Fundamental Bounds and Approximations for ATM Multiplexers with Applications to Video Teleconferencing

Anwar Elwalid*, Daniel Heyman†, T. V. Lakshman†, Debasis Mitra* and Alan Weiss*

∗ AT&T Bell Laboratories
600 Mountain Avenue
Murray Hill, NJ 07974
USA

† Bell Communications Research
331 Newman Springs Road
Red Bank, NJ 07701
USA

Abstract

The main contributions of this paper are two-fold. First, we prove fundamental, similarly behaving lower and upper bounds, and give an approximation based on the bounds, which is effective for analyzing ATM multiplexers, even when the traffic has many, possibly heterogeneous, sources and their models are of high dimension. Second, we apply our analytic approximation to statistical models of video teleconference traffic, obtain the multiplexing system's capacity as determined by the number of admissible sources for given cell loss probability, buffer size and trunk bandwidth, and, finally, compare with results from simulations, which are driven by actual data from coders. The results are surprisingly close. Our bounds are based on Large Deviations theory. The main assumption is that the sources are Markovian and time-reversible. Our approximation to the steady state buffer distribution is called "Chernoff-Dominant Eigenvalue" since one parameter is obtained from Chernoff's theorem and the other is the system's dominant eigenvalue. Fast, effective techniques are given for their computation. In our application we process the output of variable bit rate coders to obtain DAR(1) source models which, while of high dimension, require only knowledge of the mean, variance and correlation. We require cell loss probability not to exceed 10^{-6}, trunk bandwidth ranges from 45 to 150 Mbps, buffer sizes are such that maximum delays range from 1 to 60 msec, and the number of coder-sources ranges from 15 to 150. Even for the largest systems, the time for analysis is a fraction of a second, while each simulation takes many hours. Thus the real-time administration of admission control based on our analytic techniques is feasible.

1. INTRODUCTION

Research on the architecture and design of ATM systems has in recent times been stymied by the inability to effectively analyze multiplexers when the traffic has many, possibly heterogeneous, sources and the dimensions of their models are high. Secondly, there is a growing gap between measurements and models, more generally between real systems and their purported analyses; as a corollary, there are few checks on the efficacy of designs of real systems. The widespread acceptance of ATM and the accompanying richness of services and applications are accentuating these problems.

This paper provides some relief for both these troubling conditions. We prove fundamental upper and lower bounds on loss probabilities in buffered multiplexing systems with time-reversible Markovian sources, which provably mirror true behavior for the full spectrum of buffer sizes. We propose an approximation for all Markovian traffic sources which is based on the upper bound. We call this the Chernoff-Dominant Eigenvalue (CDE) method since it has only two parameters, one of which is obtained from Chernoff's theorem and the other is the multiplexing system's dominant eigenvalue. Both these quantities have separately been studied extensively in the past. Fast, effective techniques are available for their computation, even for heterogeneous, high-dimensioned source models. In the case of discrete-time systems, which are of particular importance here since video teleconference traffic is framed, we fill a gap in the literature by obtaining an explicit, scalar, monotonic function whose root, which is easily calculated, is the dominant eigenvalue.

The complementary part of the paper starts with the measured output of video teleconference coders. The study then proceeds along two paths, as sketched in Figure 1.1. In

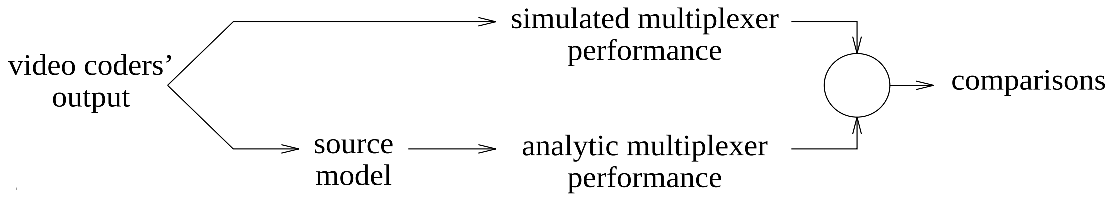

Figure 1.1: Methodology

the top "simulation" path, the output from several coders is supplied to a simulated finite multiplexing buffer, and the losses monitored. In the bottom "analytic" path the coders' output is used to define a Markovian (DAR(1)) source model, which is both high order ($\approx$ 60 states for each source) and parametrically parsimonious. Only the mean, variance, correlation of the data and the range of the number of cells per frame are required to define the source model, which can therefore be done quickly and easily. The CDE method is then used to analyze the multiplexer performance. The comparisons of the end results are in terms of system capacity. For an upper bound on the cell loss probability of about 10^{-6}, buffer size B and trunk capacity or bandwidth C, we obtain for each of the two paths the capacity of the system as measured by the maximum number of admissible video teleconference sources. The results are surprisingly close. It may therefore be reasonably inferred that our approximation technique is tight, and that our modelling of the available video teleconferencing traffic data by DAR(1) is effective.

The systems investigated have a broad range. The trunk bandwidth C ranges from 45 to 300 Mbps, the buffer sizes are such that the corresponding maximum delays range from 1 to 60 msec, and, importantly, the number of coder-sources ranges from 15 to 150. Even for the largest systems the time required on a standard workstation for analysis is a fraction of a second, while each simulation takes many hours. Thus, importantly, our analytic techniques can be implemented fast enough for real-time administration of admission control based on the techniques to be feasible.

Our new mathematical results include upper and lower bounds on the probability of buffer overflow, which have similar behavior over the full range of buffer sizes B. The techniques used to arrive at these results are from Large Deviations theory. Specifically, the upper and lower bounds correspond to lower and upper bounds on the large deviations rate function related to buffer overflows. In the homogeneous model there are K identical Markovian traffic sources of arbitrary, but finite, dimensions. The results show that if b denotes the buffer capacity per source, i.e., $b = B/K$, and $W(t)$ represents the buffer occupancy at time t, there exist easily calculated positive constants, C_1 and C_2, such that for every $b > 0$,

$$\lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}(W(t) \geq Kb) \leq -C_2 b - C_1. \tag{1.1}$$

It is also shown that the constants are the best possible since the inequality is tight at the two limits, $b \to 0$ and $b \to \infty$. Furthermore, the companion lower bound to (1.1) behaves as $-C_2 b - C_3$, for all $b \geq b_0$, where b_0 and C_3 are positive constants. The main assumption that is made in proving the above results is that the sources are Markovian and time-reversible. The sources in the video teleconference application are Markovian and time-reversible.

Consider the following conventional estimate of the stationary overflow probability of a buffer of size B derived from an infinite buffer analysis:

$$G(B) = \lim_{t\to\infty} \mathbb{P}(W(t) \geq B). \tag{1.2}$$

Based on our bounds experience with numerical experiments, we propose the following approximation for systems with general Markovian sources:

$$G(B) \approx e^{-KC_1} e^{-C_2 B}. \tag{1.3}$$

Note the connection to (1.1). We have used this approximation for systems with high dimensional Markovian sources that are time-reversible and irreversible, and found it to be effective in both cases. In (1.3) we let $L = e^{-KC_1}$ and $z = -C_2$, so that

$$G(B) \approx Le^{zB}. \tag{1.4}$$

This form has considerable appeal since we can show that L is the loss in bufferless multiplexing as estimated from Chernoff's theorem, and z is the dominant eigenvalue in buffered multiplexers, which is known to determine the large buffer behavior in the logarithmic scale. We call the approximation in (1.4) the Chernoff-Dominant Eigenvalue method of estimating overflow probability.

In Section 2.2 we give explicit procedures, which are simple and fast, for calculating z and L for stochastic fluid models. For discrete-time systems the procedure for calculating

L is unchanged, while the theory and numerical procedures for calculating the dominant eigenvalue are developed in Section 3. These procedures are used in Section 5 to calculate the CDE approximation for the video teleconference applications.

Coffman et al. [CIK91] considered on-off, 2-state sources and gave numerical evidence to support the claim $G(0) \approx L$. (It is easy to show that $G(0) \geq L$.) Simonian and Guibert [SGU94] quote the observation in [CIK91] as partial basis for a related approximation. In [EMI94] the approximation is used for the analysis and admission control of a multi-service multiplexing system in which the services are prioritized. The approximation refines the pure exponential form e^{zB} used in effective bandwidth analyses [GAN91, GHU91, EMI93, KWC93, WHI93]. Prior studies [HUI90, KEL91, ROB92] have noted that the loss in bufferless multiplexing is very well approximated by the Chernoff large deviations approximation. Note that in typical ATM applications where cell loss probabilities are in the range $10^{-6} - 10^{-9}$, a substantial contribution is derived from the mechanism underlying bufferless multiplexing; it is not atypical for L to be in the range $10^{-3} - 10^{-5}$. Hence, the prefactor L typically adds significantly to the accuracy of the effective bandwidth approximation, which otherwise can sometimes be overly conservative [CLW94]. It should also be noted that other approaches for improving the exponential bound are in [SEL91], [CLW94] and [DUF93].

The paper is organized as follows. Section 2 gives the fundamental bounds from Large Deviations theory. Section 3 considers discrete-time, discrete-state space systems. Section 4 gives the statistical model of teleconference traffic. Section 5 reports on numerical results from simulations and analyses.

2. BOUNDS AND APPROXIMATIONS FOR MULTIPLEXERS

2.1. Bounds for time-reversible buffered systems

In this section we obtain an upper bound on the probability of buffer overflow in a class of models of buffered multiplexers. This upper bound is equivalent to a lower bound on the large deviations rate function related to buffer overflows. We concentrate here on the case of homogeneous sources; the extension of the main results to heterogeneous sources is straightforward and stated in a subsequent section. The models have K traffic sources, trunk capacity or bandwidth C, a constant, and a buffer of size B. Also, b and c are respectively the buffer and trunk capacity per source; i.e., $B = bK$ and $C = cK$. Standard arguments for Markovian traffic sources show that there is a positive constant C_2 such that, if $W(t)$ represents the total buffer occupancy at time t, for each fixed K,

$$\lim_{b\to\infty} \frac{1}{Kb} \log \mathbb{P}(W(t) \geq Kb) = -C_2. \tag{2.1}$$

We show that there exists an easily calculated constant $C_1 > 0$ such that for every $b > 0$,

$$\lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}(W(t) \geq Kb) \leq -C_2 b - C_1. \tag{2.2}$$

The constants are the best possible, since we have

$$\lim_{b\downarrow 0} \lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}(W(t) \geq Kb) = -C_1 \tag{2.3}$$

and also

$$\lim_{b\to\infty} \lim_{K\to\infty} \frac{1}{Kb} \log \mathbb{P}(W(t) \geq Kb) = -C_2 \tag{2.4}$$

Furthermore, we derive a similar but less explicit lower bound:

$$\lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}(W(t) \geq Kb) \geq -f(b), \tag{2.5}$$

where $f(b)$ is given by a somewhat complicated formula, but for some fixed b_0 we have

$$f(b) = C_2 b + C_3 \tag{2.6}$$

for all $b \geq b_0$, where C_3 is a constant that is again given by a somewhat complicated formula. We can easily show the obvious bound $C_3 > C_1$.

An individual source is characterized by $(\mathbf{Q}, \mathbf{R})$ and has state space $(1, 2, \ldots, d)$. The $d \times d$ matrix generator $\mathbf{Q} = \{Q_{i,j}\}$, where $Q_{i,j}$ (for $i \neq j$) is the rate at which a source in state i jumps to state j, and $Q_{i,i} = -\sum_{j\neq i} Q_{i,j}$. The vector $\mathbf{R} = (R_1, R_2, \ldots, R_d)$, where R_i is the rate at which a source in state i generates traffic. The K traffic sources are statistically identical and independent Markov jump processes. To describe the aggregate behavior of the sources we encode each state of an individual source in a different dimension as follows: the vector $\mathbf{q}(t) \in \mathcal{Z}^d$, wherein the component $q_i(t)$ denotes the number of sources in state i at time t. A source jumping from state i to state j causes a transition of $\mathbf{q}(t)$ in direction $\mathbf{e}_j - \mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector in direction i $(i = 1, 2, \ldots, d)$. The rate of these jumps is $q_i Q_{i,j}$, because there are q_i sources in state i, each jumping at rate $Q_{i,j}$. Therefore $\mathbf{q}(t)$ is a Markov process with infinitesimal generator

$$L\phi(\mathbf{q}) = \sum_{i,j} q_i Q_{i,j} \left(\phi(\mathbf{q} + \mathbf{e}_j - \mathbf{e}_i) - \phi(\mathbf{q})\right). \tag{2.7}$$

L is an operator on functions $\phi : \mathbb{R}^d \to \mathbb{R}^1$. We make two assumptions on the process $\mathbf{q}(t)$:

1. $\mathbf{q}(t)$ is time-reversible. 2. $\mathbf{q}(t)$ is irreducible.

In fact, we can eliminate Assumption 2, which we have included only to make a few arguments simpler. We do use Assumption 1 in crucial ways for our proof, though. We do not know whether or not this assumption is necessary for our results.

The final part of the buffer model is the rate at which the buffer (whose content is denoted $W(t)$) drains. We assume that the buffer drains with rate at most C:

$$\frac{d}{dt} W(t) = \begin{cases} \langle \mathbf{R}, \mathbf{q}(t)\rangle - C & \text{if } \langle \mathbf{R}, \mathbf{q}(t)\rangle - C > 0 \text{ or } W(t) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \sum_i x_i y_i$.

We make the standard large deviations scaling of the process $\mathbf{q}(t)$ as follows.

$$\mathbf{z}_K(t) \triangleq \frac{\mathbf{q}(t)}{K}. \tag{2.9}$$

Then $\mathbf{z}_K(t)$ is a Markov process whose components represent the fraction of sources in each state. The generator of $\mathbf{z}_K(t)$ is L_K, given by

$$L_K \phi(\mathbf{x}) = \sum_{i,j} K x_i Q_{i,j} \left(\phi(\mathbf{x} + (\mathbf{e}_j - \mathbf{e}_i)/K) - \phi(\mathbf{x}) \right) \tag{2.10}$$

The generator is defined for points $\mathbf{x} \in \mathcal{S}_d$, where $\mathcal{S}_d$ is the set of probability vectors in $\mathbb{R}^d$:

$$\mathcal{S}_d \triangleq \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ x_i \geq 0, \ \sum_i x_i = 1 \right\}. \tag{2.11}$$

We now define the large deviations local rate function $\ell(\mathbf{x}, \mathbf{y})$ associated with the process $\mathbf{z}_K(t)$:

$$\ell(\mathbf{x}, \mathbf{y}) \triangleq \sup_{\mathbf{s} \in \mathbb{R}^d} \left(\langle \mathbf{s}, \mathbf{y} \rangle - \sum_{i,j} Q_{i,j} x_i \left(e^{\langle \mathbf{e}_j - \mathbf{e}_i, \mathbf{s} \rangle} - 1 \right) \right). \tag{2.12}$$

$\ell(\mathbf{x}, \mathbf{y})$ is defined for $\mathbf{x} \in \mathcal{S}_d$ and for $\mathbf{y}$ with $\sum_i y_i = 0$, which is a condition satisfied by the difference of probability vectors. Intuitively, $\ell(\mathbf{x}, \mathbf{y})$ represents the negative logarithm of the local probability of the process $\mathbf{z}_K(t)$ traveling in direction $\mathbf{y}$. For example, we can show that as $K \to \infty$,

$$\mathbb{P}_{\mathbf{x}} \left(\sup_{0 \leq t \leq \Delta} |\mathbf{z}_K(t) - (\mathbf{x} + t\mathbf{y})| < \varepsilon \right) = e^{-K\ell(\mathbf{x}, \mathbf{y}) + O(K\Delta) + o(n)}. \tag{2.13}$$

Here $\mathbb{P}_{\mathbf{x}}$ refers to sample paths $\mathbf{z}_K(t)$ that start at the point $\mathbf{x}$. A more precise and general statement than (2.13), is the following statement of the large deviations principle. For any open set of paths $\mathcal{G}$ and for any closed set of paths $\mathcal{F}$, we have the following limits:

$$\liminf_{K \to \infty} \frac{1}{K} \log \mathbb{P}(\mathbf{z}_K \in \mathcal{G}) \geq - \inf_{\mathbf{r} \in \mathcal{G}} I_0^T(\mathbf{r}) \tag{2.14}$$

$$\limsup_{K \to \infty} \frac{1}{K} \log \mathbb{P}(\mathbf{z}_K \in \mathcal{F}) \leq - \inf_{\mathbf{r} \in \mathcal{F}} I_0^T(\mathbf{r}) \tag{2.15}$$

where

$$I_0^T(\mathbf{r}) \triangleq \int_0^T \ell(\mathbf{r}(t), \mathbf{r}'(t)) dt \tag{2.16}$$

The function $I_0^T(\mathbf{r})$ is called the rate function; ℓ is the local rate function. For more information about the rate functions or the large deviations principle, see Varadhan [VAR84], Dembo and Zeitouni [DZE93], Freidlin and Wentzell [FWE84], or Shwartz and Weiss [SWE95].

We can also write the buffer content $W(t)$ as an integral. Let

$$s(t) \triangleq \arg\sup_{u \le t} \int_u^t (\langle \mathbf{R}, \mathbf{q}(t) \rangle - C)\, dt. \tag{2.17}$$

That is, $s(t)$ is the last time before t that the buffer is empty:

$$s(t) \triangleq \sup \{ u \;:\; u \le t,\; W(u) = 0 \}. \tag{2.18}$$

Then we have the following representation of $W(t)$:

$$W(t) = \int_{s(t)}^t (\langle \mathbf{R}, \mathbf{q}(t) \rangle - C)\; dt. \tag{2.19}$$

We define the operator $\mathbf{B}(\mathbf{q})(t)$ as the map giving the function $W(t)$ from a path $\mathbf{q}(t)$, using either of the equivalent definitions (2.8) or (2.19). That is,

$$W(t) = \mathbf{B}(\mathbf{q})(t). \tag{2.20}$$

We also have a scaled buffer occupancy

$$w(t) \triangleq \frac{W(t)}{K}. \tag{2.21}$$

This can be obtained by a transformation on $\mathbf{z}_K(t)$ as follows:

$$w(t) = \int_{s(t)}^t (\langle \mathbf{R}, \mathbf{z}_K(t) \rangle - c)\; dt. \tag{2.22}$$

We define the scaled operator $\mathbf{B}_s(\mathbf{z}_K)$ by (2.22), namely, $w(t) = \mathbf{B}_s(\mathbf{z}_K)(t)$.

The "center" of the process $\mathbf{p}$ is defined to be the unique limit of the solution of the fluid equation for the scaled process $\mathbf{z}_\infty(t)$:

$$\frac{d}{dt}\mathbf{z}_\infty(t) = \sum_{i,j} z_{\infty,i}(t) Q_{i,j}(\mathbf{e}_j - \mathbf{e}_i) = \mathbf{z}_\infty(t)\mathbf{Q} \tag{2.23}$$

where $z_{\infty,i}(t)$ means the i^{th} component of $\mathbf{z}_\infty(t)$. That is, we define

$$\mathbf{p} \triangleq \mathbf{z}_\infty(\infty) \tag{2.24}$$

Assumption 2 assures the uniqueness of $\mathbf{p}$. In fact, for the class of models considered here, $\mathbf{p}$ is identical to $\boldsymbol{\pi}$, where the component π_i represents the unique stationary probability that a single source is in state i. That is, $\boldsymbol{\pi} \in \mathcal{S}_d$ and

$$\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}. \tag{2.25}$$

We now prove (2.2) and give explicit expressions for the constants C_1 and C_2:

$$C_1 = \inf_{\mathbf{r},\mathbf{v},T \in F} \int_0^T \ell(\mathbf{r}, \mathbf{r}')\, dt, \qquad C_2 = \inf_{\mathbf{r} \in \mathcal{S}_d \,:\, \langle \mathbf{r}, \mathbf{R} \rangle > c} \frac{\ell(\mathbf{r}, \mathbf{0})}{\langle \mathbf{r}, \mathbf{R} \rangle - c} \tag{2.26}$$

where

$$F \triangleq \{\mathbf{r}, \mathbf{v}, T \ : \ \mathbf{r}(0) = \pi, \ \langle \mathbf{v}, \ \mathbf{R} \rangle = c, \ \mathbf{r}(T) = \mathbf{v}\} \tag{2.27}$$

and the function ℓ is defined by (2.12). Furthermore, we have a much more explicit expression for C_1:

$$C_1 = \inf_{\mathbf{x} \in H(c)} \ell_1(\mathbf{x}) \tag{2.28}$$

where $\ell_1(\mathbf{x})$ is the rate function for a multinomial random variable:

$$\ell_1(\mathbf{x}) \triangleq \sum_{i=1}^{d} \mathbf{x}_i \log \frac{\mathbf{x}_i}{\pi_i} \tag{2.29}$$

where π_i is the steady state probability that a source is in state i, and

$$H(c) \triangleq \{\mathbf{x} \in \mathcal{S}_d \ : \ \langle \mathbf{x}, \mathbf{R} \rangle = c\} . \tag{2.30}$$

Here is a precise statement of our main result.

Theorem 2.1 *Suppose that the underlying process $\mathbf{q}(t)$ is reversible and irreducible. Let C_1 be defined by (2.28), let C_2 be defined by (2.26), and let $\mathbf{p} = \boldsymbol{\pi}$, which is defined by (2.25). Define*

$$I^*(b) \triangleq \inf_{(\mathbf{r},T) \in \mathcal{G}(b)} I_0^T(\mathbf{r}), \tag{2.31}$$

where

$$\mathcal{G}(b) = \{\mathbf{r}, T \ : \ \mathbf{r}(-\infty) = \mathbf{p}, \ \mathbf{B}_s(\mathbf{r})(T) = b\} . \tag{2.32}$$

Then for each $b > 0$,

$$I^*(b) \geq C_1 + C_2 b. \tag{2.33}$$

The equivalence of C_1 as defined by (2.26) and by (2.28) comes from the same calculations that show that Chernoff's theorem is equivalent to Sanov's theorem; see, e.g., [SWE95, Chapter 2].

$I_0^T(\mathbf{r})$ should be thought of as a cost. It is the cost for the process $\mathbf{z}_K(t)$ to follow the path $\mathbf{r}(t)$. The cost is related to probability by the large deviations principle:

$$\mathbb{P}(\sup_{0 \leq t \leq T} |\mathbf{z}_K(t) - \mathbf{r}(t)| < \varepsilon) = e^{-K I_0^T(\mathbf{r}) + o(K)}. \tag{2.34}$$

The probability of an event is related to the cheapest cost of the paths $\mathbf{r}$ that cause the event to occur. That is, to calculate the frequency of an unlikely event's occurrence, think of all the different ways that it might occur, calculate the cost of each way, and take the cheapest cost I^*. Then the probability of the event is approximated by $\exp(-K I^*)$.

The idea behind the bound in Theorem 2.1 is the following. The quantity C_1 represents the lowest cost for going from the point $\mathbf{p}$ to the point, say $\mathbf{v}^*$, where the buffer begins to fill. See Figure 2.2. The quantity C_2 represents the lowest possible cost per unit buffer for a

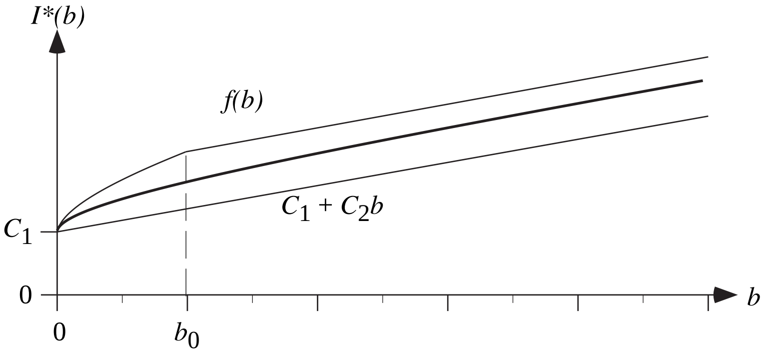

Figure 2.1: The cost function $I^*(b)$ and our bounds. $I^*(b)$ was calculated numerically for a two state model. $f(b)$ is linear and parallel to $C_1 + C_2 b$ for $b > b_0$.

path that doesn't move. The lowest cost of achieving a buffer occupancy b should be larger than $C_1 + C_2 b$, since any path that makes the buffer occupancy reach b will have to cross to the place where the buffer begins to fill, and then will have to make the buffer fill to b, but will also have to be a nearly continuous path, so it can't be near all the minima all the time.

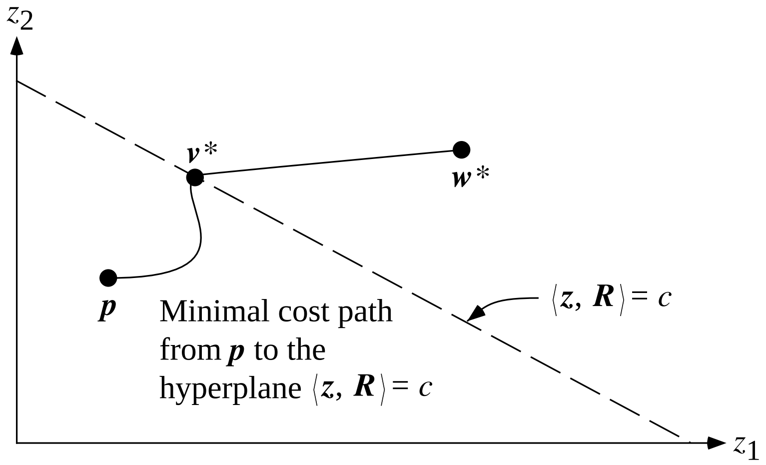

Figure 2.2: A path $\mathbf{r}_b(t)$ for Theorem 2.3.

We give a related result. Suppose that

$$C_2 = \frac{\ell(\mathbf{w}^*, \mathbf{0})}{\langle \mathbf{w}^*, \mathbf{R} \rangle - c} \tag{2.35}$$

for a unique point $\mathbf{w}^* \in \mathcal{S}_d \ : \ \langle \mathbf{w}^*, \mathbf{R} \rangle > c$. Define

$$A \triangleq \frac{1}{\langle \mathbf{w}^*, \mathbf{R} \rangle - c}. \tag{2.36}$$

Given $\varepsilon > 0$ and T, define

$$g_\varepsilon(b, T) = \frac{1}{bA} \int_{T-bA}^{T} \mathbf{1}[|\mathbf{z}_K(t) - \mathbf{w}^*| < \varepsilon] \, dt. \tag{2.37}$$

Theorem 2.2 *For each $\varepsilon > 0$ there is a $\delta > 0$ such that for any T,*

$$\lim_{K,b\to\infty} \mathbb{P}_{ss}(g_\varepsilon(b,T) > 1-\varepsilon \mid w_K(T) \geq b) = 1. \tag{2.38}$$

($\mathbb{P}_{ss}$ refers to steady state probability.)

This theorem states that, if there is a unique $\mathbf{w}^*$ such that (2.35) holds, then we know exactly how the system behaves in order that the buffer occupancy reaches a high level—the system spends almost all the time just before overflow in a small neighborhood of $\mathbf{w}^*$. The proof of this theorem does not use either of the two assumptions; that is, the theorem holds for both reducible and irreversible systems.

We have a lower bound on buffer overflow probability which is a bit harder to write explicitly, but has the same form and asymptotics as the upper bound of (2.2).

Theorem 2.3 *There is a function $f(b)$ such that for every $b > 0$,*

$$I^*(b) \leq f(b) \tag{2.39}$$

Furthermore, there is a constant b_0 such that

$$\begin{aligned} f(0) &= C_1 \\ f(b) &= C_3 + C_2 b \text{ for } b \geq b_0 \end{aligned} \tag{2.40}$$

Compare the bounds on $I^*(b)$ given by (2.40) and Theorem 2.1; the discrepancy is bounded for all b.

The proofs of these results are immediate consequences of the large deviations principle for the process $\mathbf{z}_K(t)$ and of the Freidlin-Wentzell theory, plus some new lemmas. The Freidlin-Wentzell theory equates steady state probabilities with upcrossing probabilities. The large deviations principle equates upcrossing probabilities with solutions to variational problems. The variational problems are integrals of the function $\ell(\mathbf{x},\mathbf{y})$ as in (2.26). The bound in (2.33) and the result in (2.38) follow from some new lemmas, which bound the solutions to the variational problems. The bound (2.39) follows from a specific construction: consider a path that goes from $\mathbf{p}$ to the region $\langle \mathbf{z}_K, \mathbf{R}\rangle = c$, and from there to a minimizing point $\mathbf{w}^*$, which is defined in (2.35). Then the minimum of the variational problem has to have lower cost than this particular path. The bound is then established, with the function f being the cost of the particular path. The path is depicted in Figure 2.2.

We use the following result in our analysis. It is essentially due to [FAN89]. (See also [SWE93].)

Theorem 2.4 *If the process $\mathbf{z}_K(t)$ is reversible, and if Kurtz's theorem holds, then given any $\mathbf{x} \neq \mathbf{p}$ the time reversed path $\mathbf{r}(t) = \mathbf{z}_\infty^{\mathbf{x}}(-t)$ from $\mathbf{p}$ to $\mathbf{x}$ is a minimal cost path from $\mathbf{p}$ to $\mathbf{x}$. Therefore*

$$\lim_{\varepsilon\downarrow 0} \lim_{K\to\infty} \frac{1}{K} \log \mathbb{P}_{ss}(|\mathbf{z}_K(t) - \mathbf{x}| < \varepsilon) = -I_0^T(\mathbf{r}). \tag{2.41}$$

The proofs of Theorems 2.1 – 2.3 are in the Appendix.

2.2. Approximation, Numerical Procedures

As mentioned in Section 1, see (1.2), $G(B)$ denotes the estimate of the stationary overflow probability of a buffer of size B. There the following approximation was also proposed:

$$G(B) \approx e^{-KC_1} e^{-C_2 B}. \tag{2.42}$$

This approximation was shown in Section 2.1 to have attractive asymptotic properties; in our experience it is also both conservative and close to the true overflow probability in typical applications with both reversible and irreversible sources. In (2.42) we let $L = e^{-KC_1}$ and $z = -C_2$, to obtain the Chernoff-Dominant Eigenvalue approximation to the overflow probability:

$$G(B) \approx Le^{zB}. \tag{2.43}$$

From the discussion below Theorem 2.1, L is the loss in bufferless multiplexing as estimated from either Chernoff's theorem or Sanov's theorem; z is the dominant eigenvalue of the buffered multiplexer, which is known to determine the large buffer behavior of the overflow probability. (The dominant eigenvalue in stable irreducible Markovian models is always real and negative.)

The dominant eigenvalue and its calculation have been topics of central importance in most studies of statistical multiplexing based on stochastic fluid models [AMS82, KOS84, KOS86, MIT88, NRS91, SEL91, ROB92]. Here are two results quoted from [EMI93]. Define the diagonal matrix $\mathbf{R}_d = \text{diag}(R_1, R_2, \ldots, R_d)$. Observe that for z real and negative, $\left[\mathbf{R}_d - \frac{1}{z}\mathbf{Q}\right]$ is an irreducible matrix with nonnegative off-diagonal elements. Such a matrix has a real eigenvalue, called the "maximal real eigenvalue" (MRE) that is greater than the real part of all its other eigenvalues. Let

$$g(z) \triangleq MRE\left(\mathbf{R}_d - \frac{1}{z}\mathbf{Q}\right). \tag{2.44}$$

FACT 1. The dominant eigenvalue z of the homogeneous system with K sources, each source described by $(\mathbf{Q}, \mathbf{R})$, and channel capacity C is obtained by solving the equation

$$Kg(z) = C. \tag{2.45}$$

∎

Equation (2.45) is easily solved because $g(z)$ is monotonic decreasing for $z < 0$, and c lies between $\hat{R} \triangleq \max_i R_i = g(-\infty)$, and $\bar{R} = \sum_i \pi_i R_i = \lim_{z \to 0} g(z)$. Now suppose that there are J classes of sources, where each class $j \in [1, \ldots, J]$ is comprised of K_j sources characterized by $(\mathbf{Q}^{(j)}, \mathbf{R}^{(j)})$. Then we have

FACT 2. The dominant eigenvalue z of the heterogeneous system is obtained by solving

$$\sum_{j=1}^{J} K_j g^{(j)}(z) = C, \tag{2.46}$$

where

$$g^{(j)}(z) \triangleq MRE\left(\mathbf{R}_d^{(j)} - \frac{1}{z}\mathbf{Q}^{(j)}\right). \tag{2.47}$$

■

We may now turn to the procedure for calculating L, the estimate from Chernoff's theorem of the loss in bufferless multiplexing. Let $V_{j,i}(t)$ denote the rate of traffic generation by source i of class j at time t, and let $\{V_{j,i}\}$ be a collection of independent random variables where $V_{j,i}$ has the stationary distribution of $V_{j,i}(t)$. The total traffic generation has a stationary distribution given by a random variable $V = \sum_j \sum_i V_{j,i}$. Loss occurs when the total traffic generation exceeds the level C. Therefore we estimate $\mathbb{P}(V \geq C)$.

Let $\pi^{(j)}$ denote the stationary probability vector of a class j source. Then $V_{j,i}$ has moment generating function

$$M_j(s) \triangleq \mathbb{E}\left(e^{sV_{j,i}}\right) = \sum_k \pi_k^{(j)} e^{sR_k^{(j)}}. \tag{2.48}$$

Chernoff's theorem states that

$$\log \mathbb{P}(V \geq C) \leq -F(s^*), \text{ and } \log \mathbb{P}(V \geq C) = -F(s^*)\left[1 + O\left(\frac{\log C}{C}\right)\right] \tag{2.49}$$

where

$$F(s) \triangleq sC - \sum_j K_j \log M_j(s) \tag{2.50}$$

and $F(s^*) = \sup_{s \geq 0} F(s)$. Hence the estimate of the loss $L = \exp(-F(s^*))$. For $C < \max_i R_i$ it is easy to check that $F(s)$ is a strictly concave function with a unique maximum at $s^* > 0$ that can be obtained by solving $F'(s) = 0$.

For our numerical procedures we use a refinement to the estimate of $\mathbb{P}(V \geq C)$ given by (2.49) [PET65, CSE93].

FACT 3. As $C \to \infty$ with $K_j/C = O(1)$, $j = 1, \ldots, J$,

$$\mathbb{P}(V \geq C) = \frac{\exp(-F(s^*))}{s^*\sigma(s^*)\sqrt{2\pi}}\left[1 + o(1)\right] \tag{2.51}$$

where

$$\sigma^2(s) = \frac{\partial^2 \log \mathbb{E}(s^V)}{\partial s^2}. \tag{2.52}$$

■

More specifically,

$$\sigma^2(s) = \sum_{j=1}^{J} K_j \left[\frac{M_j''(s)}{M_j(s)} - \left(\frac{M_j'(s)}{M_j(s)}\right)^2\right]. \tag{2.53}$$

To summarize, we obtain L by calculating the leading term on the right hand side of the expression for $\mathbb{P}(V \geq C)$ in (2.51).

3. DISCRETE-TIME MULTIPLEXING SYSTEMS

In this section we obtain the Chernoff-Dominant Eigenvalue approximation to discrete-time Markov models. Specifically, we let the approximation for the buffer overflow be of the form (2.43), and we develop the theory and numerical procedures for computing the dominant eigenvalue, which here is given by e^z. The bufferless multiplexing loss L is obtained from Chernoff's theorem exactly as described in Section 2.2, and hence is not considered further. Prior work on the analysis of related discrete-time Markov models are in [LI91], [ZHA92], [SOH92], [KWC93]. However, we did not find the specific result of interest here in the literature.

Consider the homogeneous model in which an infinite buffer is supplied by K independent, identical sources and is serviced by a channel which transmits at most C cells in unit time. Here time is divided into units; the natural time unit in the system model in the sequel is the frame. Each source is described by an irreducible Markov chain with transition matrix $\mathbf{P}$. When the source is in state i ($i = 1, 2, \ldots, d$) at a particular time unit, R_i cells are produced in that time unit. Thus each source is characterized by $(\mathbf{P}, \mathbf{R})$. The superposition of the K sources is characterized by $\mathbf{M}$ and $\boldsymbol{\Lambda}$, where $\mathbf{R}_d = \mathrm{diag}(\mathbf{R})$,

$$\mathbf{M} = \mathbf{P} \otimes \mathbf{P} \otimes \cdots \otimes \mathbf{P} \quad \text{and} \quad \boldsymbol{\Lambda} = \mathbf{R}_d \oplus \mathbf{R}_d \oplus \cdots \oplus \mathbf{R}_d. \tag{3.1}$$

Here K copies occur in the Kronecker product and sum, so that $\mathbf{M}$ and $\boldsymbol{\Lambda}$ are $d^K \times d^K$. Note that the above representation, while not corresponding to the minimal state representation of the superposition process, is nonetheless essential for the derivation of the decomposition obtained below. We only consider stable systems for which $\overline{R} = \sum_i \pi_i R_i < C/K$, where $\boldsymbol{\pi} \in \mathcal{S}_d$ and $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$.

Let $W(t)$ denote the number of cells in the buffer at the beginning of the t^{th} time unit. Also, let $L(\mathbf{s})$ be the number of cells generated when the source state is $\mathbf{s}$; hence, $L(\mathbf{s}) \in \{\lambda_1, \lambda_2, \ldots, \lambda_{d^K}\}$, where λ_i is the i^{th} diagonal element of $\boldsymbol{\Lambda}$. The evolution of the buffer content is described by

$$W(t+1) = [W(t) + L(\mathbf{s}(t)) - C]^+ \tag{3.2}$$

where $[\,\cdot\,]^+ = \max(\,\cdot\,, 0)$. Let $p(n,k) \triangleq \lim_{t\to\infty} \mathbb{P}(W(t) = n,\ L(\mathbf{s}(t)) = \lambda_k)$ ($k = 1, 2, \ldots, d^K$), and $\mathbf{p}(n) = [p(n,1)\, p(n,2) \cdots p(n,d^K)]$ ($n = 0, 1, 2, \cdots$). The system's steady-state balance equations are

$$\begin{aligned} \mathbf{p}(0) &= \sum_{\ell, m : \lambda_m + \ell \le C} \mathbf{p}(\ell)\mathbf{M}_m \\ \mathbf{p}(n) &= \sum_{m : \lambda_m \le n - C} \mathbf{p}(n - \lambda_m + C)\mathbf{M}_m \end{aligned} \tag{3.3}$$

where $\mathbf{M}_m$ is the matrix obtained from $\mathbf{M}$ by replacing every row except the m^{th} by a row of zeros.

Assume independent solutions of (3.3) of the form

$$\mathbf{p}(n) = e^{zn}\boldsymbol{\phi} \quad (n = 0, 1, 2, \cdots). \tag{3.4}$$

Upon substitution of (3.4) into (3.3), we obtain the following eigenvalue equation,

$$e^{-zC}\boldsymbol{\phi} = \phi e^{-z\boldsymbol{\Lambda}}\mathbf{M} \tag{3.5}$$

where e^z is an eigenvalue and $\boldsymbol{\phi}$ is the corresponding eigenvector. Now, for real z, the matrix $\mathbf{A}(z) = e^{-z\boldsymbol{\Lambda}}\mathbf{M}$ is nonnegative and irreducible, hence its (Perron-Frobenius) eigenvalue of maximum modulus, which we denote by $g(z)$, is real, positive and simple. Utilizing the structure of $\mathbf{M}$ and $\boldsymbol{\Lambda}$ in (3.1), we obtain

$$\mathbf{A}(z) = (e^{-z\mathbf{R}_d}\mathbf{P}) \otimes (e^{-z\mathbf{R}_d}\mathbf{P}) \otimes \cdots \otimes (e^{-z\mathbf{R}_d}\mathbf{P}). \tag{3.6}$$

From this structure we may infer [GRA81] that

$$g(z) = \{\mu(z)\}^K, \tag{3.7}$$

where $\mu(z)$ is the Perron-Frobenius eigenvalue of $e^{-z\mathbf{R}_d}\mathbf{P}$.

The dominant eigenvalue of the multiplexing system, which dominates the behavior of $\mathbf{p}(n)$ for large n, is the largest value of e^z which satisfies (3.5). This quantity is real, positive and, for stable systems, less than unity, i.e., $z < 0$. From (3.5) and (3.7) we have,

FACT 1. The dominant eigenvalue of the discrete-time, homogeneous multiplexing system is e^z, where z is obtained by solving

$$-K\frac{\log \mu(z)}{z} = C. \tag{3.8}$$

■

Since $-\{\log \mu(z)\}/z$ is monotonic decreasing for $z < 0$, and C is bounded by $\hat{R} = \max_i R_i$ and $\overline{R}$, and C/K lies between $\hat{R}$ and $\overline{R}$, (3.8) can be solved without difficulty.

The above analysis easily extends to the case of heterogeneous sources. In particular, if there are J classes of sources, where each class $j \in [1, 2, \ldots, J]$ is comprised of K_j sources characterized by $(\mathbf{P}^{(j)}, \mathbf{R}^{(j)})$, then we have

FACT 2. The dominant eigenvalue of the heterogeneous multiplexing system is given by e^z, where z is obtained by solving

$$\sum_{j=1}^{J} K_j \left\{ \frac{-\log \mu^{(j)}(z)}{z} \right\} = C, \tag{3.9}$$

where $\mu^{(j)}(z)$ is the Perron-Frobenius eigenvalue of $e^{-z\mathbf{R}_d^{(j)}}\mathbf{P}^{(j)}$. ■

4. A STATISTICAL MODEL OF VBR-CODED TELECONFERENCE TRAFFIC

To formulate statistical models of VBR-coded teleconference traffic, we analyzed traffic from three thirty minute long video conference sequences coded using three different methods. All of the sequences show head-and-shoulders scenes with moderate motion and scene changes, and with very little camera zoom or pan. Let us call the three coding algorithms A, B, and C. Algorithm A uses intrafield/interframe DPCM coding without DCT nor motion compensation. Algorithm B is a modified version of the H.261 video coding standard. H.261 is a hybrid DPCM/DCT coding scheme with motion compensation. The modified version uses an open loop (no rate control) coding scheme with a fixed quantizer step size ($Q = 2$). Algorithm C uses a hybrid DPCM/DCT coding algorithm and is similar to algorithm B. However, it does not use motion compensation. The three algorithms also differ in some other aspects of coding, such as picture formats and entropy coding. The key differences to note are that A uses neither motion compensation nor DCT, B uses both motion compensation and DCT, and C uses DCT without motion compensation. The traffic data that we used gives the number of cells per frame. It does not specify how the cells arrive to the network within an interframe interval. Hence, we only model the number of cells per frame.

Let X_n be the number of cells in the n^{th} frame of a VBR-coded video teleconference. In [HTL92] and [HLT94] we showed that X_n has the following properties for all three coding schemes.

1. The number of cells per frame is a stationary Markov chain.
2. The marginal distribution of X_n is negative-binomial.
3. The correlation between X_n and X_{n+k} has the form ρ^k.

The probability function for the negative-binomial distribution is

$$f_k = \begin{pmatrix} k+r-1 \\ k \end{pmatrix} p^r q^k = \begin{pmatrix} -r \\ k \end{pmatrix} p^r (-q)^k, \quad k = 0, 1, \cdots. \tag{4.1}$$

The mean and variance of this distribution are

$$m = \frac{r(1-p)}{p} \quad \text{and} \quad v = \frac{r(1-p)}{p^2} \tag{4.2}$$

respectively. Here, $0 < p < 1$, $q = 1 - p$ and $r > 0$. The method of moments gives the estimates

$$\hat{p} = \frac{m_0}{v_0} \quad \text{and} \quad \hat{r} = \frac{{m_0}^2}{v_0 - m_0} \tag{4.3}$$

for p and r in terms of the observed values of m_0 and v_0; $v_0 > m_0$ is required for the estimates to make sense.

From properties (2) and (3), the only parameters that are needed to specify the X_n are the mean and variance of the marginal distribution, and the autocorrelation coefficient. From property (1), the temporal evolution of the process is completely specified once a suitable transition matrix for the Markov chain is given. Estimating the transition matrix $\mathbf{P} = (p_{ij})$ for the Markov chain modeling X_n (or some aggregation of the X_n) using

$$\hat{p}_{ij} = \frac{\text{number of transitions } i \text{ to } j}{\text{number of transitions out of } i} \tag{4.4}$$

is not practical since this has too many parameters.

To be of practical use, we would like the model to be based only on parameters which are either known at call set-up time or can be measured without introducing too much complexity in the network. Hence, we use the discrete autoregressive process of order 1, or DAR(1) process [JLE83], because it provides an easy and practical method to compute the transition matrix and gives us a model based only on three physically meaningful parameters, the mean, variance, and correlation of the offered traffic. Let $\mathbf{Q}$ be a square stochastic matrix where each row is $\mathbf{f} = (f_0, f_1, \ldots, f_M)$, where f_i $(i = 0, 1, \ldots, M-1)$ are the negative binomial probabilities in (4.1), $f_M = \sum_{m>M-1} f_m$, and M is the peak rate in cells per frame. If the peak rate is not known, any suitably large number M can be used. The matrix $\mathbf{P}$ given by

$$\mathbf{P} = \rho\mathbf{I} + (1-\rho)\mathbf{Q}, \tag{4.5}$$

where $\mathbf{I}$ is the identity matrix, has the desired properties. The rate vector $\mathbf{R} = (R_0, R_1, \ldots, R_M)$, where $R_i = i$ $(i = 0, 1, \ldots, M)$. Later we consider aggregating these states. The transition matrix in (4.5) has the property that if the current frame has i cells say, then the next frame will have i cells with probability $\rho + (1-\rho)f_i$, and will have k cells, $k \neq i$, with probability $(1-\rho)f_k$. This makes each sample path more regular than the data traces because the number of cells per frame stays constant for a mean of $((1-\rho)(1-f_i))^{-1}$ frames; this is about 100 frames with our data. This can be seen in Figure 4.1 which plots a segment of the actual data trace and a trace of the same length produced by the DAR(1) model. However, this difference between the DAR and the actual trace attenuates for an ensemble of sources as more and more sources are superposed. This is evident from Figure 4.2 which shows multiplexed traces produced using the actual traffic and using the DAR(1) model.

The stationary probability vector of the DAR(1) process is $\mathbf{f}$, i.e., $\mathbf{f} = \mathbf{fP}$. Also, detailed balance holds, i.e., $f_i P_{ij} = f_j P_{ji}$ for all i and j. Hence,

FACT. The DAR(1) process with transition matrix $\mathbf{P}$ is reversible. ∎
Consequently, all eigenvalues of $\mathbf{P}$ are real.

This DAR model was introduced in [HTL92] and was shown to accurately predict the blocking probability for a superposition of several identical sources fed to a statistical mul-

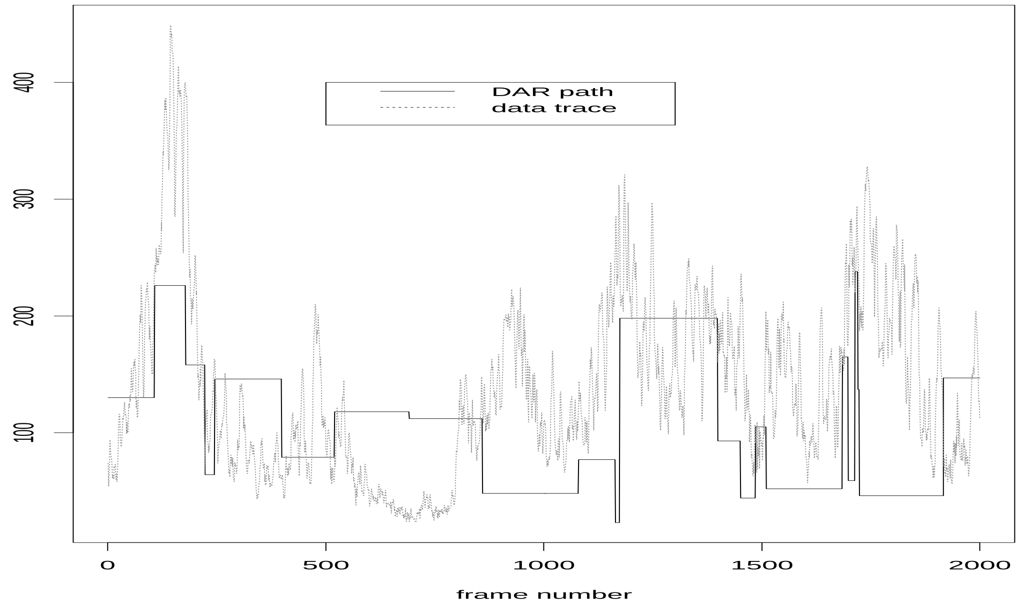

Figure 4.1: Comparison of number of cells/frame (Y-axis) for 2000 frame trace for 1 source using actual data and DAR(1) model

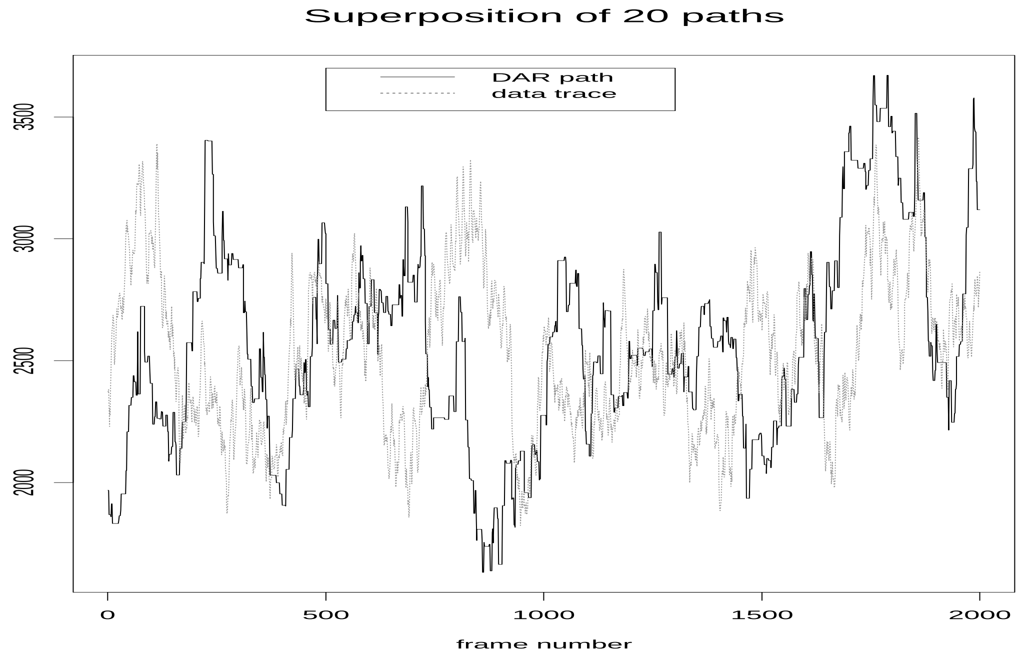

Figure 4.2: Comparison of number of cells/frame (Y-axis) for 2000 frame trace for 20 multiplexed sources using actual data and DAR(1) model

tiplexer. Algorithm C was used in that study. This result was shown to hold for algorithms A and B in [HLT94]. Further evidence that the DAR model is a good fit to video conference data is given in [LNR94] where the analyzed sequence was generated by a codec different from those used in our studies.

This three parameter model is restricted to video conferences because we have not so far been able to adequately model entertainment video sources using similar models [HLA94].

5. ANALYTIC AND SIMULATION RESULTS

We considered the following traffic engineering problem: How many simultaneous (statistically identical) video conferences can an ATM switch carry with a cell-loss rate (CLR) of about 10^{-6}? We computed this number using the Chernoff-Dominant Eigenvalue method and compared it to the number obtained by simulation. This was done for the different video sources (using the different coding schemes) described in Section 4. We found the agreement between the two numbers to be sufficiently accurate, for traffic engineering purposes, over a wide range of system characteristics.

For the analytic approximation to the buffer overflow probability with DAR(1) source models we use the expression in (2.43), where the loss in bufferless multiplexing, L, is obtained from the refinement to Chernoff's theorem, see (2.51), and the dominant eigenvalue e^z is obtained by the following result. It is assumed here that $R_0 < R_1 < \cdots < R_M$.

Theorem 5.1 *The dominant eigenvalue of the multiplexing system with K homogeneous DAR(1) sources, each described by $(\mathbf{P}, \mathbf{R})$ is e^z, where z is the unique solution to the scalar equation*

$$(1-\rho)\sum_{i=1}^{M} f_i[e^{z(R_i - C/K)} - \rho]^{-1} = 1 \tag{5.1}$$

in the interval $((\log\rho)/(R_{M-1} - C/K), (\log\rho)/(R_M - C/K))$. The function in (5.1) in this interval is monotonic decreasing.

The proof, which is omitted, exploits the structure of $\mathbf{P}$ in (4.5). For a related result see [EMI93, Section VI].

To obtain the number of admitted sources by simulation, we used the actual traffic traces giving the number of cells per frame for video teleconferences of approximately 30 minutes duration. Here, we present results for sequences A and C which are the traffic traces generated by the coding schemes A and C described in Section 4. The relevant parameters for the different sequences are shown in Table 5.1.

For sequence A, the frame rate is 30 frames per second, and the trace is 38100 frames long. Sequence C is 45000 frames long and the frame rate is 25 frames per second.

The switch is modeled as a multiplexer with a buffer whose size is determined by the maximum buffering delay. Cell arrivals from each individual source are equally spaced during the interframe interval (33.3 ms for sequence A, 40 ms for sequence B). The recorded data trace is used to generate traffic for each of the sources. Since we do not have hundreds of different 30 minute long recorded traces to simulate the different admitted sources, we use the same sequence to generate traffic for all sources. This is done by using different starting points (indices) in the trace. We found by experimentation, that with a random choice of indices the variation of the number of admitted sources with choice of indices is generally about 10 percent (excluding clearly pathological choices such as several sources having the same indices).*

Another factor which can affect cell losses is the relative phase between frames arriving from different sources. We use the same relative phases in all experiments. Also, to minimize the effect of phases found in [HTL92] we choose the phases such that the arrival instants of frames from different sources are equally spaced in the 40 ms or 33.3 ms interframe interval. (For a 40 ms interframe interval, if there are 20 sources, the first cells belonging to a new frame from each source arrive at times 2 ms, 4 ms, ...,40 ms, and every 40 ms thereafter.)

We use the following notation:

C = output rate [Mb/s]
B = buffer size [ms] (maximum delay at buffer served at rate C)
K = maximum number of sources that approximately achieve CLR = 10^{-6}

The subscripts sim and CDE denote results from simulation and results from computations using the Chernoff-Dominant Eigenvalue method.

Table 5.2 compares the number of sources using coding scheme C admitted in simulation experiment to the number admitted using the CDE method. The number admitted by simulation was determined by adding sources until the loss rate exceeded the specified bound of approximately 10^{-6}. The CDE computation was done using the DAR(1) source model with the required three parameters of the model (mean, variance, and correlation) being estimated from the traffic trace. From the table, we see that $K_{\text{CDE}} < K_{\text{sim}}$ and their ratio gets monotonically closer to one as the correctly engineered maximum number of sources gets larger. The peak-to-mean ratio for this traffic is 5, and the mean bit rate is 1.668 Mb/s. Hence, from the table it can be seen that the statistical multiplexing gain is in the 3 to 4 range.

Table 5.3 presents results using traffic coded by scheme A. We see that the number admitted using the CDE method is a very close approximation to the "true" number obtained by simulation.

*The cell loss probability is sensitive to choice of indices. However, this does not translate to large variations in the number of admitted sources because of the relatively large increments in the offered traffic when new sources are admitted.

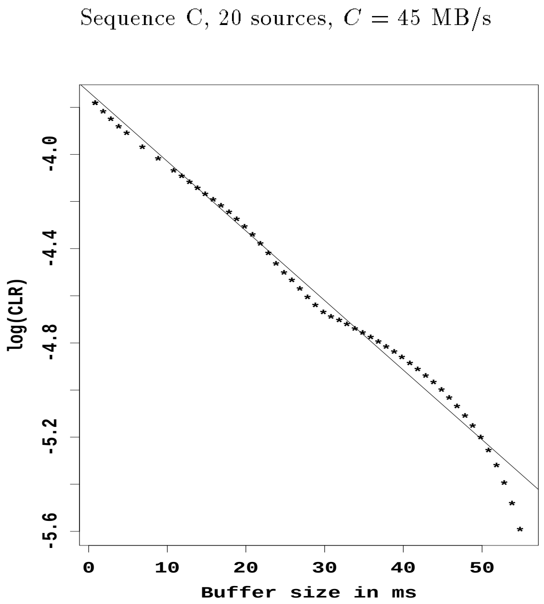

Figure 5.1: Log(CLR) vs. buffer size

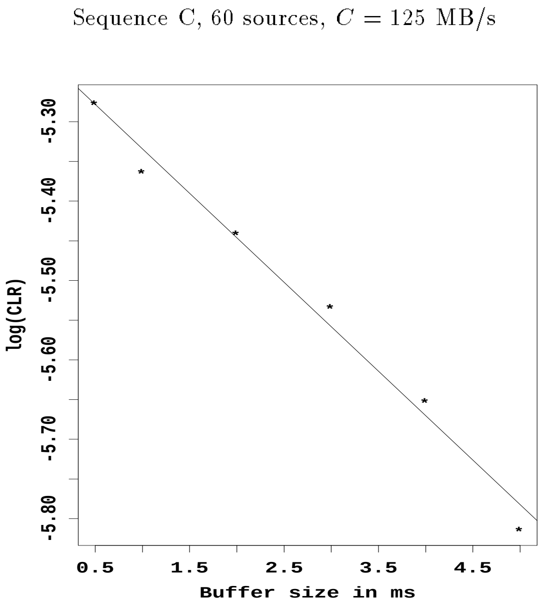

Figure 5.2: Log(CLR) vs. buffer size

We also tested by simulation one of the basic hypothesis underlying the use of the Chernoff-Dominant Eigenvalue method. The hypothesis is that $\log(\mathrm{CLR}) \approx \ell + zB$, where ℓ and z are constants. Figures 5.1 and 5.2 plot on a log-linear scale the buffer overflow probabilities for various buffer sizes. The overflow probabilities were obtained by simulation using sequence C for the parameters indicated in the figure. The plots shows that the hypothesis is well founded for this set of parameters. Similar results were obtained for many other parameter settings. For 60 sources, the traffic information in sequence C allows us to simulate 3.5×10^8 cell arrivals to the switch. This does not permit us to reliably estimate CLRs much lower than the ranges shown in the figure.

Figure 5.3 shows the results from experiments with sequence A. For the 50 source example shown in the figure, the information in sequence A allows us to simulate 2.8×10^9 cell arrivals at the switch allowing us to obtain reliable estimates of CLR for the ranges shown in the figure. Again, we find that the buffer overflow probability decreases exponentially with buffer size. Beran et al. [BST94] state that our traffic data exhibit long-range dependence. The DAR(1) model has a geometrically declining autocorrelation function. So it takes into accounts only short range dependence. Leland et al. [LTW93] argue (in the context of Ethernet traffic) that when traffic is long range dependent "overall packet loss decreases very slowly with increasing buffer capacity". If these assertions were both valid for the system we model, then the Chernoff-Dominant Eigenvalue method would not be applicable to video teleconference traffic. Results from our simulation studies using actual data traces (summarized in Figures 5.1, 5.2, and 5.3) do not conform to these assertions. Furthermore, the underlying hypothesis necessary for using the Chernoff-Dominant Eigenvalue method seems to hold for video teleconferences and the results in Tables 5.2 and 5.3 indicate that

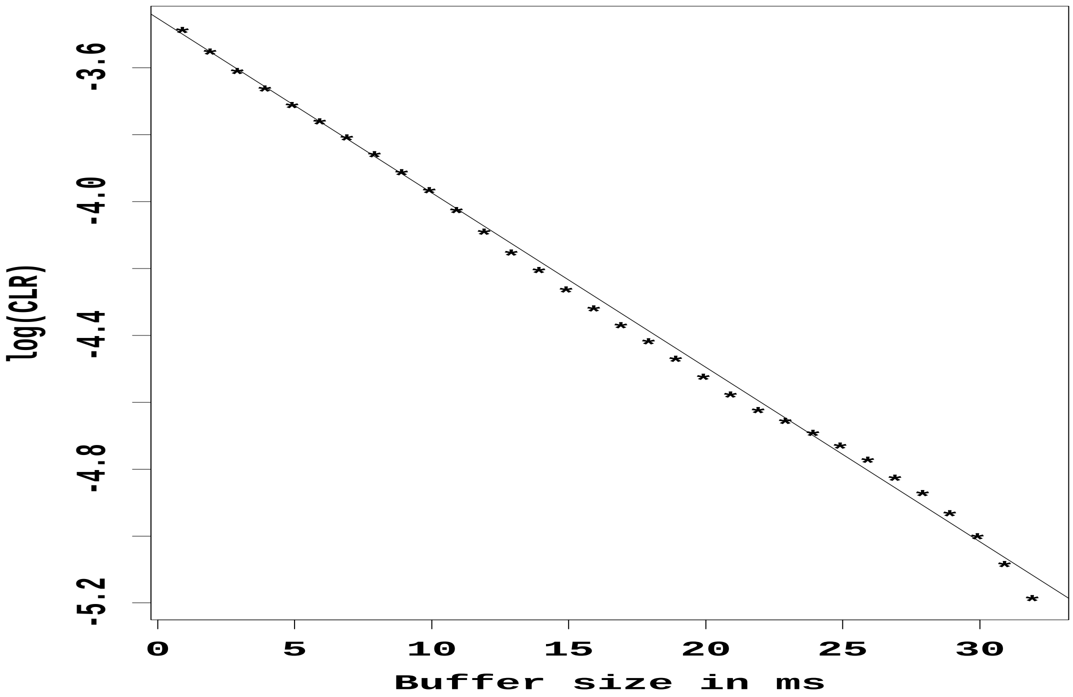

Figure 5.3: Log(CLR) vs. buffer size for sequence A, 50 sources and C = 280 MB/s

this method is accurate enough to be used for admission control and bandwidth allocation of video teleconferences.

Sequence	Mean Cells/frame	Peak Cells/frame	Variance	Correlation 1-frame lag	Cell Size Bytes
A	1506.4	4818.0	262827.1	0.9731	14
B	104.8	220.0	882.5	0.9795	48
C	130.3	630.0	5536.9	0.9846	64

Table 5.1: Parameters of traffic sequences studied

C	45	67	81	103	125	145	195	245	270	310
B	57	9	50	44	5	0.5	9	23	7	1
K_{sim}	20	30	40	50	60	70	98	128	139	156
K_{CDE}	16	25	33	44	53	63	90	120	130	150
$\frac{K_{\text{CDE}}}{K_{\text{sim}}}$	.80	.83	.83	.88	.88	.90	.92	.94	.94	.96

Table 5.2: Comparison of number of coding scheme C sources admitted using simulation and the Chernoff-Dominant Eigenvalue (CDE) method

C	110	185	280	375
B	8.5	9.5	10	11
K_{sim}	16	30	49	66
K_{CDE}	15	30	50	70
$\frac{K_{\text{CDE}}}{K_{\text{sim}}}$	.94	1.0	1.02	1.06

Table 5.3: Comparison of number of coding scheme A sources admitted using simulation and using the Chernoff-Dominant Eigenvalue method

APPENDIX: PROOFS of THEOREMS 2.1, 2.2 and 2.3

We need two lemmas. The proof of Lemma 1 is in [SWE95, Chapter 13].

Lemma 1 *For any path* $\mathbf{r}(t)$ *and time* $T > 0$ *define*

$$\bar{\mathbf{r}}(T) \triangleq \frac{1}{T}\int_0^T \mathbf{r}(t)\,dt \tag{A.1}$$

$$\bar{\mathbf{r}}'(T) \triangleq \frac{1}{T}\int_0^T \mathbf{r}'(t)\,dt = \frac{\mathbf{r}(T)-\mathbf{r}(0)}{T} \tag{A.2}$$

Then

$$I_0^T(\mathbf{r}) \geq T\ell\left(\bar{\mathbf{r}}(T), \bar{\mathbf{r}}'(T)\right). \tag{A.3}$$

Lemma 2

$$I_{-\infty}^T(\mathbf{r}) = I_{-\infty}^T(\mathbf{s}) = I_{-\infty}^\infty(\mathbf{r}) = I_{-\infty}^\infty(\mathbf{s}). \tag{A.4}$$

Proof. The definition of reversibility is

$$\pi_x Q_{x,y} = \pi_y Q_{y,x} \text{ for every pair } x, y. \tag{A.5}$$

Extending this by iteration we find that for any sequence of states $x(1), \ldots, x(n)$ we have

$$\begin{array}{c} \pi_{x(1)}Q_{x(1),x(2)}Q_{x(2),x(3)}\cdots Q_{x(n-1),x(n)} = \\ \pi_{x(n)}Q_{x(n),x(n-1)}Q_{x(n-1),x(n-2)}\cdots Q_{x(2),x(1)}. \end{array} \tag{A.6}$$

This means that for cost functions we obtain

$$\mathbb{P}(\mathbf{r}(0))\exp\left(-KI_0^T(\mathbf{r})\right) = \mathbb{P}(\mathbf{r}(T))\exp\left(-KI_0^T(\mathbf{r}(T-t))\right). \tag{A.7}$$

But for reversible systems it is easy to show (essentially from Theorem 2.4) that

$$\begin{array}{c} \mathbb{P}(\mathbf{r}(0)) = \exp\left(-KI_{-\infty}^0(\mathbf{r}) + o(K)\right) \\ \mathbb{P}(\mathbf{r}(T)) = \exp\left(-KI_{-\infty}^0(\mathbf{s}) + o(K)\right). \end{array}$$

This finishes the proof. ∎

Proof of Theorem 2.1. Recall the definition of $\mathcal{G}(b)$ in equation (2.32). Choose a $b > 0$. Suppose that $(\mathbf{r}, T) \in \mathcal{G}(b)$, and that $\mathbf{r}$ is a minimal cost trajectory. We can shift time by any amount τ and keep $(\mathbf{r}(t+\tau), T+\tau) \in \mathcal{G}(b)$. Therefore we are free to state that 0 is the last time before T that $\mathbf{B}_s\mathbf{r}(t) = 0$. We can extend $\mathbf{r}$ to times larger than T by setting it equal to $\mathbf{z}_\infty$. It is easy to see that $\mathbf{B}_s r(t) < b$ for every $t > T$, since the path $\mathbf{z}_\infty \to \mathbf{p}$ and since otherwise we could achieve the buffer content b with lower cost. Therefore the path

$$\mathbf{s}(t) \triangleq \mathbf{r}(T-t) \tag{A.8}$$

has the property that $\mathbf{B}_s\mathbf{s}(0) = 0$. Now the reversibility of $\mathbf{z}_K(t)$ implies that the cheapest path from $\mathbf{p}$ to any point $\mathbf{y}$ is the time reversal of $\mathbf{z}_\infty(t)$ starting at $\mathbf{y}$; that is, the cheapest way to go from the center $\mathbf{p}$ to any point $\mathbf{y}$ is the time reversal of the most likely way to go from $\mathbf{y}$ to $\mathbf{p}$.

Thus the path $\mathbf{s}(t)$ has the same properties as $\mathbf{r}$: $(\mathbf{s}, T)$ is a member of $\mathcal{G}$, and $\mathbf{s}$ has 0 as the last time before T when $\mathbf{B}_s\mathbf{s}(t) = 0$. Note that since the cost of $\mathbf{z}_\infty(t)$ is zero, $I_T^\infty(\mathbf{r}) = I_T^\infty(\mathbf{s}) = 0$.

Now define

$$\begin{array}{ll} \beta_1 = I_{-\infty}^0(\mathbf{r}) & \beta_2 = I_0^T(\mathbf{r}) \\ \beta_3 = I_{-\infty}^0(\mathbf{s}) & \beta_4 = I_0^T(\mathbf{s}) \end{array} \tag{A.9}$$

Lemma 2 states that $\beta_1 + \beta_2 = \beta_3 + \beta_4$. Now use Lemma 1 to obtain

$$\beta_2 \geq T\ell\left(\bar{\mathbf{r}}(T), \bar{\mathbf{r}}'(T)\right). \tag{A.10}$$

Similarly,

$$\beta_4 \geq T\ell\left(\bar{\mathbf{r}}(T), -\bar{\mathbf{r}}'(T)\right), \tag{A.11}$$

since the average of $\mathbf{s}'$ over $(0, T)$ is $-\bar{\mathbf{r}}'(T)$, and the average of $\mathbf{s}$ over the same interval is $\bar{\mathbf{r}}(T)$. Now $\ell(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{y}$, so

$$\frac{1}{2}(\beta_2 + \beta_4) \geq T\ell\left(\bar{\mathbf{r}}(T), \mathbf{0}\right). \tag{A.12}$$

Furthermore, since β_1 and β_3 are each costs of going from the point $\mathbf{p}$ to the hyperplane $\langle \mathbf{R}, \mathbf{z}_K \rangle = c$, each of them is larger than C_1, which is defined to be the minimal cost of all paths to go from $\mathbf{p}$ to that hyperplane. Now we use Lemma 2:

$$\begin{aligned} I_{-\infty}^T(\mathbf{r}) &= \beta_1 + \beta_2 = \frac{\beta_1 + \beta_2 + \beta_3 + \beta_4}{2} = \frac{\beta_1 + \beta_3}{2} + \frac{\beta_2 + \beta_4}{2} \\ &\geq C_1 + T\ell\left(\bar{\mathbf{r}}(T), \mathbf{0}\right) \\ &\geq C_1 + bC_2, \end{aligned} \tag{A.13}$$

since C_2 was defined to be the minimum cost per unit buffer for a path that does not move.
■

Proof of Theorem 2.2. By the lower semicontinuity of $\ell(\mathbf{x}, \mathbf{y})$ and the assumed uniqueness of $\mathbf{w}^*$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{w}^*| > \delta$ then

$$\frac{\ell(\mathbf{x}, \mathbf{0})}{\langle \mathbf{x}, \mathbf{R} \rangle - c} > C_2 + \varepsilon \triangleq \frac{\ell(\mathbf{w}^*, \mathbf{0})}{\langle \mathbf{x}, \mathbf{R} \rangle - c} + \varepsilon. \tag{A.14}$$

Lemma 1 shows that any path has a cost that is lower bounded by a constant plus the cost of its center. As $b \to \infty$, the point $\bar{\mathbf{r}}' \to \mathbf{0}$, since the time tends to infinity and $\mathbf{r}(t)$ is bounded. Therefore

$$I_0^T(\mathbf{r})/T \geq \ell(\bar{\mathbf{r}}, \bar{\mathbf{r}}') > \ell(\mathbf{w}^*, \mathbf{0}) + \varepsilon \tag{A.15}$$

unless $|\bar{\mathbf{r}} - \mathbf{w}^*| < \delta$. Now the inequality of Lemma 1 is strict unless $\mathbf{r}(t)$ is a constant. This proves that the fraction of time over which $\mathbf{r}(t)$ is close to $\mathbf{w}^*$ tends to one as $T \to \infty$ (that is, as $b \to \infty$). ■

Proof of Theorem 2.3. There are many ways of arriving at a function $f(b)$ such that (2.39) and (2.40) hold. We simply have to find a set of functions $\{\mathbf{r}_b(t)\}$ with associated $T(b)$ satisfying the following conditions (see Figure 2.2):

$$\begin{aligned} \mathbf{r}_b(0) &= \mathbf{v}^* = \mathbf{r}_b(T(b)) && \text{(A.16)} \\ \int_0^{T(b)} (\langle \mathbf{r}_b(t), \mathbf{R}\rangle - c)\, dt &= b && \text{(A.17)} \end{aligned}$$

Then we take

$$f(b) = I_0^{T(b)}(\mathbf{r}_b). \tag{A.18}$$

It is not hard to show that there is a choice of the family $\{\mathbf{r}_b(t)\}$ such that $f(b) = I^*(b)$. However, for many models it is difficult to find $I^*(b)$ analytically. We therefore propose a set of paths that give only an approximation, but one that has bounded error as $b \to \infty$, and that is tight as $b \to 0$.

For every $b > 0$ we define $\mathbf{r}_b(t)$ for $t < 0$ to be the minimal cost path from $\mathbf{p}$ to $\mathbf{v}^*$ that reaches $\mathbf{v}^*$ at time 0. This is, by the reversibility assumption, the time reversal of $\mathbf{z}_\infty(t)$ starting at $\mathbf{v}^*$ at time 0; see Theorem 2.4. Let $\mathbf{w}^*$ represent any minimizing point of the fraction that defines C_2, see (2.35). Then let ω represent the denominator of that fraction:

$$\omega \triangleq \langle \mathbf{w}^*, \mathbf{R}\rangle - c. \tag{A.19}$$

Now define $b_0 = \omega$, and define

$$u(b) \triangleq \min\left(1, \sqrt{\frac{2b}{\omega}}\right). \tag{A.20}$$

($u(b)$ is the first time when either $\mathbf{r}_b(t) = \mathbf{w}^*$ or when the buffer reaches $b/2$; see (A.23) below.) Furthermore define

$$U(b) \triangleq \begin{cases} u(b) & \text{if } u(b) < 1 \\ \dfrac{b - 2\omega}{\omega} & \text{if } u(b) = 1 \end{cases} \tag{A.21}$$

($U(b)$ is the time when the path $\mathbf{r}_b(t)$ starts going back to $\mathbf{v}^*$.) We now define $\mathbf{r}_b(t)$ for $t > 0$ as follows:

$$\mathbf{r}_b(t) = \begin{cases} (1-t)\mathbf{v}^* + t\mathbf{w}^* & \text{if } 0 < t < u(b) \\ \mathbf{w}^* & \text{if } 1 \le u(b) \le t \le U(b) \\ (1-(t-U(b)))\mathbf{w}^* + (t-U(b))\mathbf{v}^* & \text{if } U(b) < t < U(b)+1 \\ \mathbf{z}_\infty(t - U(b) - 1) & \text{if } t > U(b)+1 \end{cases} \tag{A.22}$$

In words, this makes $\mathbf{r}_b(t)$ linear between $\mathbf{v}^*$ and $\mathbf{w}^*$ for time up to 1, then $\mathbf{r}_b(t)$ is equal to $\mathbf{w}^*$ for a while, then it goes linearly back to $\mathbf{v}^*$ and from there follows $\mathbf{z}_\infty(t)$ back to $\mathbf{p}$. With these definitions it is easy to see that $T(b) = u(b) + U(b)$.

First we check that $\int_0^{T(b)}(\langle \mathbf{r}_b(t), \mathbf{R}\rangle - c)\,dt = b$. For $0 \leq s \leq 1$,

$$\int_0^s (\langle \mathbf{r}_b(t), \mathbf{R}\rangle - c)\,dt = \int_0^s t\omega\,dt = \omega\frac{s^2}{2}. \tag{A.23}$$

This shows why we chose $u(b)$ as we did. From here it is clear that $\int_0^{T(b)}(\langle \mathbf{r}_b(t), \mathbf{R}\rangle - c)\,dt = b$.

If $b > b_0$ then we have $f(b) = f(b_0) + C_2(b - b_0)$ (recall that we define $f(b)$ via (A.18), since the integral of $\ell(\mathbf{r}_b, \mathbf{r}_b')$ can be broken up into the pieces where $\mathbf{r}_b = \mathbf{w}^*$ and $\mathbf{r}_b \neq \mathbf{w}^*$. Those portions where $\mathbf{r}_b \neq \mathbf{w}^*$ are just the times when $\mathbf{r}_b = \mathbf{r}_{b_0}$, and those where $\mathbf{r}_b = \mathbf{w}^*$ give C_2 increase in $I_0^T(\mathbf{r}_b)$ per unit increase in b. ∎

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