Refined Asymptotic Approximations to Loss Probabilities and Their Sensitivities in Shared Unbuffered Resources

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Abstract

We consider an unbuffered resource having capacity C which is shared by several different services. Calls of each service arrive in a Poisson stream and request a fixed, integral amount of capacity, which may depend on the service. An arriving call is blocked and lost if there is not enough capacity. Otherwise, the capacity of the call is held for the duration of the call, and the holding period is generally distributed. It is assumed that C and the traffic intensities of the services are commensurately large, and asymptotic approximations are obtained for the loss probabilities, and their sensitivities to the traffic intensity of each service. These sensitivities are important in optimizing the performance of multiservice, multirate, loss networks. Numerical results illustrate the accuracy of the asymptotic approximations. These results show that while prior asymptotic approximations to the loss probabilities are quite accurate, the new approximations are very accurate. Moreover, while the prior asymptotic approximations to the sensitivities are overall rather poor, the new approximations are very good. Experience with case studies has shown that computationally efficient asymptotic techniques are crucial to cope with the size and the number of service offerings of emerging broadband networks in their design and optimization.

Key words. implied costs, loss probabilities, multirate, network traffic engineering, uniform asymptotic approximations

AMS subject classifications. 60K30, 90B12

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1 Introduction

We consider an unbuffered resource having capacity C, which is shared by S different services. Calls of service $s(s = 1, 2, \ldots, S)$ arrive in a Poisson stream with mean rate λ_s, and request capacity d_s. An arriving call is blocked and lost if there is less than d_s free capacity. Otherwise, the call capacity d_s is held for the duration of the call, and the holding period is generally distributed with mean $1/\mu_s$, and is independent of earlier arrival and holding times. The traffic intensity of calls of service s is $\nu_s = \lambda_s/\mu_s$, and the product form and the insensitivity property hold [9], [13], i.e. the joint stationary distribution of the number of active calls of each service depends on the distributions only through ν_s.

A primary performance measure is the loss probability $\mathbb{L}_s$ for each service s. In an earlier paper, Mitra and Morrison [15] derived a uniform asymptotic approximation (UAA) to $\mathbb{L}_s$, when C and the traffic intensities $\nu_s(s = 1, 2, \ldots, S)$ are commensurately large. The UAA is quite accurate, whether the resource is underloaded, critically loaded or overloaded.

In a subsequent paper, Mitra, Morrison and Ramakrishnan [16] investigated route optimization in multirate, circuit-switched, loss networks. The loss probabilities were obtained by solving fixed point equations [7], [10], [19], [21], which are based on the approximation of link independence. A central element of the analysis in [16] is the systematic use of the UAA to handle networks with links of large capacities and with many services. The UAA is used to compute the loss probabilities for each link in the network, and also to eliminate the dependence on the number of services in the dimension of the system of linear equations for the network's implied costs or performance sensitivities. Experience with case studies have provided strong evidence that such asymptotic techniques are crucial to cope with the size and number of service offerings of emerging broadband networks. The savings in computations at the level of individual links are hugely leveraged in the analysis and optimization of realistically sized networks.

Prior work on multiservice, multirate loss networks is extensive. See, for instance, the recent book by Ross [19] and references therein. There are also several works on the asymptotic treatment of multirate links subject to Poisson traffic. Evans [4] and Reiman [18] consider such systems in the critical regime. Gazdzicki, Lambadaris and Mazumdar [6] use local limit theorems and large deviations to obtain results in the overloaded and underloaded regimes, and at the critical loading point. Labourdette and Hart [12] give a combined asymptotic and heuristic approximation. We also note that Whittle [22], Kogan [11], Birman and Kogan [1] and Simonian [20] have used the saddlepoint technique to investigate loss networks.

Network optimization relies on the sensitivity of the network revenue, which is a proxy for performance, to the offered traffic. These sensitivities depend on certain implied costs. The equations for the network's implied costs involve the similar sensitivities of the loss probabilities for each link in the network. For a single link, [16] differentiates the UAA to $\mathbb{L}_s$ with respect to the traffic intensities ν_t. As pointed out in [16], the resulting approximation to $\partial\mathbb{L}_s/\partial\nu_t$ is not useful in the overloaded regime. Moreover, numerical examples show that the approximation is not very accurate in the important critically loaded regime. Consequently, it is necessary to refine the procedure to obtain a uniform approximation to $\partial\mathbb{L}_s/\partial\nu_t$, which we do in this paper. The results are applied to network analysis and optimization in a subsequent paper [17].

In §2 we formulate the problem, and give a contour integral representation for the normalization constant in the joint stationary distribution of the number of active calls. The loss probabilities $\mathbb{L}_s$, and their derivatives $\partial\mathbb{L}_s/\partial\nu_t$, are all expressed in terms of the normalization constant, evaluated at various values of the capacity. In §3 we introduce the asymptotic scaling, in which the capacity and the traffic intensities are commensurately large, and we summarize the UAA. In §4 we derive a refined uniform asymptotic approximation (RUAA) to the contour integral for the normalization constant, using a technique due to Bleistein [2]. In §5 we derive the RUAA to $\mathbb{L}_s$ and $\partial\mathbb{L}_s/\partial\nu_t$. Importantly, the complexity of these calculations remains $O(1)$.

The expression for a quantity which arises in the UAA (and RUAA) to the normalization constant remains finite as critical loading is approached, but it contains two terms which become singular, so that its numerical evaluation loses accuracy. In §6 we derive an alternate representation for this quantity which gives numerically accurate results, whether or not the resource is critically loaded. An alternate representation is also given for an additional quantity which arises in the RUAA, the expression for which contains even more singular terms as critical loading is approached. In §7 we present some numerical results. These illustrate that while the UAA to $\mathbb{L}_s$ is quite accurate, the RUAA is very accurate. Moreover, while the UAA to $\partial\mathbb{L}_s/\partial\nu_t$ is overall rather poor, the RUAA is very good.

2 Formulation

Let $n_s (1 \leq s \leq S)$ be the number of active calls of service s. Then [9], [13], [15], the stationary distribution is

$$\pi(n_1, \ldots, n_S) = \frac{1}{G(\mathbf{d}, \boldsymbol{\nu}, C)} \prod_{s=1}^{S} e^{-\nu_s} \frac{\nu_s^{n_s}}{n_s!}, \quad \sum_{s=1}^{S} d_s n_s \leq C, \tag{2.1}$$

where the normalization constant is

$$G(\mathbf{d}, \boldsymbol{\nu}, C) = \sum_{\sum_{s=1}^{S} d_s n_s \leq C} \prod_{s=1}^{S} e^{-\nu_s} \frac{\nu_s^{n_s}}{n_s!}. \tag{2.2}$$

Also [10], the loss probability of service s is

$$\mathbb{L}_s(\mathbf{d}, \boldsymbol{\nu}, C) = 1 - \frac{G(\mathbf{d}, \boldsymbol{\nu}, C - d_s)}{G(\mathbf{d}, \boldsymbol{\nu}, C)}. \tag{2.3}$$

We are also interested in the partial derivatives of $\mathbb{L}_s$ with respect to ν_t. From (2.2) we obtain

$$\frac{\partial G}{\partial \nu_t}(\mathbf{d}, \boldsymbol{\nu}, C) = G(\mathbf{d}, \boldsymbol{\nu}, C - d_t) - G(\mathbf{d}, \boldsymbol{\nu}, C), \tag{2.4}$$

and it follows from (2.3) that

$$\frac{\partial \mathbb{L}_s}{\partial \nu_t}(\mathbf{d}, \boldsymbol{\nu}, C) = \frac{G(\mathbf{d}, \boldsymbol{\nu}, C - d_s) G(\mathbf{d}, \boldsymbol{\nu}, C - d_t)}{[G(\mathbf{d}, \boldsymbol{\nu}, C)]^2} - \frac{G(\mathbf{d}, \boldsymbol{\nu}, C - d_s - d_t)}{G(\mathbf{d}, \boldsymbol{\nu}, C)}, \tag{2.5}$$

which is symmetric in s and t.

Kaufman [8] gave a one-dimensional recursion for the distribution of capacity being used, from which the loss probabilities $\mathbb{L}_s$, and hence from (2.5) the partial derivatives with respect to ν_t, may be obtained. However, for commensurately large values of C and $\boldsymbol{\nu}$, the complexity of the recursive procedure is $O(C)$.

Now, from (2.2),

$$\sum_{C=0}^{\infty} G(\mathbf{d}, \boldsymbol{\nu}, C) z^C = (1 - z)^{-1} \exp\left[\sum_{s=1}^{S} \nu_s \left(z^{d_s} - 1\right)\right]. \tag{2.6}$$

We henceforth suppress the dependence of $G(\mathbf{d}, \boldsymbol{\nu}, C)$ on $\mathbf{d}$ and $\boldsymbol{\nu}$. Then, applying Cauchy's formula [3] to (2.6), we obtain, as in [15],

$$G(C) = \frac{1}{2\pi i} \int_{|z|<1} \frac{\exp\left[\sum_{s=1}^{S} \nu_s (z^{d_s} - 1)\right]}{(1 - z) z^{C+1}} \, dz, \tag{2.7}$$

where the integral is taken in a counter-clockwise direction around a circle of radius less than 1. Hence,

$$G(C) - G(C - d_s) = \frac{1}{2\pi i} \int_{|z|<1} \frac{(1 - z^{d_s}) \exp\left[\sum_{u=1}^{S} \nu_u (z^{d_u} - 1)\right]}{(1 - z) z^{C+1}} \, dz \,. \tag{2.8}$$

If e_s is the revenue earned per carried call per unit time by calls of service type s, then [16] the implied cost for a call of service type t is

$$c_t = \frac{1}{(1 - \mathbb{L}_t)} \sum_{s=1}^{S} e_s \nu_s \frac{\partial \mathbb{L}_s}{\partial \nu_t} \,. \tag{2.9}$$

Expressions for $\mathbb{L}_t$ and $\partial \mathbb{L}_s / \partial \nu_t$ are given by (2.3) and (2.5).

3 Uniform asymptotic approximations

We first summarize the results of Mitra and Morrison [15], as they pertain to the case of Poisson streams of calls. We assume that

$$C \gg 1; \; \nu_s = \alpha_s C, \; (s = 1, 2, \ldots, S) \,, \tag{3.1}$$

where $\alpha_s > 0$ is $O(1)$, and bounded away from zero. We also assume that $d_s (s = 1, 2, \ldots, S)$ are positive integers, not large relative to C, and (without loss of generality) that the greatest common divisor (g.c.d.) of $d_1, \ldots, d_S$ is 1. We define

$$f(z) \triangleq \sum_{s=1}^{S} \alpha_s (z^{d_s} - 1) - \log z \,. \tag{3.2}$$

Then, from (2.7) and (2.8),

$$G(C) = \frac{1}{2\pi i} \int_{|z|<1} \frac{e^{C f(z)}}{(1 - z) z} \, dz \,, \tag{3.3}$$

and

$$G(C) - G(C - d_s) = \frac{1}{2\pi i} \int_{|z|<1} \frac{(1 - z^{d_s}) e^{C f(z)}}{(1 - z) z} \, dz \,. \tag{3.4}$$

Although $f(z)$ is not single-valued, $\exp[C f(z)]$ is, since C and d_s, $s = 1, \ldots, S$, are positive integers.

Asymptotic approximations to the contour integrals in (3.3) and (3.4), obtained by the saddle-point method, are given in [15]. There is a unique positive solution z^* of $f'(z) = 0$, where the prime denotes derivative, so that

$$\sum_{s=1}^{S} \alpha_s d_s (z^*)^{d_s} = 1, \; z^* > 0 \,. \tag{3.5}$$

A simple procedure, such as bisection, quickly yields z^*. Since $f'(z^*) = 0$,

$$v \triangleq (z^*)^2 f''(z^*) = \sum_{s=1}^{S} \alpha_s d_s^2 (z^*)^{d_s} > 0 \,. \tag{3.6}$$

Hence $f(z)$ has a minimum on the real axis at $z = z^*$, and it was shown [15] that $|z| = z^*$ is a saddle-point contour [5].

A uniform asymptotic approximation to the contour integral in (3.3) was derived by means of the technique given by Bleistein [2]. The approximation remains valid when the stationary point at $z = z^*$ is close to the pole at $z = 1$, and it is also valid when z^* is not close to 1. We have [15]

Proposition 3.1 Let

$$M = \frac{1}{2} \,\mathrm{Erfc}\, \left[\,\mathrm{sgn}\,(1 - z^*) \sqrt{-Cf(z^*)} \right] + \frac{K e^{Cf(z^*)}}{\sqrt{2\pi C v}} \,, \tag{3.7}$$

where

$$K = \frac{1}{(1 - z^*)} - \frac{\sqrt{v} \;\,\mathrm{sgn}\,(1 - z^*)}{\sqrt{-2f(z^*)}} \,, \; z^* \neq 1 \,, \tag{3.8}$$

$$K = \frac{1}{2} + \frac{1}{6v} \sum_{s=1}^{S} \alpha_s d_s^3 , \; z^* = 1 \,, \tag{3.9}$$

and the complementary error function is given by

$$\mathrm{Erfc}\,(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\xi^2} d\xi \,. \tag{3.10}$$

Then, asymptotically,

$$G(C) \sim M \,. \tag{3.11}$$

$\square$

We note that, since $f(1) = 0$, $f'(z^*) = 0$ and $(z^*)^2 f''(z^*) = v$, the expression for K in (3.8) remains finite as $z^* \to 1$, and its limiting value is given by (3.9). Since the numerical evaluation of (3.8) loses accuracy when z^* is close to 1, in Section 6 we derive a representation for K which gives numerically accurate results, whether or not z^* is close to 1, or even equal to 1. We also derive an alternate expression for $f(z^*)$, which leads to a more accurate numerical evaluation than (3.2) does when z^* is close to 1.

Since the integrand in (3.4) is analytic at $z = 1$, an asymptotic approximation to $G(C) - G(C - d_s)$ was obtained by the standard saddle-point method. Then, from (2.3) and (3.11), we have [15]

Proposition 3.2 Let

$$H_s^{(1)}(z) \triangleq \sum_{n=0}^{d_s-1} z^n = \begin{cases} (1-z^{d_s})/(1-z), & z \neq 1, \\ d_s, \quad z = 1, \end{cases} \tag{3.12}$$

and

$$B = \frac{e^{Cf(z^*)}}{M\sqrt{2\pi C v}}. \tag{3.13}$$

Then, a uniform asymptotic approximation to the loss probability for service s is

$$\mathbb{L}_s \sim BH_s^{(1)}(z^*). \tag{3.14}$$

$\square$

Numerical results [15] illustrate that the approximation (3.14) is quite accurate in the overloaded, critically loaded and underloaded regimes.

From (3.2), (3.5), (3.7)–(3.10) and Proposition 3.2, we obtain [15]

Proposition 3.3

(i). In the overloaded regime, corresponding to $\sum_{s=1}^{S} \alpha_s d_s > 1$, $0 < z^* < 1$, $B \sim 1 - z^*$ and $\mathbb{L}_s \sim 1 - (z^*)^{d_s}$.

(ii). In the critically loaded regime, corresponding to $\sum_{s=1}^{S} \alpha_s d_s - 1 = O(1/\sqrt{C})$, $1 - z^* = O(1/\sqrt{C})$, $B = O(1/\sqrt{C})$ and $\mathbb{L}_s \sim Bd_s$.

(iii). In the underloaded regime, corresponding to $\sum_{s=1}^{S} \alpha_s d_s < 1$, $z^* > 1$ and B and $\mathbb{L}_s$ are exponentially small. $\square$

Differentiating the uniform asymptotic approximation (3.14) to $\mathbb{L}_s$ with respect to ν_t gives, as reported in [16],

$$\partial \mathbb{L}_s / \partial \nu_t \sim B(B - 1 + z^*) H_s^{(1)}(z^*) H_t^{(1)}(z^*). \tag{3.15}$$

The same result may be obtained from the exact expression in (2.5). However, since $B \sim 1 - z^*$ in the overloaded regime, a refinement is required. Moreover, numerical examples show that the approximation (3.15) is not very accurate in the important critically loaded regime, where $B - 1 + z^* = O(1/\sqrt{C})$. Consequently, in this paper, we derive a refined uniform asymptotic approximation to the contour integral representation (3.3) for $G(C)$, and then a refined approximation to the expression (2.5) for $\partial \mathbb{L}_s / \partial \nu_t$, with the help of (3.4) and the corresponding refined saddle-point approximation.

4 Refined uniform asymptotic approximations

We now derive a refined uniform asymptotic approximation to the contour integral in (3.3). We assume for the time being that $z^* \neq 1$, and shift the contour of integration to $|z| = z^*$, so that

$$G(C) = \frac{1}{2}[1 - \text{ sgn } (1 - z^*)] + \frac{1}{2\pi i} \int_{|z|=z^*} \frac{e^{Cf(z)}}{(1-z)z} dz, \; z^* \neq 1 \,. \tag{4.1}$$

Following [15], with a slight change in notation, we let

$$f(z^*) - f(z) = u^2 \,. \tag{4.2}$$

Now $f(1) = 0$ and $f(z^*) < 0$, for $z^* \neq 1$. From the sign rule in [2, Sect. 6], we take $z = 1$ to correspond to $u = -i\kappa$, where

$$\kappa = \text{ sgn } (1 - z^*)\sqrt{-f(z^*)} \,. \tag{4.3}$$

Hence, for $|z - 1| \ll 1$,

$$-(u^2 + \kappa^2) = f'(1)(z-1) + O(|z-1|^2) \,, \tag{4.4}$$

so that

$$z - 1 = \frac{2i\kappa}{f'(1)}(u + i\kappa) + O(|u + i\kappa|^2) \,. \tag{4.5}$$

It follows that

$$\frac{1}{(1-z)z} \frac{dz}{du} = \frac{-1}{(u + i\kappa)} + \ell(u) \,, \tag{4.6}$$

where $\ell(u)$ is analytic at $u = -i\kappa$.

From (4.2), since $f'(z^*) = 0$,

$$-u^2 = \frac{1}{2}f''(z^*)(z-z^*)^2 + \frac{1}{6}f^{(3)}(z^*)(z-z^*)^3 + \frac{1}{24}f^{(4)}(z^*)(z-z^*)^4 + \cdots \,, \tag{4.7}$$

for $|z - z^*| \ll 1$. Since $f''(z^*) > 0$, z is an analytic function of u at $u = 0$, and hence, from (4.6), so is $\ell(u)$. Reversion of the series in (4.7) leads to

$$z - z^* = i\beta_1 u + \beta_2 u^2 + i\beta_3 u^3 + \cdots \,, \tag{4.8}$$

for $|u| \ll 1$, where

$$\beta_1 = \frac{\sqrt{2}}{\sqrt{f''(z^*)}}, \; \beta_2 = \frac{f^{(3)}(z^*)}{3[f''(z^*)]^2}, \; \beta_3 = \frac{1}{6\sqrt{2f''(z^*)}} \left\{ \frac{f^{(4)}(z^*)}{[f''(z^*)]^2} - \frac{5[f^{(3)}(z^*)]^2}{3[f''(z^*)]^3} \right\} \,. \tag{4.9}$$

Let

$$\tau \triangleq (z^*)^3 f^{(3)}(z^*), \quad y \triangleq (z^*)^4 f^{(4)}(z^*)\,. \tag{4.10}$$

It is shown in Appendix A that

$$\tau = \sum_{s=1}^{S} \alpha_s d_s^2 (d_s - 3)(z^*)^{d_s}\,, \tag{4.11}$$

and

$$y = \sum_{s=1}^{S} \alpha_s d_s^2 (d_s^2 - 6d_s + 11)(z^*)^{d_s}\,. \tag{4.12}$$

From (3.6), (4.9) and (4.10), we have

$$\frac{\beta_1}{z^*} = \frac{\sqrt{2}}{\sqrt{v}}, \quad \frac{\beta_2}{z^*} = \frac{\tau}{3v^2}, \quad \frac{\beta_3}{z^*} = \frac{1}{6v\sqrt{2v}}\left(\frac{y}{v} - \frac{5\tau^2}{3v^2}\right)\,. \tag{4.13}$$

If $h(z)$ is analytic at $z = z^*$, then, for $|z - z^*| \ll 1$,

$$h(z) = h(z^*) + h'(z^*)(z - z^*) + \frac{1}{2}h''(z^*)(z - z^*)^2 + \cdots\,. \tag{4.14}$$

It follows from (4.8) that

$$h(z) = h(z^*) + i\beta_1 h'(z^*)u + \left[\beta_2 h'(z^*) - \frac{1}{2}\beta_1^2 h''(z^*)\right] u^2 + \cdots\,, \tag{4.15}$$

and

$$\begin{aligned} h(z)\frac{dz}{du} &= i\beta_1 h(z^*) + \left[2\beta_2 h(z^*) - \beta_1^2 h'(z^*)\right] u \\ &\quad + i\left[3\beta_3 h(z^*) + 3\beta_1\beta_2 h'(z^*) - \frac{1}{2}\beta_1^3 h''(z^*)\right] u^2 + \cdots\,. \end{aligned} \tag{4.16}$$

Also, for $|u| \ll 1$,

$$\frac{1}{(u + i\kappa)} = \frac{-i}{\kappa}\left(1 - \frac{iu}{\kappa}\right)^{-1} = \frac{-i}{\kappa}\left(1 + \frac{iu}{\kappa} - \frac{u^2}{\kappa^2} + \cdots\right)\,. \tag{4.17}$$

We let

$$g(z) \triangleq \frac{1}{(1 - z)z}\,. \tag{4.18}$$

Then, from (4.6), (4.16) and (4.17),

$$\begin{aligned} \ell(u) &= i\left[\beta_1 g(z^*) - \frac{1}{\kappa}\right] + \left[2\beta_2 g(z^*) - \beta_1^2 g'(z^*) + \frac{1}{\kappa^2}\right] u \\ &\quad + i\left[3\beta_3 g(z^*) + 3\beta_1\beta_2 g'(z^*) - \frac{1}{2}\beta_1^3 g''(z^*) + \frac{1}{\kappa^3}\right] u^2 + \cdots\,. \end{aligned} \tag{4.19}$$

Now, from (4.2) and (4.6),

$$\int_{|z|=z^*} \frac{e^{Cf(z)}}{(1-z)z} dz = e^{Cf(z^*)} \int_\Gamma e^{-Cu^2} \left[\ell(u) - \frac{1}{(u+i\kappa)} \right] du \,, \tag{4.20}$$

where the contour Γ corresponds to $|z| = z^*$. The main contribution to the integral arises from the neighborhood of the saddle-point $z = z^*$, that is $u = 0$. Moreover, the limits of $\sqrt{C}\, u$ may be taken as $\pm\infty$, with an exponentially small error. If we let $\xi = \sqrt{C}\, u$, we obtain

$$\int_\Gamma e^{-Cu^2} \left[\ell(u) - \frac{1}{(u+i\kappa)} \right] du \sim \int_{-\infty}^{\infty} e^{-\xi^2} \left[\frac{1}{\sqrt{C}} \ell\left(\frac{\xi}{\sqrt{C}}\right) - \frac{1}{(\xi + i\kappa\sqrt{C}]} \right] d\xi \,. \tag{4.21}$$

From (4.19), we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\xi^2} \ell\left(\frac{\xi}{\sqrt{C}}\right) d\xi &= i\sqrt{\pi} \left\{ \left[\beta_1 g(z^*) - \frac{1}{\kappa} \right] \right. \\ &\left. + \frac{1}{2C} \left[3\beta_3 g(z^*) + 3\beta_1\beta_2 g'(z^*) - \frac{1}{2}\beta_1^3 g''(z^*) + \frac{1}{\kappa^3} \right] + \cdots \right\} . \end{aligned} \tag{4.22}$$

Also [14],

$$\int_{-\infty}^{\infty} \frac{e^{-\xi^2}}{(\xi + i\kappa\sqrt{C})} d\xi = -i\pi \text{ sgn } (\kappa) e^{\kappa^2 C} \text{ Erfc } (|\kappa|\sqrt{C}) \,, \tag{4.23}$$

and

$$\text{Erfc } (-x) = 2 - \text{ Erfc } (x) \,. \tag{4.24}$$

We note, from (3.8), (4.3), (4.13) and (4.18), that

$$K = \frac{\sqrt{v}}{\sqrt{2}} \left[\beta_1 g(z^*) - \frac{1}{\kappa} \right] \,, \tag{4.25}$$

and we define

$$E \triangleq -\frac{v\sqrt{v}}{2\sqrt{2}} \left[3\beta_3 g(z^*) + 3\beta_1\beta_2 g'(z^*) - \frac{1}{2}\beta_1^3 g''(z^*) + \frac{1}{\kappa^3} \right] . \tag{4.26}$$

Then, from (4.1) and (4.20)–(4.26), we obtain

Proposition 4.1 Asymptotically,

$$G(C) \sim M - \frac{E e^{Cf(z^*)}}{Cv\sqrt{2\pi C v}} \,, \tag{4.27}$$

where M is given by (3.7) and (3.8), v is given by (3.6) and

$$\begin{aligned} E &= \frac{[1 - 3z^* + 3(z^*)^2]}{(1-z^*)^3} + \frac{\tau(1-2z^*)}{2v(1-z^*)^2} \\ &+ \frac{1}{8}\left(\frac{5\tau^2}{3v^2} - \frac{y}{v}\right) \frac{1}{(1-z^*)} - \frac{v\sqrt{v} \text{ sgn } (1-z^*)}{[-2f(z^*)]^{3/2}}, \; z^* \neq 1 \,, \end{aligned} \tag{4.28}$$

where τ and y are given by (4.11) and (4.12). □

It is shown in Appendix A that the expression for E in (4.28) remains finite as $z^* \to 1$, and its limiting value is

$$E = 1 + \frac{\tau}{2v} + \frac{5\tau^2}{24v^2} + \frac{35\tau^3}{432v^3} - \frac{y}{8v}\left(1 + \frac{5\tau}{6v}\right) + \frac{w}{40v}, \ z^* = 1\,, \tag{4.29}$$

where

$$w \triangleq (z^*)^5 f^{(5)}(z^*) = \sum_{s=1}^{S} \alpha_s d_s^2 \left(d_s^3 - 10d_s^2 + 35d_s - 20\right)(z^*)^{d_s}\,. \tag{4.30}$$

Hence the expression in (4.27) provides a refined uniform asymptotic approximation to $G(C)$. However, the numerical evaluation of (4.28) loses accuracy significantly when z^* is close to 1, on account of the leading two terms. In Section 6 we give a representation for E which gives numerically accurate results whether or not z^* is close to 1, or even equal to 1.

If we make the transformation of variables in (4.2), and let $\xi = \sqrt{C}\,u$ as before, then, with the help of (4.16) we obtain

Proposition 4.2 If $h(z)$ is analytic in a domain containing $|z| = z^*$, then asymptotically

$$\begin{aligned} &\frac{1}{2\pi i}\int_{|z|=z^*} h(z)e^{Cf(z)}dz \\ &\sim \frac{e^{Cf(z^*)}}{2\sqrt{\pi C}}\left\{\beta_1 h(z^*) + \frac{1}{2C}\left[3\beta_3 h(z^*) + 3\beta_1\beta_2 h'(z^*) - \frac{1}{2}\beta_1^3 h''(z^*)\right]\right\}\,. \end{aligned} \tag{4.31}$$

□

In preparation for the next section, we let

$$h_s(z) \triangleq H_s^{(1)}(z)/z\,, \tag{4.32}$$

where $H_s^{(1)}(z^*)$ is defined in (3.12). If we deform the contour of integration in (3.4) to $|z| = z^*$, we obtain

$$\begin{aligned} &G(C) - G(C - d_s) \\ &\sim \frac{e^{Cf(z^*)}}{2\sqrt{\pi C}}\left\{\beta_1 h_s(z^*) + \frac{1}{2C}\left[3\beta_3 h_s(z^*) + 3\beta_1\beta_2 h_s'(z^*) - \frac{1}{2}\beta_1^3 h_s''(z^*)\right]\right\}\,. \end{aligned} \tag{4.33}$$

Also, from (2.7), (3.1) and (3.2),

$$\begin{aligned} &G(C) - G(C - d_s) - G(C - d_t) + G(C - d_s - d_t) \\ &= \frac{1}{2\pi i}\int_{|z|<1} \frac{(1 - z^{d_s})(1 - z^{d_t})}{(1 - z)z} e^{Cf(z)}dz\,. \end{aligned} \tag{4.34}$$

We define

$$h_{s,t}(z) \triangleq (1 - z^{d_s})h_t(z) = (1 - z^{d_t})h_s(z)\,. \tag{4.35}$$

Then, if we deform the contour of integration in (4.34) to $|z| = z^*$, we find that

$$\begin{aligned} &G(C) - G(C - d_s) - G(C - d_t) + G(C - d_s - d_t) \\ &\sim \frac{e^{Cf(z^*)}}{2\sqrt{\pi C}} \left\{ \beta_1 h_{s,t}(z^*) + \frac{1}{2C}\left[3\beta_3 h_{s,t}(z^*) + 3\beta_1\beta_2 h'_{s,t}(z^*) - \frac{1}{2}\beta_1^3 h''_{s,t}(z^*)\right]\right\}. \end{aligned} \tag{4.36}$$

We use (4.33) and (4.36) in the next section.

5 Refined approximations to $\mathbb{L}_s$ and $\partial\mathbb{L}_s/\partial\nu_t$

We first derive a refined asymptotic approximation to $\partial\mathbb{L}_s/\partial\nu_t$. We write (2.5) in the equivalent form

$$\begin{aligned} \partial\mathbb{L}_s/\partial\nu_t &= [G(C) - G(C - d_s)][G(C) - G(C - d_t)]\,/[G(C)]^2 \\ &\quad - [G(C) - G(C - d_s) - G(C - d_t) + G(C - d_s - d_t)]/G(C)\,. \end{aligned} \tag{5.1}$$

But, asymptotically, from (4.27) and (3.13), neglecting smaller order terms,

$$\frac{1}{G(C)} \sim \frac{1}{M}\left(1 + \frac{BE}{Cv}\right),\ \frac{1}{[G(C)]^2} \sim \frac{1}{M^2}\left(1 + \frac{2BE}{Cv}\right). \tag{5.2}$$

Hence, from (4.33) and (4.36), again neglecting smaller order terms,

$$\begin{aligned} \partial\mathbb{L}_s/\partial\nu_t &\sim \beta_1^2 \frac{e^{2Cf(z^*)}}{4\pi M^2 C}\left(1 + \frac{2BE}{Cv}\right) h_s(z^*)h_t(z^*) \\ &\quad + \beta_1 \frac{e^{2Cf(z^*)}}{8\pi M^2 C^2} k_{s,t}(z^*) - \beta_1 \frac{e^{Cf(z^*)}}{2M\sqrt{\pi C}}\left(1 + \frac{BE}{Cv}\right) h_{s,t}(z^*) \\ &\quad - \frac{e^{Cf(z^*)}}{4MC\sqrt{\pi C}}\left[3\beta_3 h_{s,t}(z^*) + 3\beta_1\beta_2 h'_{s,t}(z^*) - \frac{1}{2}\beta_1^3 h''_{s,t}(z^*)\right], \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} k_{s,t}(z) &\triangleq h_s(z)\left[3\beta_3 h_t(z) + 3\beta_1\beta_2 h'_t(z) - \frac{1}{2}\beta_1^3 h''_t(z)\right] \\ &\quad + h_t(z)\left[3\beta_3 h_s(z) + 3\beta_1\beta_2 h'_s(z) - \frac{1}{2}\beta_1^3 h''_s(z)\right]. \end{aligned} \tag{5.4}$$

Now, from (3.12) and (4.32),

$$H_t^{(1)}(z) = zh_t(z) = \sum_{n=0}^{d_t - 1} z^n\,. \tag{5.5}$$

We define

$$H_t^{(2)}(z) \triangleq z^2 h_t'(z) = \sum_{n=0}^{d_t-1} (n-1) z^n \,, \tag{5.6}$$

and

$$H_t^{(3)}(z) \triangleq z^3 h_t''(z) = \sum_{n=0}^{d_t-1} (n-1)(n-2) z^n \,. \tag{5.7}$$

From the first expression for $h_{s,t}(z)$ in (4.35), and (5.5)–(5.7), we obtain

$$z h_{s,t}(z) = (1 - z^{d_s}) H_t^{(1)}(z) \,, \tag{5.8}$$

$$z^2 h_{s,t}'(z) = (1 - z^{d_s}) H_t^{(2)}(z) - d_s z^{d_s} H_t^{(1)}(z) \,, \tag{5.9}$$

and

$$z^3 h_{s,t}''(z) = (1 - z^{d_s}) H_t^{(3)}(z) - 2 d_s z^{d_s} H_t^{(2)}(z) - d_s(d_s - 1) z^{d_s} H_t^{(1)}(z) \,. \tag{5.10}$$

From (3.12) or (5.5),

$$1 - z^{d_s} = (1 - z) H_s^{(1)}(z) \,, \tag{5.11}$$

for all z, and substitution in (5.8) exhibits the symmetry of $h_{s,t}(z)$ in s and t. Expressions for $d_s z^{d_s}$ and $d_s(d_s-1)z^{d_s}$, which lead to symmetric expressions for $h_{s,t}'(z)$ and $h_{s,t}''(z)$, may also be obtained from (5.5)–(5.7). However, it is simpler not to use those expressions, and it is unnecessary in view of the expression in (2.9) for the implied costs.

From (5.3)–(5.11), with the help of (4.13), we obtain

Proposition 5.1 Let

$$\begin{aligned} A_s^{(1)} \;&\triangleq\; \left\{ B(B - 1 + z^*) + \frac{B}{Cv}(2B - 1 + z^*) \left[\frac{1}{8} \left(\frac{y}{v} - \frac{5\tau^2}{3v^2} \right) + BE \right] \right\} H_s^{(1)}(z^*) \\ &\quad + \frac{B}{2Cv} \left\{ B \left[\frac{\tau}{v} H_s^{(2)}(z^*) - H_s^{(3)}(z^*) \right] + d_s \left(\frac{\tau}{v} + 1 - d_s \right) (z^*)^{d_s} \right\} \,, \end{aligned} \tag{5.12}$$

$$A_s^{(2)} \triangleq \frac{B}{Cv} \left[\frac{\tau}{2v}(B - 1 + z^*) H_s^{(1)}(z^*) - d_s (z^*)^{d_s} \right] \,, \tag{5.13}$$

and

$$A_s^{(3)} \triangleq -\frac{B}{2Cv}(B - 1 + z^*) H_s^{(1)}(z^*) \,, \tag{5.14}$$

where B and E are given by (3.13) and (4.28), and v, τ, and y by (3.6), (4.11) and (4.12). Then, we have the refined uniform asymptotic approximation

$$\partial \mathbb{L}_s / \partial \nu_t \sim \sum_{i=1}^{3} A_s^{(i)} H_t^{(i)}(z^*) \,. \tag{5.15}$$

□

Explicit expressions for $H_t^{(2)}(z)$ and $H_t^{(3)}(z)$, analogous to those for $H_t^{(1)}(z)$ given by (3.12), may be obtained from (5.6) and (5.7). However, although $H_t^{(i)}(z)$, $i = 1, 2, 3$, remain finite as $z \to 1$, it is better to use the finite sums in (5.5)–(5.7) for the purpose of numerical evaluation.

We now consider $\mathbb{L}_s$ itself. From (2.3), (4.33) and (5.2), neglecting smaller order terms, we have

$$\begin{aligned} \mathbb{L}_s \quad \sim \quad & \frac{\beta_1 e^{Cf(z^*)}}{2M\sqrt{\pi C}} h_s(z^*) \left(1 + \frac{BE}{Cv}\right) \\ & + \frac{e^{Cf(z^*)}}{4MC\sqrt{\pi C}} \left[3\beta_3 h_s(z^*) + 3\beta_1\beta_2 h_s'(z^*) - \frac{1}{2}\beta_1^3 h_s''(z^*)\right]. \end{aligned} \tag{5.16}$$

Hence, from (3.13), (4.13) and (5.5)–(5.7), we obtain

Proposition 5.2 A refined uniform asymptotic approximation to the loss probability for service s is

$$\begin{aligned} \mathbb{L}_s \quad \sim \quad & B\left\{1 + \frac{1}{Cv}\left[\frac{1}{8}\left(\frac{y}{v} - \frac{5\tau^2}{3v^2}\right) + BE\right]\right\} H_s^{(1)}(z^*) \\ & + \frac{B}{2Cv}\left[\frac{\tau}{v} H_s^{(2)}(z^*) - H_s^{(3)}(z^*)\right]. \end{aligned} \tag{5.17}$$

□

6 Uniform representations

We first derive a representation for K which gives numerically accurate results, whether or not z^* is close to 1. From (3.8) we obtain

$$K = 1 - \frac{[2(z^*)^2 f(z^*) + (1-z^*)^2 v]}{(1-z^*)\sqrt{-2f(z^*)}\left[z^*\sqrt{-2f(z^*)} + |1-z^*|\sqrt{v}\right]}. \tag{6.1}$$

Since $z^* > 0$, $f(1) = 0$ and $f'(z^*) = 0$, it follows from (3.6) that $f(z^*) = 0(|1-z^*|^2)$ for $|1-z^*| \ll 1$, and that both the numerator and the denominator in the fraction in (6.1) are $O(|1-z^*|^3)$. But,

$$m! \sum_{r=0}^{m} f^{(r)}(z) \frac{(1-z)^r}{r!} = (1-z)^{m+1} \frac{d^m}{dz^m}\left[\frac{f(z)}{(1-z)}\right], \tag{6.2}$$

and, from (3.2),

$$\frac{f(z)}{(1-z)} = -\left[\sum_{s=1}^{S} \alpha_s \sum_{n=0}^{d_s-1} z^n + \frac{\log z}{(1-z)}\right]. \tag{6.3}$$

We let

$$\Gamma^{\{m\}}(z) \triangleq z^m \frac{d^m}{dz^m}\left[\frac{\log z}{(1-z)}\right]. \tag{6.4}$$

Then, from (6.2)–(6.4), for $m = 1, 2, \ldots,$ we obtain

$$m!\; z^m \sum_{r=0}^{m} f^{(r)}(z)\frac{(1-z)^r}{r!} = -(1-z)^{m+1}\left[\sum_{s=1}^{S}\alpha_s \sum_{n=0}^{d_s-1}\prod_{\ell=0}^{m-1}(n-\ell)z^n + \Gamma^{\{m\}}(z)\right]. \tag{6.5}$$

Since $f'(z^*) = 0$, (6.5) implies that

$$f(z^*) = -(1-z^*)^2\phi\,, \tag{6.6}$$

where

$$z^*\phi = \sum_{s=1}^{S}\alpha_s \sum_{n=0}^{d_s-1} n(z^*)^n + \Gamma^{\{1\}}(z^*)\,, \tag{6.7}$$

and, from (3.6), that

$$2(z^*)^2 f(z^*) + (1-z^*)^2 v = -(1-z^*)^3\psi\,, \tag{6.8}$$

where

$$\psi = \sum_{s=1}^{S}\alpha_s \sum_{n=0}^{d_s-1} n(n-1)(z^*)^n + \Gamma^{\{2\}}(z^*)\,. \tag{6.9}$$

Explicit expressions for $\Gamma^{\{m\}}(z)$ for $z \neq 1$ may be obtained by substituting $\log z$ for $f(z)$ in (6.2). Also, it is shown in Appendix B that

$$\Gamma^{\{m\}}(z) = (-1)^{m+1}(m!)^2 \sum_{i=0}^{\infty}\frac{i!(1-z)^i}{(i+m+1)!}, \quad |1-z| < 1\,. \tag{6.10}$$

From (6.1), (6.6), (6.8) and (6.10), we obtain

Proposition 6.1

$$K = 1 + \frac{\psi}{\sqrt{2\phi}\;(z^*\sqrt{2\phi}+\sqrt{v})}\,, \tag{6.11}$$

where v, ϕ and ψ are given by (3.6), (6.7) and (6.9), and

$$\Gamma^{\{1\}}(z) = \begin{cases} (z\log z + 1 - z)/(1-z)^2, & |1-z| > \epsilon_1, \\ \displaystyle\sum_{i=0}^{\infty}\frac{(1-z)^i}{(i+1)(i+2)}, & |1-z| \le \epsilon_1 < 1, \end{cases} \tag{6.12}$$

and

$$\Gamma^{\{2\}}(z) = \begin{cases} [2z^2\log z + 2z(1-z) - (1-z)^2]/(1-z)^3, & |1-z| > \epsilon_2, \\ \displaystyle -4\sum_{i=0}^{\infty}\frac{(1-z)^i}{(i+1)(i+2)(i+3)}, & |1-z| \le \epsilon_2 < 1. \end{cases} \tag{6.13}$$

□

Since $z^* > 0$, it follows from (6.7) and (6.12) that $\phi > 0$.

Accurate numerical evaluations of $\Gamma^{\{1\}}(z)$ and $\Gamma^{\{2\}}(z)$ are obtained by choosing $\epsilon_1 = 0.1 = \epsilon_2$ in (6.12) and (6.13), and truncating the sums at $i = 12$. Thus (6.11) leads to accurate numerical values of K, whether or not z^* is close to 1. Also, (6.6) leads to more accurate values of $f(z^*)$ than (3.2) does when z^* is close to 1.

We now give a representation for E which gives numerically accurate results, whether or not z^* is close to 1. We omit the details of the derivation. We have

Proposition 6.2

$$
\begin{aligned}
E \;=\; & \frac{(\theta + y)}{8z^* v} + K^3 - \frac{9z^*\psi^2}{8v^2} - \frac{\chi}{v}\left[1 + \frac{5}{24v}(\tau + 3z^*\psi)\right] \\
& + 3\psi^2(K-1)\frac{[3(z^*\sqrt{2\phi})^2 + 9\sqrt{v}(z^*\sqrt{2\phi}) + 8v]}{8v^2(z^*\sqrt{2\phi} + \sqrt{v})^2} \\
& + \frac{3\psi(z^*\sqrt{2\phi} + 2\sqrt{v})}{2v(z^*\sqrt{2\phi} + \sqrt{v})}(K-1)\left(K + 1 + \frac{z^*\psi}{2v}\right),
\end{aligned} \tag{6.14}
$$

where τ and y are given by (4.11) and (4.12), and

$$
\chi = \sum_{s=1}^{S} \alpha_s \sum_{n=0}^{d_s - 1} n(n-1)(n-2)(z^*)^n + \Gamma^{\{3\}}(z^*), \tag{6.15}
$$

$$
\theta = \sum_{s=1}^{S} \alpha_s \sum_{n=0}^{d_s - 1} n(n-1)(n-2)(n-3)(z^*)^n + \Gamma^{\{4\}}(z^*), \tag{6.16}
$$

and, from (6.2) with $\log z$ substituted for $f(z)$, and (6.10),

$$
\Gamma^{\{3\}}(z) = \begin{cases} [6z^3\log z + 6z^2(1-z) - 3z(1-z)^2 + 2(1-z)^3]/(1-z)^4, & |1-z| > \epsilon_3, \\[2ex] 36\displaystyle\sum_{i=0}^{\infty}\frac{(1-z)^i}{(i+1)(i+2)(i+3)(i+4)}, & |1-z| \le \epsilon_3 < 1, \end{cases} \tag{6.17}
$$

and

$$
\Gamma^{\{4\}}(z) = \begin{cases} [24z^4\log z + 24z^3(1-z) - 12z^2(1-z)^2 + 8z(1-z)^3 - 6(1-z)^4]/(1-z)^5, \\ \hfill |1-z| > \epsilon_4, \\[2ex] -576\displaystyle\sum_{i=0}^{\infty}\frac{(1-z)^i}{(i+1)(i+2)(i+3)(i+4)(i+5)}, \quad |1-z| \le \epsilon_4 < 1. \end{cases} \tag{6.18}
$$

□

Accurate numerical evaluations of $\Gamma^{\{3\}}(z)$ and $\Gamma^{\{4\}}(z)$ are obtained by choosing $\epsilon_3 = 0.2$ and $\epsilon_4 = 0.275$, and by truncating the sums in (6.17) and (6.18) at $i = 15$ and $i = 18$, respectively. Thus (6.14) leads to accurate numerical values of E, whether or not z^* is close to 1.

7 Numerical results

A uniform asymptotic approximation (UAA) to the loss probability $\mathbb{L}_s$ for service s is given by (3.14), see Proposition 3.2, where z^* satisfies (3.5), and v and M are given by (3.6) and (3.7). Numerically accurate values of $f(z^*)$ and K are obtained from (6.6) and (6.11), see Proposition 6.1. A refined uniform asymptotic approximation (RUAA) to $\mathbb{L}_s$ is given by (5.17), where τ and y are given by (4.11) and (4.12), and $H_s^{(i)}(z)$, $i = 1, 2, 3$, by (5.5)–(5.7). Numerically accurate values of E are obtained from (6.14), see Proposition 6.2. Exact values of $\mathbb{L}_s$ are obtained from the one-dimensional recursion of Kaufman [8]. We compare the UAA and RUAA to the loss probabilities with the exact results, for two particular examples.

We first consider an example which was investigated by Reiman [18], and has capacity $C = 96$ and two services, with capacity per call, $d_1 = 1$ and $d_2 = 6$. The corresponding traffic intensities are $\nu_1 = 0.86\nu/1.7$ and $\nu_2 = 0.14\nu/1.7$, where $\nu = \nu_1 d_1 + \nu_2 d_2$ is the total traffic intensity. Reiman compared the approximations which he derived for the loss probabilities in the critically loaded regime with the exact results for $\nu = 96 + \beta\sqrt{96}$ and $-1 \le \beta \le 1$. In Table 1 we compare the UAA and RUAA to the loss probabilities with the exact results for $\beta = -6, -5, \ldots, 3$, which includes the underloaded, critically loaded and overloaded regimes. It is seen that while the UAA is quite accurate, the RUAA is very accurate.

The second example has capacity $C = 130$ and three services, with capacity per call, $d_1 = 1$, $d_2 = 3$ and $d_3 = 10$. The corresponding traffic intensities are $\nu_1 = 2\nu/13 = \nu_2$ and $\nu_3 = \nu/26$, where $\nu = \nu_1 d_1 + \nu_2 d_2 + \nu_3 d_3$ is the total traffic intensity. In Table 2 we compare the UAA and RUAA to the loss probabilities with the exact results for $\nu = 46.8(20.8)171.6$, which includes the underloaded, critically loaded and overloaded regimes. Again the UAA is quite accurate, and the RUAA is very accurate.

A UAA to $\partial\mathbb{L}_s/\partial\nu_t$ is given by (3.15), and a RUAA by (5.15), see Proposition 5.1. It may be shown that the expression in (5.15) is symmetric in s and t. Exact values of $\partial\mathbb{L}_s/\partial\nu_t$ are obtained from (2.5), with the help of the one-dimensional recursion of Kaufman [8] for the aggregate state distribution. In Table 3 we compare the UAA and RUAA to $\partial\mathbb{L}_1/\partial\nu_1$,

$\partial\mathbb{L}_1/\partial\nu_2 = \partial\mathbb{L}_2/\partial\nu_1$ and $\partial\mathbb{L}_2/\partial\nu_2$ with the exact results for the first example. It is seen that overall the UAA is rather poor, but the RUAA is very good. In Table 4 we compare the UAA and RUAA to $\partial\mathbb{L}_1/\partial\nu_1$, $\partial\mathbb{L}_1/\partial\nu_2 = \partial\mathbb{L}_2/\partial\nu_1$, $\partial\mathbb{L}_1/\partial\nu_3 = \partial\mathbb{L}_3/\partial\nu_1$, $\partial\mathbb{L}_2/\partial\nu_2$, $\partial\mathbb{L}_2/\partial\nu_3 = \partial\mathbb{L}_3/\partial\nu_2$ and $\partial\mathbb{L}_3/\partial\nu_3$ with the exact results for the second example. Again the UAA is rather poor, but the RUAA is very good.

Appendix A

We first establish (4.11), (4.12) and (4.30). From (3.2) it follows that

$$zf'(z) = \sum_{s=1}^{S} \alpha_s d_s z^{d_s} - 1\,. \tag{A1}$$

Successive differentations imply that

$$zf''(z) + f'(z) = \sum_{s=1}^{S} \alpha_s d_s^2 z^{d_s-1}\,, \tag{A2}$$

$$zf^{(3)}(z) + 2f''(z) = \sum_{s=1}^{S} \alpha_s d_s^2 (d_s-1) z^{d_s-2}\,, \tag{A3}$$

$$zf^{(4)}(z) + 3f^{(3)}(z) = \sum_{s=1}^{S} \alpha_s d_s^2 (d_s-1)(d_s-2) z^{d_s-3}\,, \tag{A4}$$

and

$$zf^{(5)}(z) + 4f^{(4)}(z) = \sum_{s=1}^{S} \alpha_s d_s^2 (d_s-1)(d_s-2)(d_s-3) z^{d_s-4}\,. \tag{A5}$$

But, $f'(z^*) = 0$ and $f''(z^*)$ is given by (3.6). Some straightforward algebra leads to (4.11), (4.12) and (4.30).

We now consider $|1-z^*| \ll 1$. Since $f(1) = 0$ and $f'(z^*) = 0$, we have

$$\begin{aligned} 0 &= f(z^*) + \frac{1}{2} f''(z^*)(1-z^*)^2 + \frac{1}{6} f^{(3)}(z^*)(1-z^*)^3 \\ &+ \frac{1}{24} f^{(4)}(z^*)(1-z^*)^4 + \frac{1}{120} f^{(5)}(z^*)(1-z^*)^5 + O(|1-z^*|^6)\,. \end{aligned} \tag{A6}$$

Hence, from (3.6), (4.10) and (4.30),

$$\begin{aligned} -2f(z^*) &= v\left(\frac{1}{z^*}-1\right)^2 + \frac{\tau}{3}\left(\frac{1}{z^*}-1\right)^3 + \frac{y}{12}\left(\frac{1}{z^*}-1\right)^4 \\ &+ \frac{w}{60}\left(\frac{1}{z^*}-1\right)^5 + O(|1-z^*|^6)\,. \end{aligned} \tag{A7}$$

It follows, for $z^* \neq 1$, that

$$\begin{aligned} &\frac{\sqrt{v}\ \mathrm{sgn}\ (1-z^*)}{\sqrt{-2f(z^*)}} = \frac{z^*}{(1-z^*)} \\ &\cdot \left[1 + \frac{\tau}{3v}\left(\frac{1}{z^*}-1\right) + \frac{y}{12v}\left(\frac{1}{z^*}-1\right)^2 + \frac{w}{60v}\left(\frac{1}{z^*}-1\right)^3 + O(|1-z^*|^4)\right]^{-1/2}\,, \end{aligned} \tag{A8}$$

and hence that

$$\frac{v\sqrt{v}\,\mathrm{sgn}\,(1-z^*)}{[-2f(z^*)]^{3/2}} = \frac{(z^*)^3}{(1-z^*)^3} - \frac{\tau(z^*)^2}{2v(1-z^*)^2} + \frac{1}{8}\left(\frac{5\tau^2}{3v^2} - \frac{y}{v}\right)\frac{z^*}{(1-z^*)}$$

$$+ \frac{1}{48}\left(\frac{5y\tau}{v^2} - \frac{35\tau^3}{9v^3} - \frac{6w}{5v}\right) + O(|1-z^*|)\,. \tag{A9}$$

The limiting value in (4.29) follows from (4.28) and (A9).

Appendix B

We here establish (6.10). First, if we substitute $\log z$ for $f(z)$ in (6.2), it follows, for $z \neq 1$, that

(B1) $$\frac{d^{m+1}}{dz^{m+1}}\left[\frac{\log z}{(1-z)}\right] - \frac{(m+1)}{(1-z)}\,\frac{d^m}{dz^m}\left[\frac{\log z}{(1-z)}\right] = \frac{(-1)^m m!}{z^{m+1}(1-z)},$$

for $m = 0, 1, 2, \cdots$. Hence, from (6.4), we have

(B2) $$(1-z)\Gamma^{\{m+1\}}(z) = (m+1)z\,\Gamma^{\{m\}}(z) + (-1)^m m!$$

Now (6.10) holds for $m = 0$. We assume that it is true for some nonnegative integer m, and show that it is true for $m+1$. Thus, from (6.10),

$$\begin{aligned}
&(m+1)[1-(1-z)]\Gamma^{\{m\}}(z) + (-1)^m m! \\
&\quad = (-1)^m m!(m+1)!\left[\sum_{i=0}^{\infty}\frac{i!(1-z)^{i+1}}{(i+m+1)!} - \sum_{i=1}^{\infty}\frac{i!(1-z)^i}{(i+m+1)!}\right] \\
&\quad = (-1)^m[(m+1)!]^2\sum_{i=0}^{\infty}\frac{i!(1-z)^{i+1}}{(i+m+2)!}. \qquad \text{(B3)}
\end{aligned}$$

Hence, from (B2), (6.10) holds with m replaced by $(m+1)$, and the result is proved by induction.

Table 1: Comparison of UAA and RUAA to the loss probabilities $\mathbb{L}_1$ and $\mathbb{L}_2$ with the exact results for two services with $C = 96$, $d_1 = 1$, $d_2 = 6$, $\nu_1 = 0.86\nu/1.7$, $\nu_2 = 0.14\nu/1.7$ and $\nu = 96 + \beta\sqrt{96}$.

β	UAA	RUAA	Exact
-6	5.684×10^{-6}	5.6517×10^{-6}	5.6516×10^{-6}
	6.504×10^{-5}	6.5061×10^{-5}	6.5063×10^{-5}
-5	1.213×10^{-4}	1.2065×10^{-4}	1.2065×10^{-4}
	1.189×10^{-3}	1.1898×10^{-3}	1.1898×10^{-3}
-4	1.017×10^{-3}	1.0110×10^{-3}	1.0110×10^{-3}
	8.780×10^{-3}	8.7827×10^{-3}	8.7828×10^{-3}
-3	4.439×10^{-3}	4.4130×10^{-3}	4.4129×10^{-3}
	3.440×10^{-2}	3.4413×10^{-2}	3.4414×10^{-2}
-2	1.227×10^{-2}	1.2199×10^{-2}	1.2198×10^{-2}
	8.651×10^{-2}	8.6539×10^{-2}	8.6541×10^{-2}
-1	2.489×10^{-2}	2.4749×10^{-2}	2.4748×10^{-2}
	1.613×10^{-1}	1.6133×10^{-1}	1.6133×10^{-1}
0	4.123×10^{-2}	4.0993×10^{-2}	4.0991×10^{-2}
	2.474×10^{-1}	2.4745×10^{-1}	2.4745×10^{-1}
1	5.984×10^{-2}	5.9508×10^{-2}	5.9504×10^{-2}
	3.346×10^{-1}	3.3468×10^{-1}	3.3469×10^{-1}
2	7.961×10^{-2}	7.9176×10^{-2}	7.9171×10^{-2}
	4.168×10^{-1}	4.1697×10^{-1}	4.1697×10^{-1}
3	9.982×10^{-2}	9.9290×10^{-2}	9.9282×10^{-2}
	4.915×10^{-1}	4.9165×10^{-1}	4.9166×10^{-1}

Table 2: Comparison of UAA and RUAA to the loss probabilities $\mathbb{L}_1, \mathbb{L}_2$ and $\mathbb{L}_3$ with the exact results for three services with $C = 130$, $d_1 = 1$, $d_2 = 3$, $d_3 = 10$, $\nu_1 = 2\nu/13 = \nu_2$ and $\nu_3 = \nu/26$.

ν	UAA	RUAA	Exact
46.8	4.260×10^{-6}	4.2324×10^{-6}	4.2324×10^{-6}
	1.508×10^{-5}	1.5049×10^{-5}	1.5049×10^{-5}
	9.565×10^{-5}	9.5638×10^{-5}	9.5641×10^{-5}
67.6	2.933×10^{-4}	2.9144×10^{-4}	2.9143×10^{-4}
	9.826×10^{-4}	9.8054×10^{-4}	9.8049×10^{-4}
	4.963×10^{-3}	4.9621×10^{-3}	4.9263×10^{-3}
88.4	3.235×10^{-3}	3.2152×10^{-3}	3.2151×10^{-3}
	1.038×10^{-2}	1.0364×10^{-2}	1.0363×10^{-2}
	4.431×10^{-2}	4.4302×10^{-2}	4.4304×10^{-2}
109.2	1.272×10^{-2}	1.2647×10^{-2}	1.2647×10^{-2}
	3.939×10^{-2}	3.9329×10^{-2}	3.9327×10^{-2}
	1.469×10^{-1}	1.4689×10^{-1}	1.4690×10^{-1}
130	2.858×10^{-2}	2.8425×10^{-2}	2.8423×10^{-2}
	8.575×10^{-2}	8.5659×10^{-2}	8.5653×10^{-2}
	2.858×10^{-1}	2.8573×10^{-1}	2.8574×10^{-1}
150.8	4.785×10^{-2}	4.7601×10^{-2}	4.7598×10^{-2}
	1.395×10^{-1}	1.3944×10^{-1}	1.3943×10^{-1}
	4.222×10^{-1}	4.2193×10^{-1}	4.2195×10^{-1}
171.6	6.825×10^{-2}	6.7922×10^{-2}	6.7917×10^{-2}
	1.939×10^{-1}	1.9386×10^{-1}	1.9385×10^{-1}
	5.388×10^{-1}	5.3841×10^{-1}	5.3842×10^{-1}

Table 3: Comparison of UAA and RUAA to $\partial \mathbb{L}_1/\partial \nu_1$, $\partial \mathbb{L}_1/\partial \nu_2 = \partial \mathbb{L}_2/\partial \nu_1$ and $\partial \mathbb{L}_2/\partial \nu_2$ with the exact results for two services with $C = 96$, $d_1 = 1$, $d_2 = 6$, $\nu_1 = 0.86\nu/1.7$, $\nu_2 = 0.14\nu/1.7$ and $\nu = 96 + \beta\sqrt{96}$.

β	UAA	RUAA	Exact
-6	1.46×10^{-6}	1.486×10^{-6}	1.486×10^{-6}
	1.67×10^{-5}	1.655×10^{-5}	1.655×10^{-5}
	1.91×10^{-4}	1.800×10^{-4}	1.799×10^{-4}
-5	2.37×10^{-5}	2.430×10^{-5}	2.429×10^{-5}
	2.32×10^{-4}	2.297×10^{-4}	2.297×10^{-4}
	2.27×10^{-3}	2.122×10^{-3}	2.121×10^{-3}
-4	1.48×10^{-4}	1.537×10^{-4}	1.536×10^{-4}
	1.28×10^{-3}	1.264×10^{-3}	1.264×10^{-3}
	1.10×10^{-2}	1.015×10^{-2}	1.015×10^{-2}
-3	4.72×10^{-4}	4.970×10^{-4}	4.968×10^{-4}
	3.65×10^{-3}	3.604×10^{-3}	3.604×10^{-3}
	2.83×10^{-2}	2.548×10^{-2}	2.548×10^{-2}
-2	9.39×10^{-4}	1.011×10^{-3}	1.011×10^{-3}
	6.62×10^{-3}	6.504×10^{-3}	6.505×10^{-3}
	4.67×10^{-2}	4.069×10^{-2}	4.068×10^{-2}
-1	1.38×10^{-3}	1.530×10^{-3}	1.530×10^{-3}
	8.96×10^{-3}	8.746×10^{-3}	8.746×10^{-3}
	5.80×10^{-2}	4.847×10^{-2}	4.846×10^{-2}
0	1.70×10^{-3}	1.945×10^{-3}	1.943×10^{-3}
	1.02×10^{-2}	9.873×10^{-3}	9.874×10^{-3}
	6.12×10^{-2}	4.847×10^{-2}	4.846×10^{-2}
1	1.89×10^{-3}	2.239×10^{-3}	2.238×10^{-3}
	1.05×10^{-2}	1.010×10^{-2}	1.010×10^{-2}
	5.89×10^{-2}	4.387×10^{-2}	4.386×10^{-2}
2	1.97×10^{-3}	2.441×10^{-3}	2.440×10^{-3}
	1.03×10^{-2}	9.769×10^{-3}	9.770×10^{-3}
	5.41×10^{-2}	3.756×10^{-2}	3.755×10^{-2}
3	2.00×10^{-3}	2.580×10^{-3}	2.580×10^{-3}
	9.85×10^{-3}	9.164×10^{-3}	9.164×10^{-3}
	4.85×10^{-2}	3.116×10^{-2}	3.115×10^{-2}

Table 4: Comparison of UAA and RUAA to $\partial\mathbb{L}_1/\partial\nu_1$, $\partial\mathbb{L}_1/\partial\nu_2$, $\partial\mathbb{L}_1/\partial\nu_3$, $\partial\mathbb{L}_2/\partial\nu_2$, $\partial\mathbb{L}_2/\partial\nu_3$ and $\partial\mathbb{L}_3/\partial\nu_3$ with the exact results for three services with $C = 130$, $d_1 = 1$, $d_2 = 3$, $d_3 = 10$, $\nu_1 = 2\nu/13 = \nu_2$ and $\nu_3 = \nu/26$.

ν	UAA	RUAA	Exact
46.8	7.27×10^{-7}	7.426×10^{-7}	7.424×10^{-7}
	2.57×10^{-6}	2.620×10^{-6}	2.620×10^{-6}
	1.63×10^{-5}	1.612×10^{-5}	1.612×10^{-5}
	9.11×10^{-6}	9.235×10^{-6}	9.236×10^{-6}
	5.78×10^{-5}	5.651×10^{-5}	5.650×10^{-5}
	3.66×10^{-4}	3.373×10^{-4}	3.373×10^{-4}
67.6	3.31×10^{-5}	3.429×10^{-5}	3.428×10^{-5}
	1.11×10^{-4}	1.141×10^{-4}	1.141×10^{-4}
	5.60×10^{-4}	5.507×10^{-4}	5.507×10^{-4}
	3.71×10^{-4}	3.788×10^{-4}	3.789×10^{-4}
	1.87×10^{-3}	1.819×10^{-3}	1.818×10^{-3}
	9.47×10^{-3}	8.514×10^{-3}	8.515×10^{-3}
88.4	2.31×10^{-4}	2.454×10^{-4}	2.453×10^{-4}
	7.41×10^{-4}	7.768×10^{-4}	7.769×10^{-4}
	3.16×10^{-3}	3.090×10^{-3}	3.090×10^{-3}
	2.38×10^{-3}	2.455×10^{-3}	2.456×10^{-3}
	1.02×10^{-2}	9.706×10^{-3}	9.703×10^{-3}
	4.33×10^{-2}	3.732×10^{-2}	3.734×10^{-2}
109.2	5.64×10^{-4}	6.228×10^{-4}	6.223×10^{-4}
	1.75×10^{-3}	1.881×10^{-3}	1.882×10^{-3}
	6.52×10^{-3}	6.276×10^{-3}	6.275×10^{-3}
	5.41×10^{-3}	5.672×10^{-3}	5.677×10^{-3}
	2.02×10^{-2}	1.879×10^{-2}	1.878×10^{-2}
	7.52×10^{-2}	6.030×10^{-2}	6.033×10^{-2}
130	8.17×10^{-4}	9.497×10^{-4}	9.491×10^{-4}
	2.45×10^{-3}	2.738×10^{-3}	2.740×10^{-3}
	8.17×10^{-3}	7.702×10^{-3}	7.699×10^{-3}
	7.35×10^{-3}	7.878×10^{-3}	7.890×10^{-3}
	2.45×10^{-2}	2.199×10^{-2}	2.197×10^{-2}
	8.17×10^{-2}	5.906×10^{-2}	5.914×10^{-2}
150.8	9.41×10^{-4}	1.161×10^{-3}	1.160×10^{-3}
	2.74×10^{-3}	3.193×10^{-3}	3.198×10^{-3}
	8.30×10^{-3}	7.589×10^{-3}	7.583×10^{-3}
	8.00×10^{-3}	8.768×10^{-3}	8.787×10^{-3}
	2.42×10^{-2}	2.066×10^{-2}	2.063×10^{-2}
	7.32×10^{-2}	4.649×10^{-2}	4.660×10^{-2}
171.6	9.83×10^{-4}	1.291×10^{-3}	1.291×10^{-3}
	2.79×10^{-3}	3.390×10^{-3}	3.397×10^{-3}
	7.76×10^{-3}	6.820×10^{-3}	6.811×10^{-3}
	7.93×10^{-3}	8.883×10^{-3}	8.910×10^{-3}
	2.20×10^{-2}	1.770×10^{-2}	1.766×10^{-2}
	6.12×10^{-2}	3.340×10^{-2}	3.356×10^{-2}

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