denote the stationary buffer content distribution in the feedback model by $\mathbb{P}(W < x)$. A busy period in this model is distributed as the random variable B', and an idle period as I'. The sequence of busy periods is i.i.d., and so is the sequence of idle periods, as can be seen easily. The distribution of $Y(t)$ at the end of the busy period is denoted by (the vector) P'. The next lemma links B' and P' to the corresponding quantities in the model without feedback.

Lemma 2.1 *Busy periods B and B' have the same distribution. Also, the distributions P and P' are identical.*

Proof. Both in the models with (Section 2.1) and without (Section 2.2) feedback, during busy periods on-periods terminate at a rate $ir\mu$ when there are i sources in their on-state. In both models the busy period starts when there are $N''+1 = \lceil C/r\rceil$ sources transmitting. Hence, the buffer dynamics in both models have the same probabilistic properties during a busy period. This immediately implies both assertions. □

Corollary 2.2 *With the same argument as in the proof of the previous lemma, we find*

$$\mathbb{P}(W \leq x | W > 0) = \mathbb{P}(V \leq x | V > 0), \;\; x > 0.$$

This immediately implies that $\mathbb{P}(W \leq x)$ equals

$$\mathbb{P}(V \leq x | V > 0)\mathbb{P}(W > 0) + \mathbb{P}(W = 0) =$$

$$\frac{\sum_i F_i(x) - \sum_i F_i(0)}{1 - \sum_i F_i(0)} \mathbb{P}(W > 0) + \mathbb{P}(W = 0).$$

As $\mathbb{P}(W = 0) = 1 - \mathbb{P}(W > 0)$, the only quantity that is left to compute is the probability of an empty buffer in our feedback model. This is given by

$$\mathbb{P}(W = 0) = \frac{\mathbb{E}I'}{\mathbb{E}I' + \mathbb{E}B'} = \frac{\mathbb{E}I'}{\mathbb{E}I' + \mathbb{E}B},$$

where we apply Lemma 2.1 and use the fact that the process regenerates at the start of each busy period. As we know $\mathbb{E}B$ from Section 2.2, we only have to find $\mathbb{E}I'$. This will be done in the next lemma, but first we introduce some required notation.

Q'_{DD} is a square matrix of dimension $N'' + 1$. For $i \neq j$:

$$Q'_{DD}(i,j) := \begin{cases} (N-i)\lambda & \text{if } j = i+1, \\ ip\mu & \text{if } j = i-1, \\ 0 & \text{otherwise,} \end{cases}$$

if $i \leq N'$, and

$$Q'_{DD}(i,j) := \begin{cases} (N-i)\lambda & \text{if } j = i+1, \\ C\mu & \text{if } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

if i is between $N' + 1$ and N''. The diagonal elements are such that the rowsums are zero, except for $Q'_{DD}(N'', N'')$, which equals $-C\mu - (N - N'')\lambda$. Notice that, as long as the buffer is empty, $Y(t)$ is a Markov chain which obeys the transition rates of Q'_{DD}.

Lemma 2.3 *With* $\mathbf{P}$ *is defined in (3), the mean idle time in the feedback-based model is given by*

$$\mathbb{E}I' = \langle -\mathbf{P}(Q'_{DD})^{-1}, \mathbf{1} \rangle. \tag{5}$$

Proof. This follows directly from standard results of mean passage times [15]. It says that the mean time spent by $Y(t)$ in j before the set D is left, given that the process starts in i, is given by the (i,j) entry of $-(Q'_{DD})^{-1}$.

Then the reasoning is analogous to Equations (5.11) and (5.14) of [10], as follows. The vector $-\mathbf{P}(Q'_{DD})^{-1}$ gives the mean time spent in all states in D during an idle period of the buffer, applying Lemma 2.1. The sum of its entries is the mean length of the idle period. □

We arrive at the following result for the buffer content distribution in the feedback-based model. A similar proportionality result was found, although not explicitly mentioned, in Adan *et al.* [1], where another feedback fluid queue is analyzed that has two types of behavior depending on whether the buffer is empty or not.

Theorem 2.4 *In the feedback-based model, the stationary buffer content distribution is given by*

$$\mathbb{P}(W \leq x) = \frac{\sum_i F_i(x) - \sum_i F_i(0)}{1 - \sum_i F_i(0)} \frac{\mathbb{E}B}{\mathbb{E}I' + \mathbb{E}B} + \frac{\mathbb{E}I'}{\mathbb{E}I' + \mathbb{E}B},$$

where the vector $\mathbf{F}(x)$ *is given by (2),* $\mathbb{E}B$ *by (4), and* $\mathbb{E}I'$ *by (5). Equivalently,*

$$\mathbb{P}(W > x) = \mathbb{P}(V > x) \frac{\mathbb{E}I + \mathbb{E}B}{\mathbb{E}I' + \mathbb{E}B}.$$

An important interpretation of the above theorem is the following: the gain with respect to the model without feedback (where the sources always send at rate r when active) is expressed by the term

$$\frac{\mathbb{E}I + \mathbb{E}B}{\mathbb{E}I' + \mathbb{E}B} \leq 1. \tag{6}$$

The fact that this ratio is less than 1 is due to the fact that $\mathbb{E}I' \geq \mathbb{E}I$, which can be understood as follows. Clearly, I' can be interpreted as a first entrance time in the birth-death process with generator Q'_{DD}, namely as the first entrance time to state $N'' + 1$, starting from a state $i \leq N''$ that is drawn from the distribution P'. Similarly I is the corresponding entrance time in the birth-death process with generator Q, due to the fact that $P = P'$. Since the death rates in Q'_{DD} are larger than those in Q, while the birth rates are equal, it follows that $\mathbb{E}I' \geq \mathbb{E}I$.
In many situations, particularly when the number of sources is large, $\mathbb{E}B$ will be much smaller than $\mathbb{E}I$ and $\mathbb{E}I'$. In that case (6) is well approximated by $\mathbb{E}I \cdot (\mathbb{E}I')^{-1}$. That is, the ratio of mean idle times of the buffer in the models without and with feedback effectively quantifies the performance gain from feedback.

3 Feedback model with threshold

3.1 Model

In this section we consider a generalization of the feedback model presented in Section 2.1. As before, $Y(t)$ is the number of active users at time t (with state space $\{0, \ldots, N\}$) and $W(t)$ is the buffer content at time t. We now introduce a threshold level $B^* > 0$ such that the sources are allowed to send at peak rate p as long as $W(t) < B^*$. When $W(t) > B^*$ the sources are allowed to transmit data only at the guaranteed rate r. When $W(t) = B^*$ the processor sharing policy applies. In that case the buffer content process will 'stick' at level B^* for as long as the number of active users lies in the set $\{N' + 1, \ldots, N''\}$. The algorithm is summarized in Table 2. Notice that the model described in Section 2.1 is a limiting case of this model, as it is obtained by letting $B^* \downarrow 0$. We will describe the exact solution of the stationary buffer content distribution.

Number of active sources	$W(t) < B^*$	$W(t) = B^*$	$W(t) > B^*$
$0 \leq Y(t) \leq N'$	$p \quad (-)$	NA	$r \quad (-)$
$N' < Y(t) \leq N''$	$p \quad (+)$	$C/Y(t) \quad (0)$	$r \quad (-)$
$N'' < Y(t) \leq N$	$p \quad (+)$	NA	$r \quad (+)$

Table 2: *Allowed user transmission rates (and sign of the buffer drift), as functions of number of active sources and buffer content.*

Notice that the stability condition for this model is the same as (1). Define the matrix $Q^{(r)}$ to be Q (as defined in Section 2.2); generator $Q^{(p)}$ is also defined as Q, but with rate r replaced by rate p. Finally, $Q^{(*)}$ is similarly defined, but now $Q(i, i-1) = C\mu$. The idea is that $Y(t)$ behaves like a Markov chain with generator $Q^{(p)}, Q^{(*)}, Q^{(r)}$, whenever the buffer content process $W(t)$ is respectively below, at, or above level B^*. Furthermore $R^{(r)}$ is defined as in Section 2.2, i.e. as a diagonal matrix of dimension $N+1$, its ith diagonal element being given by $ir - C$. $R^{(p)}$ is similarly defined, except that its ith diagonal element is given by $ip - C$. The entries in these matrices are the net fluid rates into the buffer for the guaranteed rate ($W(t) > B^*$) and peak rate ($W(t) < B^*$) regimes respectively.

3.2 Analysis

Our purpose is to find the joint distribution $G_i(x)$ defined as $G_i(x) = \lim_{t\to\infty} \mathbb{P}(Y(t) = i, W(t) \leq x)$. To do this we determine the Kolmogorov forward equations for

$$\begin{aligned} G_i^{(p)}(t,x) &:= \mathbb{P}(Y(t) = i, W(t) \leq x), \quad 0 \leq x < B^*; \\ G_i^{(r)}(t,x) &:= \mathbb{P}(Y(t) = i, W(t) \leq x), \quad x \geq B^*. \end{aligned}$$

For $x < B^*$,

$$\begin{aligned} G_i^{(p)}(t+h,x) &= (1 - h\,(Q^{(p)}(i,i-1) + Q^{(p)}(i,i+1)))\, G_i^{(p)}(t, x - hr_i^{(p)}) \\ &\quad + h\, Q^{(p)}(i-1,i)\, G_{i-1}^{(p)}(t,x) + \; h\, Q^{(p)}(i+1,i)\, G_{i+1}^{(p)}(t,x) + o(h), \end{aligned}$$

cf. [2]. By taking $h \downarrow 0$ we find, in matrix form,

$$\frac{\partial}{\partial t}\mathbf{G}^{(p)}(t,x) + \frac{\partial}{\partial x}\mathbf{G}^{(p)}(t,x)R^{(p)} = \mathbf{G}^{(p)}(t,x)Q^{(p)},$$

where $\mathbf{G}^{(p)}(t,x)$ is an N-dimensional row vector.
However, for $x > B^*$ the Kolmogorov equations take a less simple form. With $G_i^{(p)}(t, B^*-) :=$

$\lim_{x\uparrow B^*} G_i^{(p)}(t,x)$, we find that

$$
\begin{aligned}
G_i^{(r)}(t+h,x) \;=\; & G_i^{(r)}(t,x-hr_i^{(r)}) \\
& -h\left(Q^{(r)}(i,i-1)+Q^{(r)}(i,i+1)\right)\left(G_i^{(r)}(t,x-hr_i^{(r)})-G_i^{(r)}(t,B^*)\right) \\
& -h\left(Q^{(*)}(i,i-1)+Q^{(*)}(i,i+1)\right)\left(G_i^{(r)}(t,B^*)-G_i^{(p)}(t,B^*-)\right) \\
& -h\left(Q^{(p)}(i,i-1)+Q^{(p)}(i,i+1)\right)\;G_i^{(p)}(t,B^*-) \\
& +h\,Q^{(r)}(i-1,i)\left(G_{i-1}^{(r)}(t,x)-G_{i-1}^{(r)}(t,B^*)\right)+h\,Q^{(r)}(i+1,i)\left(G_{i+1}^{(r)}(t,x)-G_{i+1}^{(r)}(t,B^*)\right) \\
& +h\,Q^{(*)}(i-1,i)\left(G_{i-1}^{(r)}(t,B^*)-G_{i-1}^{(p)}(t,B^*-)\right)+h\,Q^{(*)}(i+1,i)\left(G_{i+1}^{(r)}(t,B^*)-G_{i+1}^{(p)}(t,B^*-)\right) \\
& +h\,Q^{(p)}(i-1,i)\;G_{i-1}^{(p)}(t,B^*-)\;+\;h\,Q^{(p)}(i+1,i)\,G_{i+1}^{(p)}(t,B^*-)\;+\;o(h).
\end{aligned}
$$

After letting $h\to 0$, this leads to

$$
\begin{aligned}
& \frac{\partial}{\partial t}\mathbf{G}^{(r)}(t,x) \quad + \quad \frac{\partial}{\partial x}\mathbf{G}^{(r)}(t,x)\,R^{(r)} = \\
& \qquad (\mathbf{G}^{(r)}(t,x)-\mathbf{G}^{(r)}(t,B^*))\,Q^{(r)} + \\
& \qquad (\mathbf{G}^{(r)}(t,B^*)-\mathbf{G}^{(p)}(t,B^*-))\,Q^{(*)} + \mathbf{G}^{(p)}(t,B^*-)\,Q^{(p)}.
\end{aligned}
$$

Assuming stationarity, we now set $G_i^{(r)}(t,x)\equiv G_i^{(r)}(x)$ and $G_i^{(p)}(t,x)\equiv G_i^{(p)}(x)$ for $i=0,\ldots,N$, and take all derivatives with respect to t equal to 0. In matrix form,

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{G}^{(p)}(x)R^{(p)} = \mathbf{G}^{(p)}(x)\,Q^{(p)}, \;\text{ and} \tag{7}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{G}^{(r)}(x)\,R^{(r)} = (\mathbf{G}^{(r)}(x)-\mathbf{G}^{(r)}(B^*))\,Q^{(r)} + \;(\mathbf{G}^{(r)}(B^*)-\mathbf{G}^{(p)}(B^*-))\,Q^{(*)} + \;\mathbf{G}^{(p)}(B^*-)\,Q^{(p)}. \tag{8}
$$

Solving (7) is immediate and leads to

$$
\mathbf{G}^{(p)}(x) = \sum_{j=0}^{N} a_j^{(p)}\mathbf{v}_j^{(p)}\exp\left[z_j^{(p)}x\right], \tag{9}
$$

where $(z_j^{(p)},\mathbf{v}_j^{(p)})$ is a pair satisfying $z_j^{(p)}\mathbf{v}_j^{(p)}R^{(p)} = \mathbf{v}_j^{(p)}Q^{(p)}$, and the $a_j^{(p)}$ are coefficients. The solution of (8) can be found as follows. To deal with the inhomogeneous terms we first differentiate with respect to x, so that we find homogeneous equations for

$$
\mathbf{g}^{(r)}(x) \equiv \frac{\mathrm{d}\mathbf{G}^{(r)}(x)}{\mathrm{d}x}.
$$

We write down the solution for the resulting system of equations using the spectral method.
In the case that all eigenvalues are different, we find the solution to the differentiated system to be of the form

$$
\mathbf{g}^{(r)}(x) = \sum_{j=1}^{N} \tilde{a}_j^{(r)}\mathbf{v}_j^{(r)}\exp\left[z_j^{(r)}x\right],
$$

where $(z_j^{(r)}, \mathbf{v}_j^{(r)})$ is an eigenvalue-eigenvector pair of $z_j^{(r)} \mathbf{v}_j^{(r)} R^{(r)} = \mathbf{v}_j^{(r)} Q^{(r)}$, and the $\tilde{a}_j^{(r)}$ are coefficients. As $Q^{(r)}$ is a generator, it has an eigenvalue 0, and hence one of the eigenvalues $z_j^{(r)}$ is zero, say $z_{j_*}^{(r)} = 0$, cf. [23]. With this in mind integration immediately yields that

$$\mathbf{G}^{(r)}(x) = \sum_{j \neq j_*} a_j^{(r)} \mathbf{v}_j^{(r)} \exp\left[z_j^{(r)} x\right] + a_{j_*}^{(r)} \mathbf{v}_{j_*}^{(r)} x + \mathbf{w}, \tag{10}$$

where $a_j^{(r)} = \tilde{a}_j^{(r)} / z_j^{(r)}$ for $j \neq j_*$, $a_{j_*}^{(r)} = \tilde{a}_{j_*}^{(r)}$, and the components w_i of $\mathbf{w}$ are integration constants.

Now the vectors $\mathbf{a^{(p)}}$, $\mathbf{a^{(r)}}$, and $\mathbf{w}$ can be found by considering the following boundary conditions:

(i) $G_i^{(p)}(0) = 0$ for all $i \geq N' + 1$ (i.e., the buffer cannot be empty when it fills), leading to $N - N'$ equations.

(ii) Similarly, $G_i^{(p)}(B^*-) = G_i^{(r)}(B^*)$ for all $i \in \{N'' + 1, \ldots, N'\}$. This gives $N + N' - N'' + 1$ equations.

(iii) For all $z_j^{(r)}$ with a non-negative real part, the corresponding $a_j^{(r)}$ is zero, since the $G_i^{(r)}(x)$ should remain bounded for $x \to \infty$. There are $N'' + 1$ such eigenvalues [23]. Notice that this also entails that the equilibrium distribution of $Y(t)$ is given by $\mathbf{w}$.

(iv) By letting $x \to \infty$ in (8), setting the left hand side equal to zero, we find the $N + 1$ global balance equations for $\mathbf{w} = \lim_{x\to\infty} \mathbf{G}^{(r)}(x)$. What we do here in fact, is to substitute the integrated solution (10) back into the inhomogeneous (undifferentiated) equations (8) to find the integration constants. From the global balance equations we can derive the N local balance equations for $i \in \{1, \ldots, N\}$,

$$(N - i + 1)\lambda w_{i-1} = ir\mu\left(w_i - G_i^{(r)}(B^*)\right) + C\mu\left(G_i^{(r)}(B^*) - G_i^{(p)}(B^*-)\right) + ip\mu G_i^{(p)}(B^*-).$$

(v) Finally we normalize: $\sum_{i=0}^{N} w_i = 1$.

Noticing that there are just as many boundary conditions as coefficients (namely $3N+3$), we conclude that the system is solvable. We have proved:

Theorem 3.1 *The above procedure gives the exact solution to the buffer content distribution.*

The above solution enables the computation of several key quantities. Denoting the throughput per user by τ, it is straightforward to obtain that

$$N\tau = \sum_{i=0}^{N} ir(w_i - G_i^{(r)}(B^*)) + \sum_{i=0}^{N} ipG_i^{(p)}(B^*-) + \sum_{i=N'+1}^{N''} C(G_i^{(r)}(B^*) - G_i^{(p)}(B^*-)). \tag{11}$$

The mean file transfer delay $\mathbb{E}T$ is found from

$$\tau = \frac{1/\mu}{\mathbb{E}T + 1/\lambda}. \tag{12}$$

In Section 6 we give some numerical results using these and other expressions.

4 General feedback fluid model with multiple thresholds

4.1 Model

In this section we consider a general framework for feedback fluid queues involving multiple thresholds, while the buffer may be finite or infinite. The interest lies primarily in showing that the mathematical analysis carries through, so that also in these cases a solution can be found. We use the results for the single threshold model with finite buffer in Section 6. The multi-threshold models may be useful to analyze the performance of more complicated feedback mechanism, but we do not investigate this in the current paper.

We let $W(t)$ denote the fluid level in the buffer at time t and $Y(t)$ be the state of the regulating process at time t. The size of the fluid reservoir is denoted by B, which may be finite or infinite. Furthermore we identify *thresholds* $B^{(k)}, k = 0, \ldots, K$, such that

$$0 = B^{(0)} < B^{(1)} < \cdots < B^{(K-1)} < B^{(K)} = B \leq \infty.$$

At times t when $B^{(k-1)} < W(t) < B^{(k)}, k = 1, \ldots, K$, we say the system is *in regime* k, while we say it is *at threshold* k at time t when $W(t) = B^{(k)}, k = 0, \ldots, K$. In the remainder any superscript $^{(k)}$ will refer to regime and/or threshold k.

We assume that the state space of the process $Y(t)$ is finite, and we denote it by $S = \{1, \ldots, N\}$. The dynamics of the system are given as follows: When the system is in regime k (at threshold k), the process $Y(t)$ behaves like an irreducible Markov process with generator $Q^{(k)}(\tilde{Q}^{(k)})$. Furthermore, the net rate of change of the process $W(t)$ is given by $r_i^{(k)}$ at times when the system is in regime k and $Y(t) = i$. For each regime k, these rates are collected in a diagonal matrix $R^{(k)}$ with elements $(R^{(k)})_{ii} = r_i^{(k)}$.

In order to state some assumptions on the rates $r_i^{(k)}$, we define for all regimes the subsets of S consisting of up-states respectively down-states, as

$$S_+^{(k)} = \{i \in S | r_i^{(k)} > 0\} \quad (13)$$
$$S_-^{(k)} = \{i \in S | r_i^{(k)} < 0\}. \quad (14)$$

From now on we will assume the following to hold:

(1) $S_+^{(k)} \cup S_-^{(k)} = S$ for all k, i.e. each fluid rate is nonzero. This assumption is made for the sake of brevity.

(2) $S_+^{(k+1)} \cap S_-^{(k)} = \varnothing$ for all k, i.e. there is no state $i \in S$ for which $r_i^{(k)} < 0$ and $r_i^{(k+1)} > 0$ for some k. Not only does this assumption seem natural to make from an applications point of view, it also excludes certain ambiguity problems that are discussed shortly.

(3) $S_-^{(k+1)} \cap S_-^{(k)} \neq \varnothing$ for all k, i.e. each threshold can be crossed downwards in at least one state $i \in S$.

(4) $S_+^{(k+1)} \cap S_+^{(k)} \neq \varnothing$ for all k, i.e. each threshold can be crossed upwards in at least one state $i \in S$.

(5) In the case when $B = \infty$, we assume the following stability condition,

$$\sum_{i=1}^{N} \pi_i^{(K)} r_i^{(K)} < 0, \tag{15}$$

where $\pi_i^{(K)}$ is the stationary distribution of a Markov process with generator $Q^{(K)}$. Hence this assumption entails that the expected rate of change of the buffer content process, conditional on the process being above level $B^{(K-1)}$, is negative.

Notice that, when $Y(t) \in S_-^{(k+1)} \cap S_+^{(k)}$ (when this set is nonempty), there is a confluence of drifts in $B^{(k)}$, so that the content process may stay at $B^{(k)}$ for a while, until $Y(t)$ jumps to a state $j \notin S_-^{(k+1)} \cap S_+^{(k)}$. This clarifies the ambiguity problem that is solved by the second assumption, since this assumption ensures that it is clearly determined what happens to the buffer content immediately after $Y(t)$ jumps from $j \in S_-^{(k+1)} \cap S_+^{(k)}$ to $i \notin S_-^{(k+1)} \cap S_+^{(k)}$.
Let (Y, W) be a pair of random variables distributed according to the stationary distribution of $(Y(t), W(t))$. Hence we are looking for

$$G_i(x) = \mathbb{P}(Y = i, W \leq x), \qquad i = 1, \ldots, N, \quad 0 \leq x \leq B.$$

4.2 Analysis

Our goal is to find explicit expressions for the distribution $G_i(x)$. We start off by defining for $k = 1, \ldots, K$, the following functions,

$$G_i^{(k)}(x) = \lim_{t\to\infty} \mathbb{P}(Y(t) = i,\ W(t) \leq x), \qquad B^{(k-1)} \leq x < B^{(k)},\ i \in S. \tag{16}$$

When we derive the Kolmogorov forward equations for the system, similarly as in Section 3, and then assume stationarity, we find that the functions $G_i^{(k)}(x)$ satisfy a system of KN differential equations, that can be represented, in matrix form, as

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{G}^{(k)}(x)}{\mathrm{d}x} R^{(k)} &= \mathbf{G}^{(k)}(x)\ Q^{(k)} \\
&+ \mathbf{G}^{(k)}(B^{(k-1)})\ (\tilde{Q}^{(k-1)} - Q^{(k)}) \\
&+ \mathbf{G}^{(k-1)}(B^{(k-1)}-)\ (Q^{(k-1)} - \tilde{Q}^{(k-1)}) \\
&\ \ \vdots \\
&+ \mathbf{G}^{(1)}(B^{(1)}-)\ (Q^{(1)} - \tilde{Q}^{(1)}) \\
&+ \mathbf{G}^{(1)}(0)\ (\tilde{Q}^{(0)} - Q^{(1)}), \qquad\qquad k = 1, \ldots, K,
\end{aligned} \tag{17}$$

where $\mathbf{G}^{(k)}(B^{(k)}-) = \lim_{x\uparrow B^{(k)}-} \mathbf{G}^{(k)}(x)$. Unlike in the Markov modulated setting, these equations are inhomogeneous (apart from the one for $k = 1$ if $\tilde{Q}^{(0)} = Q^{(1)}$, as was the case in Section 3). Once we solve the K matrix differential equations (17) using some appropriate boundary conditions, we are done in the infinite buffer case. The probabilities $G_i(x)$ that we are looking for are then given by the components of $\mathbf{G}(x) = \mathbf{G}^{(k)}(x)$, where k is such that $B^{(k-1)} \leq x < B^{(k)}$.

In the finite buffer case the same is true for $x < B$. However we now also have to find $p_i \equiv G_i(B), i = 1, \ldots, N$, the stationary probability that the regulating process is in state i. Due to the presence of feedback these probabilities cannot be found beforehand as in the traditional Markov modulated fluid models. However, we can write down the (global) balance equations for them in terms of the functions $G_i^{(k)}$ via the forward Kolmogorov equations for the process $Y(t)$. We find in matrix form

$$\begin{aligned}
\mathbf{0} &= \mathbf{p}\, \tilde{Q}^{(K)} \\
&+ \mathbf{G}^{(K)}(B-)\, (Q^{(K)} - \tilde{Q}^{(K)}) \\
&+ \mathbf{G}^{(K)}(B^{(K-1)})\, (\tilde{Q}^{(K-1)} - Q^{(K)}) \\
&+ \mathbf{G}^{(K-1)}(B^{(K-1)}-)\, (Q^{(K-1)} - \tilde{Q}^{(K-1)}) \\
&\quad \vdots \\
&+ \mathbf{G}^{(1)}(B^{(1)}-)\, (Q^{(1)} - \tilde{Q}^{(1)}) \\
&+ \mathbf{G}^{(1)}(0)\, (\tilde{Q}^{(0)} - Q^{(1)}).
\end{aligned} \tag{18}$$

4.3 Solution for the finite buffer

Our next step is to solve the matrix differential equations (17) for $k = 1, \ldots, K$, in the case that $B < \infty$. To deal with the inhomogeneous terms we first differentiate with respect to x, so that we find homogeneous equations for

$$\mathbf{g}^{(k)}(x) \equiv \frac{\mathrm{d}\mathbf{G}^{(k)}(x)}{\mathrm{d}x}.$$

We write down the solution for the resulting system of equations using the spectral method. Therefore we consider the eigensystems

$$\mathbf{v}_j^{(k)}\, Q^{(k)} = z_j^{(k)} \mathbf{v}_j^{(k)}\, R^{(k)}, \qquad j = 1, \ldots, N, \quad k = 1, \ldots, K, \tag{19}$$

where $z_j^{(k)}$ and $\mathbf{v}_j^{(k)}, j = 1, \ldots, N$ are the j-th eigenvalue and left-eigenvector corresponding to the matrix $Q^{(k)}(R^{(k)})^{-1}$. Since for each k the matrix $Q^{(k)}$, has an eigenvalue 0, it follows that the kth system has an eigenvalue 0, which we denote as $z_{j_*^{(k)}}^{(k)} = 0$.

In the case that all eigenvalues are different, we find the solution to the differentiated system to be of the form

$$\mathbf{g}^{(k)}(x) = \sum_{j=1}^{N} \tilde{a}_j^{(k)} \exp\left[z_j^{(k)} x\right] \mathbf{v}_j^{(k)},$$

so that integration immediately yields

$$\mathbf{G}^{(k)}(x) = \sum_{j \neq j_*^{(k)}} a_j^{(k)} \exp\left[z_j^{(k)} x\right] \mathbf{v}_j^{(k)} + a_{j_*^{(k)}}^{(k)} \mathbf{v}_{j_*^{(k)}}^{(k)} x + \mathbf{w}^{(k)}, \tag{20}$$

with

$$a_j^{(k)} = \frac{\tilde{a}_j^{(k)}}{z_j^{(k)}}, \quad j \neq j_*^{(k)}, \text{ and } a_{j_*^{(k)}}^{(k)} = \tilde{a}_{j_*^{(k)}}^{(k)}.$$

In total there are $2KN+N$ unknowns in this expression, namely $a_j^{(k)}, w_i^{(k)}$, and p_i for $i,j=1,\dots,N$ and $k=1,\dots,K$.
We now turn to the boundary conditions that we have at our disposal. In order to count them as we go along, we will denote the cardinalities of $S_+^{(k)}$ and $S_-^{(k)}$ by $N_+^{(k)}$ and $N_-^{(k)}$, respectively.

(i) $G_i^{(1)}(0)=0$ for all $i \in S_+^{(1)}$. This gives $N_+^{(1)}$ equations.

(ii) $G_i^{(k)}(B^{(k)}-) = G_i^{(k+1)}(B^{(k)})$ for all $i \in S_+^{(k+1)} \cup S_-^{(k)}$ and $k=1,\dots,K-1$. This gives $\sum_{k=1}^{K-1}\left(N_-^{(k)}+N_+^{(k+1)}\right)$ equations.

(iii) $G_i^{(K)}(B-)=p_i$ for all $i \in S_-^{(K)}$, giving $N_-^{(K)}$ equations.

(iv) The balance equations in (18) yield $N-1$ linearly independent equations. Notice that while there are N equations in (18), one of them must be redundant.

(v) One other equation is given by normalization, $\sum_{i=1}^{N} p_i = 1$.

(vi) Last but not least, substitution of (20) into the original, inhomogeneous differential equations (17) yields KN relations that have to be satisfied. In matrixform these are given as

$$\begin{aligned} a_{j_*^{(k)}}^{(k)} \mathbf{v}_{j_*^{(k)}}^{(k)} R^{(k)} &= \mathbf{w}^{(k)} Q^{(k)} \\ &+ \mathbf{G}^{(k)}(B^{(k-1)})\,(\tilde{Q}^{(k-1)} - Q^{(k)}) \\ &+ \mathbf{G}^{(k-1)}(B^{(k-1)}-)\,(Q^{(k-1)} - \tilde{Q}^{(k-1)}) \\ &\vdots \\ &+ \mathbf{G}^{(1)}(B^{(1)}-)\,(Q^{(1)} - \tilde{Q}^{(1)}) \\ &+ \mathbf{G}^{(1)}(0)\,(\tilde{Q}^{(0)} - Q^{(1)}) \qquad k=1,\dots,K. \end{aligned} \tag{21}$$

As an aside we mention that when $\tilde{Q}^{(0)} = Q^{(1)}$, it follows that $a_{j_*^{(1)}}^{(1)} = 0$ and that $\mathbf{w}^{(1)}$ is a multiple of $\mathbf{v}_{j_*^{(1)}}^{(1)}$, as should be expected.

Notice that conditions (i) and (ii) can be explained similarly as in Section 3, while condition (iii) states that the buffer is full with probability 0 when it is being drained.
Using Assumption (2) we now find the total number of boundary conditions to be $2KN+N$. From these equations all $2KN+N$ unknowns can be determined.

Theorem 4.1 (Finite buffer size B)
If the eigenvalues $z_j^{(k)}, j=1,\dots,N$ of the matrices $Q^{(k)}(R^{(k)})^{-1}, k=1,\dots,K$ are simple, the stationary distribution of the process $(Y(t),W(t))$ is given by (20) where the $2KN+N$ unknowns $a_j^{(k)}, w_i^{(k)}$, and p_i, $i,j=1,\dots,N$, $k=1,\dots,K$, can be found by the $2KN+N$ boundary conditions in (i)–(vi).

When not all eigenvalues are simple, the form of the solution is different, but the number of parameters that have to be solved by boundary conditions remains the same.

4.4 Solution for the infinite buffer

In the case $B = B^{(K)} = \infty$ we again find

$$\mathbf{G}^{(k)}(x) = \sum_{j \neq j_*^{(k)}} a_j^{(k)} \exp\left[z_j^{(k)} x\right] \mathbf{v}_j^{(k)} + a_{j_*^{(k)}}^{(k)} \mathbf{v}_{j_*^{(k)}}^{(k)} x + \mathbf{w}^{(k)}, \tag{22}$$

As before, $j_*^{(k)}$ is the index for which $z_{j_*^{(k)}}^{(k)} = 0$ holds. It is convenient to note that we have some extra a priori knowledge about the eigenvalues corresponding to the top regime $k = K$: ordering them according to their real values we have

$$\mathrm{Re}\left(z_1^{(K)}\right) \leq \cdots \leq \mathrm{Re}\left(z_{N_+^{(K)}}^{(K)}\right) < 0 = z_{N_+^{(K)}+1}^{(K)} < \mathrm{Re}\left(z_{N_+^{(K)}+2}^{(K)}\right) \leq \cdots \leq \mathrm{Re}\left(z_N^{(K)}\right), \tag{23}$$

which is a consequence of the negative drift in this regime, due to (15); see also [23]. In particular we find that $j_*^{(K)} = N_+^{(K)} + 1$, and that exactly $N_+^{(K)}$ eigenvalues have strictly negative real part. The unknown parameters in solution (22) are $a_j^{(k)}$, $w_i^{(k)}$ and p_i, where $i = 1, \ldots, N$, $j = 1, \ldots, N$, and $k = 1, \ldots, K$. Next we determine the boundary conditions, in the same order as before.

(i) and (ii) are the same as for the finite buffer case, leading to $N_+^{(1)} + \sum_{k=1}^{K-1} \left(N_-^{(k)} + N_+^{(k+1)}\right) = KN - N_-^{(K)}$ equations.

(iii) Due to our stability assumption, see (15), we must have $\lim_{x\to\infty} \mathbf{G}^{(K)}(x) = \mathbf{p}$. Hence it follows that $\mathbf{w}^{(K)} = \mathbf{p}$ and $a_j^{(K)} = 0$ for all j for which $\mathrm{Re}(z_j^{(K)}) \geq 0$. Using (23), we find that these conditions add up to $N + N_-^{(K)}$ equations.

(iv) The balance equations, that now take the form

$$\begin{aligned}
\mathbf{0} \;=\; & \mathbf{p}\, Q^{(K)} \\
+\; & \mathbf{G}^{(K)}(B^{(K-1)})\, (\tilde{Q}^{(K-1)} - Q^{(K)}) \\
+\; & \mathbf{G}^{(K-1)}(B^{(K-1)}-)\, (Q^{(K-1)} - \tilde{Q}^{(K-1)}) \\
& \vdots \\
+\; & \mathbf{G}^{(1)}(B^{(1)}-)\, (Q^{(1)} - \tilde{Q}^{(1)}) \\
+\; & \mathbf{G}^{(1)}(0)\, (\tilde{Q}^{(0)} - Q^{(1)}),
\end{aligned} \tag{24}$$

again yield $N - 1$ linearly independent equations.

(v) One other equation is given by normalization, $\sum_{i=1}^{N} p_i = 1$.

(vi) Finally, substitution of (22) into the original, inhomogeneous differential equation (17) for $k = 1, \ldots, K-1$, we find the corresponding $KN - N$ equations from (21). However, for $k = K$ we do not find any extra relations, since for this case (21) simplifies to (24), with $\mathbf{p}$ replaced by $\mathbf{w}^{(K)}$. This is of course consistent with (24) itself and the relation $\mathbf{w}^{(K)} = \mathbf{p}$ under (iii), and therefore does not add any information.

Again, the number of boundary conditions adds up to $2KN + N$, as it should.
To find the following theorem, we note that the conditions in (iii) can be applied beforehand to find the actual form of the solution in regime K,

$$\mathbf{G}^{(K)}(x) = \sum_{j=1}^{N_+^{(K)}} a_j^{(K)} \exp\left[z_j^{(K)} x\right] \mathbf{v}_j^{(K)} + \mathbf{p}. \tag{25}$$

This means that we have $N + N_-^{(K)}$ unknowns less than in the finite buffer case.

Theorem 4.2 (Infinite buffer size)
If the eigenvalues $z_j^{(k)}, j = 1, \ldots, N$ of the matrices $Q^{(k)}\,(R^{(k)})^{-1}, k = 1, \ldots, K$ are simple, the stationary distribution of the process $(Y(t), W(t))$ is given by (22) for $k = 1, \ldots, K-1$ and by (25) for regime K, which holds above the last (finite) threshold $B^{(K-1)}$. The $2KN - N_-^{(K)}$ unknowns in this solution can be found by the equal number of boundary conditions in (i), (ii) and (iv)–(vi).

Again, when not all eigenvalues are simple, the general form of the solution is different, but the number of parameters that have to be solved by boundary conditions, remains the same.

5 Many sources

The intrinsic drawback of the technique of the previous sections is its computational complexity. When the size of the system (i.e., the number of sources) grows, a large eigensystem needs to be solved. This explains the interest in simpler asymptotic approaches. In this section we will focus on the so-called 'many-sources scaling', which was introduced by Weiss [33]. In this regime, we derive explicit results on the overflow probability.
In the many-sources scaling, buffer and bandwidth resources are scaled by the number of users N. In other words, if we scale $C \equiv Nc$, the exponential decay rate of $\mathbb{P}(W \geq Nx)$ can be determined *explicitly* in terms of x and model parameters r, p, λ, μ, and c. Because W is now implicitly parametrized by N, we write W_N. The random variable V_N is defined as the buffer content in the corresponding model without feedback (as in Section 2.2).

5.1 Feedback model without threshold

Asymptotics

First we estimate the average behavior of the number of active sources in the asymptotic limit (large N). It is not hard to show that there are two regimes.
(A) In the first regime,

$$\frac{\lambda p}{\lambda + \mu p} < c,$$

cf. (1). In this case, in the asymptotic limit, on average the sources are allowed to transmit at peak rate; the buffer will be empty nearly always. The number of active sources on average will be Nm, with

$$m := \frac{\lambda}{\lambda + \mu p}.$$

(B) In the second regime,

$$\frac{\lambda r}{\lambda + \mu r} < c \leq \frac{\lambda p}{\lambda + \mu p}.$$

In this case, the network will in general be in the processor sharing regime. The average number of active users simultaneously in the system is Nm with

$$m := 1 - \frac{\mu c}{\lambda}. \tag{26}$$

The sources are allowed to transmit at a rate m' between r and p, where

$$m' := \frac{c}{m} = \frac{\lambda c}{\lambda - \mu c}. \tag{27}$$

The following lemma gives the decay rate of the probability of a non-empty buffer in each of the regimes. A similar result can be found in Ramanan and Weiss [25].

Lemma 5.1 *The decay rate of the probability of a non-empty buffer is given by*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(W_N > 0\right) = I(0).$$

In regime (A), i.e. when $\lambda p/(\lambda + \mu p) < c$, we have that $I(0)$ is given by

$$I(0) = \left(1 - \frac{c}{r}\right) \log\left(\frac{1 - c/r}{c\mu/\lambda}\right) + \log\left(\frac{c(\lambda + p\mu)}{p\lambda}\right) + \frac{c}{r} - \frac{c}{p},$$

and for regime (B), when $\lambda r/(\lambda + \mu r) < c \leq \lambda p/(\lambda + \mu p)$, we have

$$I(0) = \left(1 - \frac{c}{r}\right) \log\left(\frac{1 - c/r}{c\mu/\lambda}\right) + \frac{c\mu}{\lambda} - \left(1 - \frac{c}{r}\right).$$

Proof. Directly from Theorem 11.15 of [31], the decay rate of the probability of a non-empty buffer equals

$$\int_m^{c/r} \log\left(\frac{\mu_x x}{\lambda(1-x)}\right) \mathrm{d}x.$$

Here μ_x is the (downward) probability flux per source, when the number of sources in the on state is Nx. In other words,

$$\mu_x := \begin{cases} p\mu & \text{if } xp < c, \\ c\mu \cdot x^{-1} & \text{otherwise.} \end{cases}$$

Direct calculation yields the stated expression. □

Define $A(t)$ as the amount of fluid generated in the interval $[0, t)$ by one source with off-periods $\mathrm{Exp}(\lambda)$, on-periods of length $\mathrm{Exp}(r\mu)$, and constant generation rate r. Let $\mathbb{E}_0$ ($\mathbb{E}_1$, respectively) denote expectation given that the source start in the off (on) state at time 0.

Proposition 5.2 *The decay rate for positive buffer content values is given by*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(W_N > Nx\right) =: I(x) = J(x) + I(0),$$

with

$$J(x) := \inf_{t>0} \sup_{\theta} \left(\theta(x+ct) - \frac{c}{r} \log \mathbb{E}_1 e^{\theta A(t)} - \left(1 - \frac{c}{r}\right) \log \mathbb{E}_0 e^{\theta A(t)}\right). \tag{28}$$

Proof. Evidently,

$$\mathbb{P}\left(W_N > Nx\right) = \mathbb{P}\left(W_N > Nx | W_N > 0\right) \mathbb{P}\left(W_N > 0\right).$$

As shown in Section 2.3,

$$\mathbb{P}\left(W_N > Nx | W_N > 0\right) = \mathbb{P}\left(V_N > Nx | V_N > 0\right).$$

Immediately from Theorem 1 of Duffield [8], we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(V_N > Nx | V_N > 0\right) = J(x).$$

Together with Lemma 5.1 the proof of the proposition is complete. □

The variational problem in (28) cannot be solved analytically; numerical methods have to be applied. Fortunately, for large x asymptotics are available.

Simple approximations for large buffers

Let $\theta^\star$ solve

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} e^{\theta A(t)} = c\theta.$$

Define, for $i = 0, 1$,

$$a_i := \lim_{t\to\infty} \log \mathbb{E}_i e^{\theta^\star A(t)} - c\theta^\star t.$$

In Duffield [8] it is proven that, for $x \to \infty$,

$$J(x) = \theta^\star x - \frac{c}{r} a_1 - \left(1 - \frac{c}{r}\right) a_0 + o(x).$$

Following the *Chernoff Dominant Eigenvalue method* by Elwalid, Heyman, Lakshman, Mitra, and Weiss [9], we propose an even simpler approximation:

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}\left(W_N > Nx\right) \approx \theta^\star x + I(0). \tag{29}$$

The calculation of $\theta^\star$ is extremely simple:

$$\frac{r\mu}{r-c} - \frac{\lambda}{c},$$

and $I(0)$ is given in Lemma 5.1. In [9] it is shown that this approximation is conservative for all x (in fact it is the best possible linear estimate that is conservative for all x). Notice that the analysis of [9] requires the sources to be time-reversible. This condition is met for exponentially distributed file sizes and user think times. In [9] it is concluded that the approximation is usually not overly conservative.

General think-time and file-size distributions for large buffers

The nature of approximation (29) is, for large x,

$$\mathbb{P}(V_N > Nx) \approx \mathbb{P}(V_N > 0)e^{-\theta^\star Nx}. \tag{30}$$

We may follow the same approach in case of general (rather than exponentially distributed) think-time and file-size distributions. The questions then are: how to compute the counterparts of the probability of a non-empty buffer $\mathbb{P}(V_N > 0)$ and the exponential decay rate $\theta^\star$?

The probability of a non-empty buffer. As long as the buffer is empty, the process behaves like an infinite-server queue if the number of jobs is below N', and like a processor sharing queue if the number of jobs is between N' and $N''+1$. Let D_N be the number of jobs in the system. It is easily verified that in the case of exponential think-times and file-sizes, the probability that the number of simultaneous jobs in the system reaches $N''+1$ is given by

$$\mathbb{P}(D_N = N''+1) = \frac{1}{\text{Norm}}\left(\frac{\mu}{\lambda}\right)^{N''+1}\left(r^{N''-N'+1}p^{N'}\right)\frac{N'!\cdot(N-N''-1)!}{N!}, \tag{31}$$

where the normalizing constant Norm is given by

$$\text{Norm} = \sum_{j=0}^{N'}\left(\frac{\mu}{\lambda}\right)^j\left(p^j\right)\frac{(N-j)!}{N!} + \sum_{j=N'+1}^{N''+1}\left(\frac{\mu}{\lambda}\right)^j\left(r^{j-N'}p^{N'}\right)\frac{N'!(N-j)!}{N!}.$$

It is tedious but straightforward to verify that the decay rate of (31) is indeed $I(0)$, as was defined in Lemma 5.1.

Importantly, formula (31) also holds in the case of *general* think-time and file-size distributions, with respective means λ^{-1} and μ^{-1}. This is due to *insensitivity* results for networks of generalized processor sharing queues, as was shown by Cohen [5].

Exponential decay rate $\theta^\star$. Let T be the distribution of the think-time and F the distribution of the file-size. During busy periods, the buffer is fed by a superposition of N on-off sources with off-times distributed like T, and on-times distributed like F/r (with peak rate r). Let $\bar{A}(t)$ be the amount of traffic generated by such a source, in steady state, in an interval of length t. Then (under mild technical conditions) the exponential decay rate in this model is the solution of

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}e^{\theta\bar{A}(t)} = c\theta, \tag{32}$$

see Glynn and Whitt [14]. Equation (32) does not necessarily have a solution. In the case of heavy-tailed on-times there is no solution – approximation (30) does not apply.

General think-time and file-size distributions for small buffers

Above we concentrated on loss behavior under general think-time (with mean $1/\lambda$) and file-size distributions (with mean $1/\mu$), and large buffers. For small buffers a result from Mandjes and Kim [19] is applicable: for $x \downarrow 0$,

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(W_N > Nx\right) = \frac{2\sigma}{r}\sqrt{x} + I(0) + O(x),$$

with constant σ given by

$$\sigma = \sqrt{(cr\mu + (r-c)\lambda)\log\left(\frac{cr\mu}{(r-c)\lambda}\right) - 2\,(cr\mu - (r-c)\lambda)}.$$

Strikingly, σ depends on the distributions of the think times and file sizes only through their means, making this an *insensitivity* result.

5.2 Feedback model with threshold

As before, we scale the resources buffer and bandwidth: $C \equiv Nc$, and $B^* \equiv Nb^*$. Again, this regime allows explicit results, which we obtain below. Before we give the analysis, we first review a result from Mandjes [18]; closely related versions were proven by Botvich and Duffield [3, Theorem 3] and Weiss [33]. For reasons of brevity, we choose to state the result informally; formal versions of this and a closely related statement are found in [20], [21].

Lemma 5.3 *Suppose N exponential on off-sources in the system without feedback (with link rate Nc). The mean on-time is μ^{-1}, the mean off-time is λ^{-1}, and the source sends at rate $r > c$ while on. Let π be the steady state on-probability; $\pi r < c$. Define $p_N(x, \alpha, \beta)$ as the probability that, given the number of sources in the on-state is $N\alpha > C$ at time 0, the queue reaches at least level Nx, and at the epoch this level is attained the number of sources in the on-state is $N\beta$. Then, for $x \to \infty$,*

$$\lim_{N\to\infty} \frac{1}{N} \log p_N(x, \alpha, \beta) - zx =$$

$$+\alpha \log\left(\frac{\pi y}{\rho}\right) + (1-\alpha)\log\left(\frac{\bar{y}(1-\pi)}{1-\rho}\right) + \beta\log\left(\frac{\beta}{\pi y}\right) + (1-\beta)\log\left(\frac{1-\beta}{(1-\pi)\bar{y}}\right) + o(x),$$

where

$$z := \frac{\mu}{r-c} - \frac{\lambda}{c}, \quad \begin{pmatrix} \bar{y} \\ y \end{pmatrix} := \frac{1}{(r-c)\lambda + c\mu} \begin{pmatrix} (r-c)\lambda \\ c\mu \end{pmatrix},$$

and

$$\rho := \frac{\mu c^2}{\mu c^2 + \lambda (r-c)^2}.$$

Notice that in Mandjes [18] general Markov fluid sources are treated (instead of 2-dimensional). There the solution was phrased in terms of solutions of eigensystems. For exponential on-off sources these eigensystems allow explicit solutions, which are given in Lemma 5.3.

Now we are in a position to do the large deviations analysis. As in Section 5.1, it turns out that there are two possible regimes: (A) the buffer is empty with an overwhelming probability and the active sources transmit at rate p almost all the time, and (B) the buffer occupancy is approximately Nb^* on average, and the active sources send at a rate between r and p.

(A) $\pi^{(p)}p < c$, where $\pi^{(p)}$ is the on-probability in the 'peak rate regime': $\lambda(\lambda+p\mu)^{-1}$. In this case the 'average state' of the system is a (nearly) empty buffer, and the active sources transmit at peak rate.

In order for the buffer to exceed level Nx, where x exceeds b^*, four events have to occur in order: (1) The buffer must become non-empty, i.e., the number of sources in the on-state must exceed Nc/p. (2) Given that the buffer content is at the point of becoming positive, an amount of Nb^* of fluid has to accumulate. At the epoch the buffer content reaches Nb^*, let the number of sources transmitting be $N\alpha$. (3) If α is smaller than c/r, then the number of sources in the on-state has to grow to Nc/r, in order for the buffer to exceed level Nb^*. If α is already at least c/r then in this phase nothing has to happen. Let $N\alpha'$ be the number of sources in the on-state at the end of this phase. (4) An amount of $N(x-b^*)$ of fluid has to accumulate in the buffer, where at the beginning of this phase the number of sources transmitting is $N\alpha'$. This makes the decay rate be the sum of four decay rates:

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbb{P}\left(W_N > Nx\right)\approx\sum_{i=1}^{4} I_i. \tag{33}$$

Notice that this construction (the decay rate of a steady-state probability being formulated as the solution of a transient problem, namely the decay rate of the path from equilibrium towards the rare event) is essentially of the Freidlin-Wentzell type. Details on this approach are found in [31, Chapter 6]. A further specification of the four events is:

(1) First the number of sources in the on-state must reach level Nc/p. In this regime, the sources act as on-off sources, with mean on-time $(p\mu)^{-1}$, mean off-time λ^{-1}, and traffic rate p while on. Due to Sanov's theorem [7] this equals

$$I_1 = \left(\frac{c}{p}\right)\log\left(\frac{c/p}{\pi^{(p)}}\right)+\left(1-\frac{c}{p}\right)\log\left(\frac{1-c/p}{1-\pi^{(p)}}\right).$$

(2) Second, the system has to reach buffer Nb^*. From Lemma 5.3 the corresponding decay rate (for Nb^* large) $I_2(\alpha)\approx z^{(p)}b^*+J_2(\alpha)$, with $J_2(\alpha)$ defined as

$$\begin{aligned} J_2(\alpha) := {} & \left(\frac{c}{p}\right)\log\left(\frac{y^{(p)}\pi^{(p)}}{\rho^{(p)}}\right)+\left(1-\frac{c}{p}\right)\log\left(\frac{\bar{y}^{(p)}(1-\pi^{(p)})}{1-\rho^{(p)}}\right)\\ & +\alpha\log\left(\frac{\alpha}{\pi^{(p)}y^{(p)}}\right)+(1-\alpha)\log\left(\frac{1-\alpha}{(1-\pi^{(p)})\bar{y}^{(p)}}\right),\end{aligned}$$

if level Nb^* is reached while $N\alpha$ sources are transmitting. Notice that the sources still behave as on-off sources with mean on-time $(p\mu)^{-1}$, mean off-time λ^{-1}, and source transmission rate p while on. In the above the following quantities were used:

$$z^{(p)} := \frac{p\mu}{p-c}-\frac{\lambda}{c},\quad \begin{pmatrix}\bar{y}^{(p)}\\ y^{(p)}\end{pmatrix} := \frac{1}{(p-c)\lambda+cp\mu}\begin{pmatrix}(p-c)\lambda\\ cp\mu\end{pmatrix}, \tag{34}$$

and

$$\rho^{(p)} := \frac{p\mu c^2}{p\mu c^2 + \lambda(p-c)^2}. \tag{35}$$

(3) If $\alpha > c/r$ then the buffer continues to grow immediately after level Nb^* is reached, in which case $\alpha' = \alpha$ and the corresponding decay rate is

$$I_3^+(\alpha) = 0.$$

Otherwise, the number of active sources must first increase to $\alpha' = c/r$. In this phase the buffer content will remain constant, due to the processor sharing discipline. Analogously to Section 5.1, this trajectory corresponds to decay rate

$$I_3^-(\alpha) = \int_\alpha^{c/r} \log\left(\frac{c\mu}{\lambda(1-x)}\right) \mathrm{d}x.$$

Direct calculation gives

$$I_3^-(\alpha) = \left(\frac{c}{r} - \alpha\right)\log\left(\frac{c\mu}{\lambda}\right) + \left(1 - \frac{c}{r}\right)\log\left(1 - \frac{c}{r}\right) + \frac{c}{r} - \alpha - (1-\alpha)\log(1-\alpha).$$

(4) In the fourth phase, the sources are on-off sources with mean on-time $(r\mu)^{-1}$, mean off-time λ^{-1}, and source transmission rate r while on; $\pi^{(r)} := \lambda(\lambda + r\mu)^{-1}$. Define $z^{(r)}$, $\bar{y}^{(r)}$, $y^{(r)}$, and $\rho^{(r)}$ analogously to (34) and (35).

If $\alpha < c/r$ the number of sources in the on-state at the beginning of this phase is Nc/r. Notice that at the epoch of overflow, the number of sources in the on-state will be Nc/r; if it were larger, an even larger buffer could have been built up 'for free'. By direct application Lemma 5.3 this gives decay rate $I_4^-(\alpha) \approx z^{(r)}(x - b^*) + J_4^-(\alpha)$ (for $x - b^*$ large), where

$$J_4^-(\alpha) := \left(\frac{c}{r}\right)\log\left(\frac{c/r}{\rho^{(r)}}\right) + \left(1 - \frac{c}{r}\right)\log\left(\frac{1 - c/r}{(1-\rho^{(r)})}\right).$$

Otherwise, if $\alpha \geq c/r$, the number of sources transmitting at the beginning of the phase is $N\alpha$. This results in decay rate $I_4^+(\alpha) \approx z^{(r)}(x - b^*) + J_4^+(\alpha)$, where

$$J_4^+(\alpha) := \alpha\log\left(\frac{y^{(r)}\pi^{(r)}}{\rho^{(r)}}\right) + (1-\alpha)\log\left(\frac{\bar{y}^{(r)}(1-\pi^{(r)})}{1-\rho^{(r)}}\right)$$
$$+ \left(\frac{c}{r}\right)\log\left(\frac{c/r}{\pi^{(r)}y^{(r)}}\right) + \left(1 - \frac{c}{r}\right)\log\left(\frac{1 - c/r}{(1-\pi^{(r)})\bar{y}^{(r)}}\right).$$

The parameter α has to be optimized, in that the sum of the four decay rates has to be minimized over α; the optimizer $\alpha^\star$ reflects the fraction of sources transmitting at the epoch the queue length reaches Nb^*. It is not hard to verify that $f^-(\alpha) := I_1 + J_2(\alpha) + I_3^-(\alpha) + J_4^-(\alpha)$ is minimized by

$$\alpha^- = \frac{\lambda\pi^{(p)}y^{(p)}}{c\mu(1-\pi^{(p)})\bar{y}^{(p)}}. \tag{36}$$

Also, $f^+(\alpha) := I_1 + J_2(\alpha) + I_3^+(\alpha) + J_4^+(\alpha)$ is minimized by

$$\alpha^+ = \frac{\pi^{(p)}y^{(p)}y^{(r)}}{\pi^{(p)}y^{(p)}y^{(r)} + (1-\pi^{(p)})\bar{y}^{(p)}\bar{y}^{(r)}}. \tag{37}$$

Trivially $I_3^-(c/r) = I_3^+(c/r) = 0$ and $J_4^-(c/r) = J_4^+(c/r)$, and hence $f^-(c/r) = f^+(c/r)$. Both $f^-(\cdot)$ and $f^+(\cdot)$ are convex and unimodal in α. It is also verified that either $\alpha^- \leq \alpha^+ \leq c/r$ (and hence $\alpha^\star = \alpha^-$), or $c/r \leq \alpha^+ \leq \alpha^-$ (and hence $\alpha^\star = \alpha^+$).

(B) $\pi^{(r)}r < c \leq \pi^{(p)}p$. In this case the 'average state' of the system is a buffer occupancy of Nb^*, and all active sources sending at rate m', as given by (27). In equilibrium, the system is operating in the processor sharing regime. Clearly, the first two decay rates of case A do not apply anymore, as the queue is already operating at level Nb^* on average. The other two decay rates remain:

(3') In order to reach buffer level Nx, first the number of transmitting sources has to grow from m to Nc/r, where m is defined in (26). As before, this entails decay rate

$$\begin{aligned} I_{3'} &= \int_m^{c/r} \log\left(\frac{c\mu}{\lambda(1-x)}\right) \mathrm{d}x \\ &= \left(\frac{c}{r} - m\right)\log\left(\frac{c\mu}{\lambda}\right) + \left(1-\frac{c}{r}\right)\log\left(1-\frac{c}{r}\right) \\ &\quad + \frac{c}{r} - m - (1-m)\log(1-m). \end{aligned}$$

(4') Now an additional $N(x-b^*)$ has to be built up. By Lemma 5.3 this gives decay rate (for $x-b^*$ large) $I_{4'} \approx z^{(r)}(x-b^*) + J_{4'}$, where

$$\begin{aligned} J_{4'} &:= \left(\frac{c}{r}\right)\log\left(\frac{y^{(r)}\pi^{(r)}}{\rho^{(r)}}\right) + \left(1-\frac{c}{r}\right)\log\left(\frac{\bar{y}^{(r)}(1-\pi^{(r)})}{1-\rho^{(r)}}\right) \\ &\quad + \left(\frac{c}{r}\right)\log\left(\frac{c/r}{\pi^{(r)}y^{(r)}}\right) + \left(1-\frac{c}{r}\right)\log\left(\frac{1-c/r}{(1-\pi^{(r)})\bar{y}^{(r)}}\right). \end{aligned}$$

Summarizing, we have shown:

Theorem 5.4 *Let $b^* \equiv gx$ for some $g \in (0,1)$.*
If $\pi^{(p)}p < c$, for $x \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N}\log \mathbb{P}\left(W_N > Nx\right) - z^{(p)}gx - z^{(r)}(1-g)x = \begin{cases} f^-(\alpha^-) + o(x) & \text{if } \alpha^- \leq \alpha^+; \\ f^+(\alpha^+) + o(x) & \text{if } \alpha^- \geq \alpha^+, \end{cases}$$

with α^- and α^+ defined by (36) and (37).
If $\pi^{(r)}r < c \leq \pi^{(p)}p$, for $x \to \infty$,

$$\lim_{N\to\infty} \frac{1}{N}\log \mathbb{P}\left(W_N > Nx\right) - z^{(r)}(1-g)x = I_{3'} + J_{4'} + o(x).$$

6 Numerical results

In this section we provide numerical results. We first consider an example where we maximize the system throughput, using the procedure of Section 3. The second example relies on the many sources asymptotics of Section 5.

6.1 Stationary distribution and performance measures

Infinite-buffer model. We consider an infinite buffer that is fed by 10 identical and independent on-off sources. Let $Y(t)$ denote the number of active sources at time t; we choose the process $Y(\cdot)$ as the regulating process. As before, $W(t)$ will denote the buffer content at time t. The output trunk speed is $C = 11$. The off-times of the sources are exponentially distributed with $\lambda = 3$. File sizes are exponentially distributed with $\mu = 2$. The peak transmission rate of the sources $p = 4$, which is the actual transmission rate when the buffer content is below $B^* = 2$, and the rate drops to its minimum value $r = 2$ when the content is above B^*. Notice that the buffer content may stick at B^* for as long as 3, 4, or 5 sources are active; then the total actual transmission rate is 11, i.e., the output trunk speed, equally shared among the active sources. As a consequence the transition intensity of the process $Y(\cdot)$ from state i to $i-1$, given that the buffer process remains at 2, is equal to $i \cdot 2C/i = 22$. These considerations allow us to write down the diagonal matrices $R^{(p)}$ and $R^{(r)}$, as well as the tri-diagonal matrices $Q^{(p)}$, $Q^{(r)}$ and $Q^{(*)}$. After the numerical determination of the eigensystems of the matrices $Q^{(p)}(R^{(p)})^{-1}$ and $Q^{(r)}(R^{(r)})^{-1}$, we apply the appropriate boundary conditions to find the 33 unknowns (note that $N = 10$). After solving this (linear) set of equations, the stationary distribution $\mathbf{G}$ of the joint process $(Y(t), W(t))$ can be found numerically. A graphical representation of $G(y) \equiv \mathbb{P}(W \leq y) = \sum_{i=0}^{10} G_i(y)$ is given in Figure 1. The total throughput $N\tau$ of

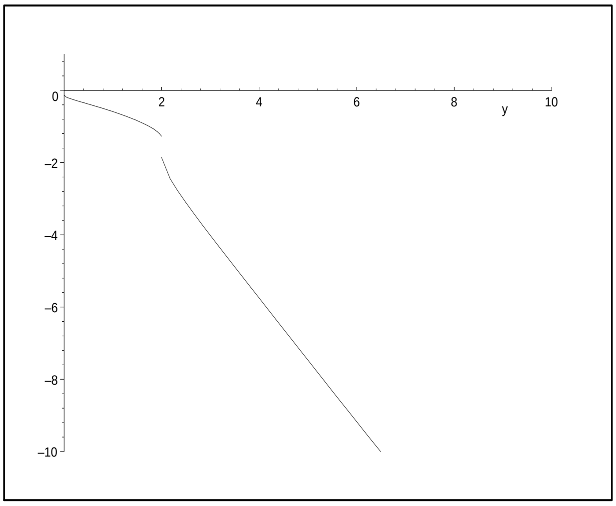


Figure 1: $B = \infty$: $\log_{10}(1 - G(y))$ as a function of y

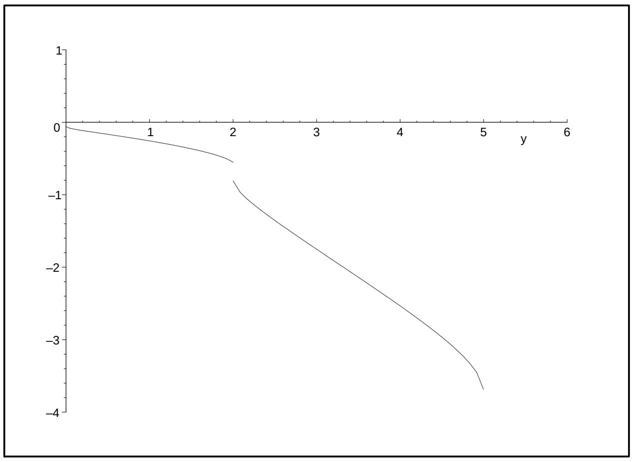
the system can be found from (11) as

$$N\tau \;=\; \sum_{j=0}^{10} jr(w_j - G_j^{(r)}(B^*)) + jpG_j^{(p)}(B^*-) + C(G_j^{(r)}(B^*) - G_j^{(p)}(B^*-)) = 10.2652. \qquad (38)$$

Finite-buffer model. We also solved the corresponding finite-buffer model. We did the calculations for $B = 5$, leaving all other parameters the same. Figure 2 gives a graphical representation of the distribution. Observe the 'jumps' of $G_3(y)$, $G_5(y)$ and $G_7(y)$ at $y = 0$, $y = B^* = 2$, and $y = B = 5$, respectively.

Clearly, the size of the jump that $G(y)$ has at $y = 5$ is exactly the (time average) probability of a full buffer:

$$\mathbb{P}(W = B) = 1 - \sum_{j=0}^{10} G_j^{(r)}(B) \;=\; 2.01 \cdot 10^{-4}.$$



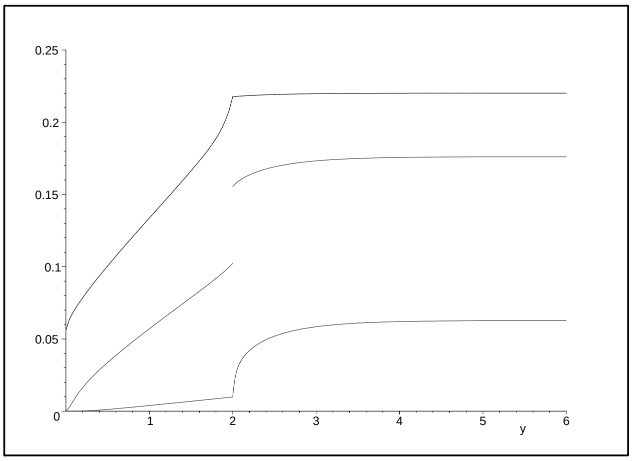


Figure 2: $B = 5$: $\log_{10}(1 - G(y))$ (top), and $G_3(y)$, $G_5(y)$ and $G_7(y)$ (bottom) as functions of y

It is also not difficult to find the average amount of fluid sent into the buffer per unit of time, or fluid input rate $N\tau^\star$:

$$N\tau^\star \;=\; \sum_{j=0}^{10} jr(p_j - G_j^{(r)}(B^*)) + jpG_j^{(p)}(B^*-) + C(G_j^{(r)}(B^*) - G_j^{(p)}(B^*-)) = 10.2656, \quad (39)$$

where p_j is defined as in Section 4, i.e. $p_j = \mathbb{P}(Y = j, W \leq B)$. Similarly we find the average amount of fluid sent over the link per unit of time, or throughput, as

$$N\tau = \sum_{j=0}^{10} C(p_j - G_j^{(p)}(0)) + jpG_j^{(p)}(0) = 10.2651. \quad (40)$$

Notice that the numerical outcomes for these quantities are close to each other and to (38), which can be explained by the fact that we chose the (finite) size of the buffer quite large. Indeed by subtracting (40) from (39) we immediately find that the average amount of lost fluid per unit of time, or fluid loss rate,

$$\text{Fluid loss rate} = N\tau^\star - N\tau = 5.83 \cdot 10^{-4}, \quad (41)$$

is small. Another way to find this result is to use

$$\text{Fluid loss rate} = \sum_{j=0}^{10} (jr - C)(p_j - G_j^{(r)}(B)). \quad (42)$$

The reason that (41) and (42) yield the same outcome is that we must have

$$\sum_{j=0}^{10} r_j^{(r)}(G_j^{(r)}(B) - G_j^{(r)}(B^*)) + r_j^{(p)}(G_j^{(p)}(B^*-) - G_j^{(p)}(0)) = 0, \quad (43)$$

expressing that the total drift in the fluid buffer should be zero. The fraction of fluid that is lost can be found as the ratio of the fluid loss rate (41) and the fluid input rate (39):

$$\text{Fraction of fluid lost} = \frac{\text{Fluid loss rate}}{\text{Fluid input rate}} = 5.67 \cdot 10^{-5}. \quad (44)$$

6.2 Maximization of system throughput

With the parameters of Section 6.1, in Figures 3 and 4 we plot the throughput and the loss fraction as functions of the threshold level for several numbers of sources: $N = 7, \ldots, 12$. We first consider the problem of selecting B^* such that the throughput is maximized, for given N, and maximum loss fraction $\epsilon = 10^{-6}$. Observe from Figure 3 that the throughput is monotonic increasing with the threshold B^* for fixed N. With Figure 4 we obtain the value of B^* that corresponds to ϵ.
As a final application we show how the threshold level B^* and the number of sources N may be jointly chosen such that the system throughput is maximized, again with the fraction of lost fluid not to exceed $\epsilon = 10^{-6}$. From Figure 4 it can be seen that only for $N \leq 9$ the loss fraction criterion can be satisfied for some value of B^*. From Figure 3 we compare the corresponding throughputs for these pairs of (N, B^*).

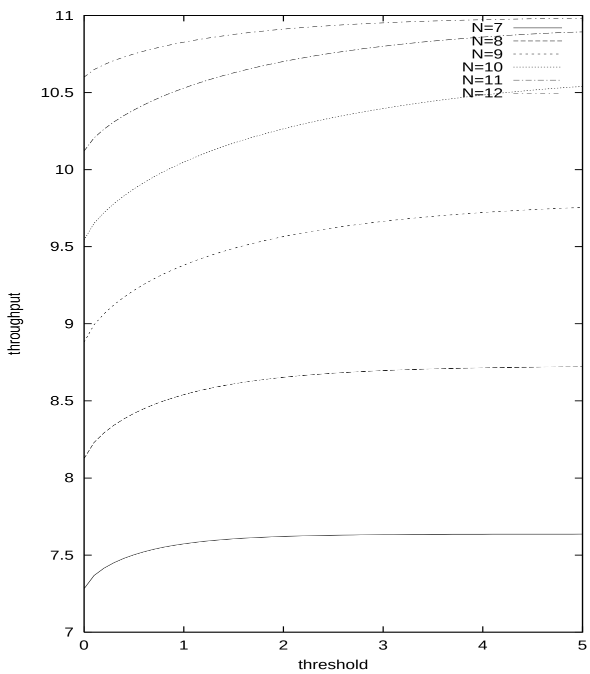


Figure 3: Throughput as a function of threshold B^* for $N = 7\,(\text{bottom}), \ldots, 12\,(\text{top})$

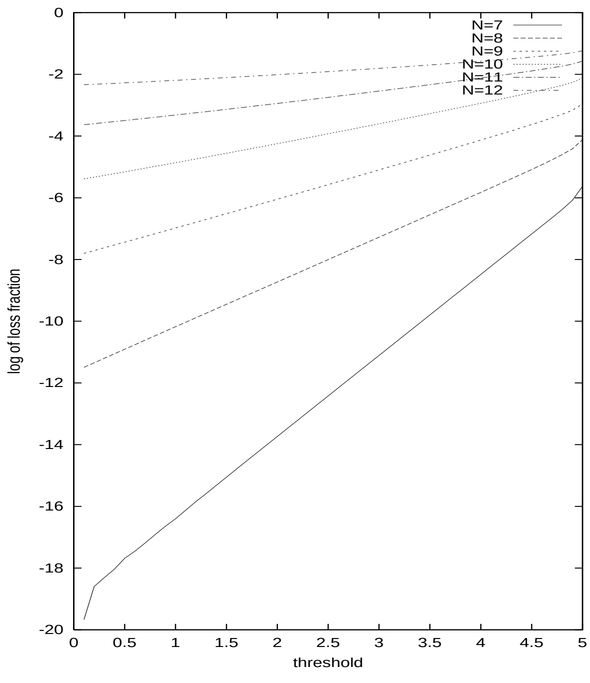


Figure 4: $\log_{10}(\text{loss fraction})$ as a function of threshold B^* for $N = 7\,(\text{bottom}), \ldots, 12\,(\text{top})$

It turns out that the system throughput is maximized by choosing $N = 9$ and $B^* = 2.05$, giving a throughput of 9.5725. This also allows us to compute the mean file transfer time, see (12): solve $\mathbb{E}T$ from

$$\frac{9.5725}{9} = \frac{0.5000}{\mathbb{E}T + 0.3333},$$

giving $\mathbb{E}T = 0.1368$. During the active period a source's throughput is $0.5000/0.1368 = 3.6526$, which is between $r = 2$ and $p = 4$.

6.3 Impact of the choice of the guaranteed rate r

The purpose of this subsection is to provide an illustration of the effect of the feedback mechanisms proposed in a more practical situation. For reasons of convenience, we assume that the number of sources is sufficiently large to use the approximations of Section 5. We consider a 45 Mbit/s link, with a buffer of 10 Mbit. The sources have peak rates of 3 Mbit/s. The users alternately send files (exponentially distributed with mean 100 kbit) and 'think' (for an exponentially distributed time with mean 10 seconds). We again require that the loss probability is below $\epsilon = 10^{-6}$. We focus on the impact of r, the guaranteed minimum throughput advertised to customers, because this is an important parameter.

Clearly, for a fixed value of B^* (possibly 0), the loss probability increases with r. On the other hand, the number of admissible sources $N_r(\epsilon)$ decreases in r, for fixed loss probability $\epsilon = 10^{-6}$. In Figure 5, $N_r(\epsilon)$ is given as a function of r, for different values of B^* ($B^* = 0, 2, \ldots, 8$ Mbit). We observe that for low values of r, $N_r(\epsilon)$ is quite sensitive to the threshold value B^*; the opposite is true for r in the neighborhood of p.

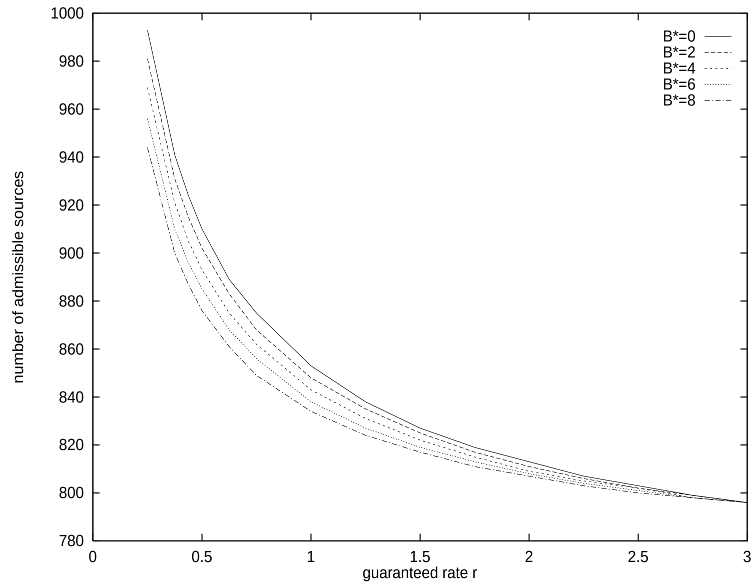


Figure 5: Number of admissible sources as a function of the guaranteed rate r.

7 Conclusions

This paper has analyzed schemes for feedback-based adaptation of transmission rates to an access multiplexer. The emphasis is on the modeling and development of computational methods. A central design problem that arises from our model of network access is the joint optimization of the feedback and admission controls for given maximum loss fraction. Our numerical investigations have shown that their are procedural simplifications and short cuts. An important direction of future theoretical research is to determine the general conditions under which the simplified procedure is optimal. Yet another important direction of future work is in generalizing the results to multiple classes of traffic sources.

References

[1] I. Adan, E. van Doorn, J. Resing, and W. Scheinhardt. Analysis of a single-server queue interacting with a fluid reservoir. *Queueing Systems*, 29: 313 – 336, 1998.

[2] D. Anick, D. Mitra, and M. Sondhi. Stochastic theory of a data-handling system with multiple sources. *The Bell System Technical Journal*, 61: 1871 – 1894, 1982.

[3] D. Botvich and N. Duffield. Large deviations, the shape of the loss curve, and economies of scale in large multiplexers. *Queueing Systems*, 20: 293 – 320, 1995.

[4] G. Choudhury, D. Lucantoni, and W. Whitt. Squeezing the most out of ATM. *IEEE Transactions on Communications,* 44: 203 – 217, 1996.

[5] J. Cohen. The multiple phase service network with generalized processor sharing. *Acta Informatica*, 12: 245 – 284, 1979.

[6] C. Courcoubetis and R. Weber. Buffer overflow asymptotics for a buffer handling many traffic sources. *Journal of Applied Probability*, 33: 886 – 903, 1996.

[7] A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications.* Jones and Bartlett, Boston, 1993.

[8] N. Duffield. Conditioned asymptotics for tail probabilities in large multiplexers. *Performance Evaluation,* 31: 281 – 300, 1998.

[9] A. Elwalid, D. Heyman, T. Lakshman, D. Mitra, and A. Weiss. Fundamental bounds and approximations for ATM multiplexers with applications to videoconferencing. *IEEE Journal on Selected Areas in Communications,* 13: 1004 – 1016, 1995.

[10] A. Elwalid and D. Mitra. Analysis and design of rate-based congestion control of high-speed networks, I: stochastic fluid models, access regulation, *Queueing Systems*, 9: 29 – 64, 1991.

[11] A. Elwalid and D. Mitra. Statistical multiplexing with loss priorities in rate-based congestion control of high-speed networks. *IEEE Transactions on Communications,* 42: 2989 – 3002, 1994.

[12] R. Gibbens and F.P. Kelly. Resource pricing and the evolution of congestion control. *Automatica*, 35: 1969 – 1985, 1999.

[13] R. Gibbens and F.P. Kelly. Distributed connection acceptance control for a connectionless network. In: *Proceedings ITC 16: Teletraffic engineering in a competitive world,* eds. P. Key and D. Smith, 941 – 952. Elsevier, Amsterdam, the Netherlands, 1999.

[14] P. Glynn and W. Whitt. Logarithmic asymptotics for steady-state tail probabilities in a single-server queue. *Journal of Applied Probability,* 31A: 131 – 156, 1994.

[15] R. Howard. *Dynamic probabilistic systems, Vol. 1.* Wiley, New York, NY, USA, 1971.

[16] F.P. Kelly. Models for a self-managed Internet. *Philosophical Transactions of the Royal Society*, A358: 2335 – 2348, 2000.

[17] L. Kosten. Stochastic theory of data-handling systems with groups of multiple sources. In: *Performance of Computer-Communication Systems*, eds. H. Rudin and W. Bux, 321 – 331. Elsevier, Amsterdam, the Netherlands, 1984.

[18] M. Mandjes. Overflow asymptotics for large communication systems with general Markov fluid sources. *Probability in the Engineering and Informational Sciences*, 10: 501 – 518, 1996.

[19] M. Mandjes and J.H. Kim. Large deviations for small buffers: an insensitivity result. *Queueing Systems*, 37: 349 – 362, 2001.

[20] M. Mandjes and A. Ridder. Optimal trajectory to overflow in a queue fed by a large number of sources. *Queueing Systems*, 31: 137 – 170, 1999.

[21] M. Mandjes and A. Weiss. Sample path large deviations of a multiple time-scale queueing model. Submitted.

[22] L. Massoulié and J. Roberts. Arguments in favour of admission control for TCP flows. In: *Proceedings ITC 16: Teletraffic engineering in a competitive world,* eds. P. Key and D. Smith, 33 – 44. Elsevier, Amsterdam, the Netherlands, 1999.

[23] D. Mitra, Stochastic theory of a fluid model of producers and consumers coupled by a buffer, *Advances in Applied Probability*, 20: 646– 676, 1988.

[24] R. Pazhyannur and R. Agrawal. Feedback-based flow control of B-ISDN/ATM networks. *IEEE Journal in Selected Areas in Communications,* 13: 1252 – 1266, 1995.

[25] K. Ramanan and A. Weiss. Sharing bandwidth in ATM. In: *Proceedings Allerton Conference*, 732 – 740, 1997.

[26] J. Roberts. Engineering for Quality of Service. In: *Self-Similar Network Traffic and Performance Evaluation*, eds. K. Park and W. Willinger. Wiley-Interscience, 2000.

[27] J. Roberts and S. Oueslati-Boulahia. Quality of Service by flow aware networking. *Philosophical Transactions of The Royal Society*, A358: 2197 – 2207, 2000.

[28] W. Scheinhardt. *Markov-modulated and feedback fluid queues.* Available at Internet page: http://www.ub.utwente.nl/webdocs/tw/1/t0000008.pdf. Ph.D. Thesis, University of Twente, the Netherlands, 1998.

[29] W. Scheinhardt. Analysis of feedback fluid queues. In: *Proceedings 14th ITC-specialists seminar on access networks and systems,* 215 – 220, Girona, Spain, April 25-27, 2001.

[30] M. Schwartz. *Broadband Integrated Networks.* Prentice Hall, Upper Saddle River, NJ, USA, 1996.

[31] A. Shwartz and A. Weiss. *Large deviations for performance analysis, queues, communication, and computing.* Chapman and Hall, New York, NY, USA, 1995.

[32] A. Simonian and J. Guibert. Large deviations approximation for fluid queues fed by a large number of on/off sources *IEEE Journal of Selected Areas in Communications*, 13: 1017 – 1027, 1995.

[33] A. Weiss. A new technique of analyzing large traffic systems. *Advances in Applied Probability*, 18: 506 – 532, 1986.