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TIME DEPENDENT RESONANCE THEORY

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Abstract

An important class of resonance problems involves the study of perturbations of systems having embedded eigenvalues in their continuous spectrum. Problems with this mathematical structure arise in the study of many physical systems, *e.g.* the coupling of an atom or molecule to a photon-radiation field, and Auger states of the helium atom, as well as in spectral geometry and number theory. We present a dynamic (time-dependent) theory of such *quantum resonances*. The key hypotheses are (i) a resonance condition which holds generically (non-vanishing of the *Fermi golden rule*) and (ii) local decay estimates for the unperturbed dynamics with initial data consisting of continuum modes associated with an interval containing the embedded eigenvalue of the unperturbed Hamiltonian. No assumption of dilation analyticity of the potential is made. Our method explicitly demonstrates the flow of energy from the resonant discrete mode to continuum modes due to their coupling. The approach is also applicable to nonautonomous linear problems and to nonlinear problems. We derive the time behavior of the resonant states for intermediate and long times. Examples and applications are presented. Among them is a proof of the instability of an embedded eigenvalue at a threshold energy under suitable hypotheses.

1 Introduction

The theory of resonances has its origins in attempts to explain the existence of metastable states in physical systems. These are states which are localized or coherent for some long time period, called the *lifetime*, and then disintegrate. Examples abound and include unstable atoms and particles.

The mathematical analysis of resonance phenomena naturally leads to the study of perturbations of self-adjoint operators which have embedded eigenvalues in their continuous spectra. An example of this is in the quantum theory of the helium atom, in which there are long-lived Auger states [RSim]. The mathematical study of this problem proceeds by viewing as the unperturbed self-adjoint operator, the Hamiltonian governing two decoupled electron-proton systems. This system has many embedded eigenvalues. The perturbed Hamiltonian is that which includes the effect of electron-electron repulsion. In Examples 3 and 4 of section 6, we discuss a class of problems with this structure. Another physical problem in which resonances play an important role is in the setting of an atom coupled to the photon-radiation field ([BFroSi], [JPi1,2], [Ki1,2]); see also Example 7 of section 6. Although initially inspired by the study of quantum phenomena, questions involving embedded eigenvalues have been seen to arise, quite naturally in spectral geometry and number theory [PhS]. The systematic mathematical study of the effects of perturbations on embedded eigenvalues was initiated by Friedrichs [F].

The method of analyzing the resonance problem we develop here is related to our work on the large time behavior of nonlinear Schrödinger and nonlinear wave equations [SoWei3-5].[1] In these problems, certain states of the system decay slowly as a result of resonant interactions generated by nonlinearity in the equations of motion. The methods required are necessarily time-dependent as the equations are nonlinear and nonintegrable. They are based on a direct approach to the study of energy transfer from discrete to continuum modes.[2]

We consider the following general problem. Suppose H_0 is a self-adjoint operator in a Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, such that H_0 has a simple eigenvalue, λ_0, which is embedded in its continuous spectrum, with associated eigenfunction, ψ_0:

$$H_0\psi_0 = \lambda_0\psi_0 , \qquad ||\psi_0||_2 = 1 .$$

We now consider the time-dependent Schrödinger equation, for the perturbed self-adjoint Hamiltonian, $H = H_0 + W$,

$$i\partial_t\phi = H\phi \tag{1.1}$$

where W is a perturbation which is small in a sense to be specified. The choice of decomposition of H into an unperturbed part, H_0, and a pertur-

[1] Some of the results of this paper were presented in the proceedings article [SoWe2] and in the preprint [SoWe3].

[2] Related to this is the observation that many nonlinear phenomena can be regarded as (generic) instabilities of embedded eigenvalues for suitable *linear* operators. This point of view is taken by I.M. Sigal in [Si1,2], who studies the non-existence of bifurcating time-periodic and spatially localized solutions of certain nonlinear wave and Schrödinger equations. The problem of absence of small amplitude *breathers* for Hamiltonian perturbations of the Sine-Gordon equation (see, for example, [SeKr] and [BiMWe]) can also be viewed in this context [Si2]. Other nonlinear wave phenomena, in which resonances have been shown to play a role, are studied in [PWe], [CrHi].

bation, W, depends on the problem at hand; see, for example, [D].

PROBLEM. *Suppose we specify initial data,* ϕ_0 *for* (1.1) *which are spectrally localized (relative to* H*) in a small interval* Δ *about* λ_0*. Describe the time-dynamics for* $t \in (-\infty, \infty)$.

We shall prove that under quite general assumptions on H_0 and W that for small perturbations W,

(i) H has absolutely continuous spectrum in an interval about λ_0,
(ii) the solution with such data decays algebraically as $t \to \pm\infty$. For the special case of initial conditions given by ψ_0, the solution is characterized by transient exponential decay. The exponential rate, Γ (reciprocal of the lifetime), can be calculated.

On the more technical side, we have imposed fairly relaxed hypotheses on the regularity of the perturbation, W; in particular we do not require any condition on its commutators. This may be useful in problems like the radiation problem and problems where Dirichlet decoupling is used.

The decay of solutions due to resonant coupling to the continuum is revealed by decomposing the solution of (1.1), with data spectrally localized (relative to H) near λ_0, in terms of the natural basis of the unperturbed problem,

$$\phi(t) = a(t)\psi_0 + \tilde{\phi}(t)\,, \quad \big(\psi_0, \tilde{\phi}(\cdot, t)\big) = 0\,. \tag{1.2}$$

After isolating the key resonant contributions, the system of equations governing $a(t)$ and $\tilde{\phi}$ is seen to have the form

$$\begin{aligned} ia' &= (\Lambda - i\Gamma)a + \mathcal{C}_1(a, \tilde{\phi}) \\ i\partial_t \tilde{\phi} &= H_0 \tilde{\phi} + \mathcal{C}_2(a, \tilde{\phi})\,, \end{aligned} \tag{1.3}$$

where the $\mathcal{C}_j$, $j = 1, 2$, denote terms which couple the dynamics of a and $\tilde{\phi}$, and $\mathcal{C}_2$ lies in the continuous spectral part of H_0. If these coupling terms are neglected, then it is clear that $a(t)$ is driven to zero provided $\Gamma > 0$. The quantity, Γ, is displayed in (2.7) and is always nonnegative. Its explicit formula, (2.7), is often referred to as the *Fermi golden rule*. Generically, Γ is strictly positive. The exponential behavior suggested by these heuristics is, in general, only a transient; in general, $e^{-iH_0 t}$ has dispersive wave solutions, and coupling to these waves leads to (weaker) algebraic decay as $t \to \pm\infty$. At this stage, we wish to point out that although presented in the setting of a Schrödinger type operator, acting in $L^2(\mathbb{R}^n)$, our results and the approach we develop below can be carried out in the setting of a general Hilbert space, $\mathcal{H}$, with appropriate modifications made in the hypotheses. These modifications are discussed in the remark following our main theorem, Theorem 2.1. Their implementation is discussed in several of the examples presented in section 6.

Historically, motivated by experimental observations, the primary focus of mathematical analyses of the resonance problem has been on establishing exponential decay at intermediate times. However, viewed as an infinite dimensional Hamiltonian system, the asymptotic ($t \to \pm\infty$) behavior of solutions is a fundamental question. Our methods address this question and are adaptable to nonautonomous linear, and nonlinear problems [SoWei4,5].

The time decay of such solutions implies that the spectrum of the perturbed Hamiltonian, in a neighborhood of λ_0, is absolutely continuous. This implies the instability of the embedded eigenvalue. More precisely, under perturbation the embedded eigenvalue moves off the real axis and becomes a pole ("resonance pole" or "resonance energy") of the resolvent analytically continued across the continuous spectrum onto a second Riemann sheet [Hu]. We will also show that in a neighborhood of such embedded eigenvalues, there are no *new* embedded eigenvalues which appear, and give an estimate on the size of this neighborhood. Most importantly, we find the time behavior of solutions of the associated Schrödinger type evolution equation for short, intermediate and long time scales. The *lifetime* of the resonant state naturally emerges from our analysis. These results are stated precisely in Theorem 2.1.

Many different approaches to the resonance problem in quantum mechanics have been developed over the last 70 years and the various characterizations of resonance energies are expected to be equivalent; see [HSj]. The first (formal) approach to the resonance problem, due to Weisskopf and Wigner [WeiWi], was introduced in their study of the phenomena of spontaneous emission and the instability of excited states; see also [L]. Their approach plays a central role in today's physics literature; see for example [AlE], [LaLi]. It is time-dependent and our approach is close in spirit to this method.

Another approach, used both by physicists and mathematicians is based on the analytic properties of the S-matrix in the energy variable; see [LaxPh]. Other approaches concentrated on the behavior of a reduced Green's function, either by direct methods, or by studying its analytic properties [Ho],[O].

The most commonly used approach is that of analytic dilation or, more generally, analytic deformation [CyFKS], [HiSi]. This method is very general, but requires a choice of deformation group adapted to the problem at hand, as well as technical analyticity conditions which do not appear to be necessary. In this approach, the Hamiltonian of interest, H, is embedded in a one-parameter family of unitarily equivalent operators, $H(\theta)$, $\theta \in \mathbb{R}$. Under analytic continuation in θ the continuous spectrum of H is seen to move and the eigenvalue, which was embedded in the continuum for the unperturbed operator, is now "uncovered" and isolated. Thus Rayleigh-Schrödinger perturbation theory for an isolated eigenvalue can be applied, and used to conclude that the embedded eigenvalue generically perturbs to a resonance. The nonvanishing of the Fermi golden rule, (2.7), arises as a nondegeneracy condition ensuring that we can see the motion of the embedded eigenvalue at second order in perturbation theory. In our work, it arises as a condition, ensuring the "damping" of states which are spectrally localized (with respect to H) about λ_0. Analytic deformation techniques do not directly address the time behavior, which require a separate argument [GeSi], [Hu], [Sk].

Additionally, "thresholds" may not be "uncovered" and therefore the method of analytic deformation is unable to address the perturbation theory of such points. Our time-dependent method can yield information about thresholds, though it may be problematic to check the local decay assumptions in intervals containing such points; see however Example 5 in section 7, concerning the instability of a threshold eigenvalue of $-\Delta+V(x)$. Finally, in many cases, previous approaches have required the potential to be dilation analytic, where we only require C^3 behavior; see the concluding remarks of Appendix D for a discussion of this point.

The paper is structured as follows. In section 2 the mathematical framework is explained and the main theorem (Theorem 2.1) is stated. In section 3 the solution is decomposed relative to the unperturbed operator, the key resonance is isolated and a dynamical system of the form (1.3) is derived. Sections 4 and 5 contained the detailed estimates of the large time behavior of solutions. In section 6 we outline examples and applications. Sections 7-11 are appendices. Section 7 (Appendix A) concerns the proof of the "singular" local decay estimate of Proposition 2.1, and section 10 (Appendix D) outlines a general approach to obtaining local decay estimates of the type assumed in the hypothesis (H4). In section 9 (Appendix C) we present the details of our expansion of the complex frequency, ω_* (see (2.12) and Proposition 3.3). In section 11 (Appendix E) we give results on boundedness of functions of self-adjoint operators in *weighted* function spaces which may be of general interest.

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2 Mathematical Framework and Statement of the Main Theorem

In this section we first introduce certain necessary terminology and notation. We then state the hypotheses (H) and (W) on the unperturbed Hamiltonian, H_0, and on the perturbation, W. The section then concludes with statements of the main results.

For an operator, L, $||L||$ denotes its norm as an operator from L^2 to itself. We interpret functions of a self adjoint operator as being defined by the spectral theorem. In the special case where the operator is H_0, we omit the argument, *i.e.* $g(H_0) = g$.

For an open interval Δ, we denote an appropriate smoothed characteristic function of Δ by $g_\Delta(\lambda)$. In particular, we shall take $g_\Delta(\lambda)$ to be a nonnegative C^∞ function, which is equal to one on Δ and zero outside a neighborhood of Δ. The support of its derivative is furthermore chosen to be small compared to the size of Δ, *e.g.* less than $\frac{1}{10}|\Delta|$. We further require that $|g_\Delta^{(n)}(\lambda)| \leq c_n\ |\Delta|^{-n}$, $n \geq 1$.

P_0 denotes the projection on ψ_0, *i.e.* $P_0 f = (\psi_0, f)\psi_0$.

P_{1b} denotes the spectral projection on $\mathcal{H}_{pp} \cap \{\psi_0\}^\perp$, the pure point spectral part of H_0 orthogonal to ψ_0. That is, P_{1b} projects onto the subspace of $\mathcal{H}$ spanned by all eigenstates other than ψ_0.

In our treatment, a central role is played by the subset of the spectrum of the operator H_0, $\mathbf{T}^\#$, on which a sufficiently rapid local decay estimate holds. For a decay estimate to hold for $e^{-iH_0 t}$, one must certainly project out the bound states of H_0, but there may be other obstructions to rapid decay. In scattering theory these are called *threshold energies* [CyFKS]. Examples of thresholds are: (i) points of stationary phase of a constant coefficient principle symbol for two-body Hamiltonians; and (ii) for N-body Hamiltonians, zero and the eigenvalues of subsystems. We will not give a precise definition of thresholds. For us it is sufficient to say that away from thresholds the favourable local decay estimates for H_0 hold.

Let Δ_* be union of intervals, disjoint from Δ, containing all thresholds of H_0, and a neighborhood of infinity. We then let

$$P_1 = P_{1b} + g_{\Delta_*}$$

where $g_{\Delta_*} = g_{\Delta_*}(H_0)$ is a smoothed characteristic function of the set Δ_*. We also define

$$\begin{aligned} \langle x\rangle^2 &= 1 + |x|^2, \\ \overline{Q} &= I - Q \qquad \text{and} \\ P_c^{\#} &= I - P_0 - P_1\,. \end{aligned} \tag{2.1}$$

Thus, $P_c^{\#}$ is a smoothed out spectral projection of the set $\mathbf{T}^{\#}$ defined as

$$\mathbf{T}^{\#} = \sigma(H_0) - \{\text{ eigenvalues, real neighborhoods of thresholds and infinity}\}\,. \tag{2.2}$$

We expect e^{-iH_0t} to satisfy good local decay estimates on the range of $P_c^{\#}$; see (H4) below.

Next we state our *hypotheses on* H_0.

(H1) H_0 is a self adjoint operator with dense domain $\mathcal{D}$, in $L^2(\mathbb{R}^n)$.

(H2) λ_0 is a simple embedded eigenvalue of H_0 with (normalized) eigenfunction ψ_0.

(H3) There is an open interval Δ containing λ_0 and no other eigenvalue of H_0.

There exists $\sigma > 0$ such that

(H4) *Local decay estimate*: Let $r \geq 2 + \varepsilon$ and $\varepsilon > 0$. If $\langle x\rangle^\sigma f \in L^2$ then

$$\left\|\langle x\rangle^{-\sigma} e^{-iH_0t} P_c^{\#} f\right\|_2 \leq C\langle t\rangle^{-r} \left\|\langle x\rangle^\sigma f\right\|_2, \tag{2.3}$$

(H5) By appropriate choice of a real number c, the L^2 operator norm of $\langle x\rangle^\sigma (H_0 + c)^{-1} \langle x\rangle^{-\sigma}$ can be made sufficiently small.

Remarks. (i) We have assumed that λ_0 is a simple eigenvalue to simplify the presentation. Our methods can be easily adapted to the case of multiple eigenvalues.

(ii) Note that Δ does not have to be small and that Δ_* can be chosen as necessary, depending on H_0.

(iii) In certain cases, the above local decay conditions can be proved even when λ_0 is a threshold; see Example 5 of section 6.

(iv) Regarding the verification of the local decay hypothesis, one approach is to use techniques based on the Mourre estimate [JeMouPe], [SiSo].

If Δ contains no threshold values then, quite generally, the bound (2.3) holds with r arbitrary and positive. See Appendix D.

We shall require the following consequence of hypothesis (H4).

PROPOSITION 2.1. *Let $r \geq 2+\varepsilon$ and $\varepsilon > 0$. Assume $\mu \in \mathbf{T}^{\#}$. Then, for $t \geq 0$*

$$\left\|\langle x\rangle^{-\sigma} e^{-iH_0 t}(H_0-\mu-i0)^{-1} P_c^{\#} f\right\|_2 \leq C\langle t\rangle^{-r+1}\left\|\langle x\rangle^{\sigma} f\right\|_2, \tag{2.4}$$

For $t<0$, estimate (2.4) *holds with $-i0$ replaced by $+i0$.*

The proof is given in Appendix A.

We now specify the conditions we require of the perturbation, W.

Conditions on W.

(W1) W is symmetric and $H = H_0 + W$ is self-adjoint on $\mathcal{D}$ and there exists $c \in \mathbb{R}$ (which can be used in (H5)), such that c lies in the resolvent sets of H_0 and H.

(W2) For some σ, which can be chosen to be the same as in (H4) and (H5),

$$|||W||| \equiv \left\|\langle x\rangle^{2\sigma} W g_\Delta(H_0)\right\| + \left\|\langle x\rangle^{\sigma} W g_\Delta(H_0)\langle x\rangle^{\sigma}\right\| + \left\|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{-\sigma}\right\| < \infty, \tag{2.5}$$

and

$$\left\|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{\sigma}\right\| < \infty, \tag{2.6}$$

(W3) *Resonance condition – nonvanishing of the Fermi golden rule*:

$$\Gamma \equiv \pi\big(W\psi_0, \delta(H_0-\tilde{\omega})(I-P_0)W\psi_0\big) \neq 0 \tag{2.7}$$

for $\tilde{\omega}$ near λ_0, and

$$\Gamma \geq \delta_0 |||W|||^2 \tag{2.8}$$

for some $\delta_0 > 0$.

(W4) $|||W||| < \theta\ |\Delta|$ for some $\theta > 0$, sufficiently small, depending on the properties of H_0, in particular the local decay constants, but not on $|\Delta|$.

REMARK. Let $\mathcal{F}_c^{H_0}$ denote the (generalized) Fourier transform with respect to the continuous spectral part of H_0. The resonance condition (2.7), can then be expressed as

$$\Gamma \equiv \pi\left|\mathcal{F}_c^{H_0}[W\psi_0](\lambda_0)\right|^2 > 0\,. \tag{2.9}$$

We can now state the main result:

Theorem 2.1. *Let H_0 satisfy the conditions* (H) *and the perturbation W satisfy the conditions* (W). *Then*

(a) *$H = H_0 + W$ has no eigenvalues in Δ.*

(b) *The spectrum of H in Δ is purely absolutely continuous; in particular local decay estimates hold for $e^{-iHt}g_\Delta(H)$. Namely, for ϕ_0 with $\langle x\rangle^\sigma \phi_0 \in L^2$, as $t \to \pm\infty$,*

$$\left\|\langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\phi_0\right\|_2 = \mathcal{O}\left(\langle t\rangle^{-r+1}\right). \tag{2.10}$$

(c) *For ϕ_0 in the range of $g_\Delta(H)$ we have (for $t \geq 0$)*

$$e^{-iHt}\phi_0 = (I + \mathcal{A}_W)\left(e^{-i\omega_* t} a(0)\psi_0 + e^{-iH_0 t}\phi_d(0)\right) + \mathcal{R}(t). \tag{2.11}$$

Here, $\|\mathcal{A}_W\|_{\mathcal{B}(L^2)} \leq C|||W|||$, $a(0)$ is a complex number and $\phi_d(0)$ is a complex function in the range of $P_c^\#$, which are determined by the initial data; see (3.1)-(3.2).

The complex frequency, ω_*, is given by

$$\omega_* = \omega - \Lambda - i\Gamma + \mathcal{O}\left(|||W|||^3\right), \text{ where} \tag{2.12}$$

$$\omega \equiv \lambda_0 + (\psi_0, W\psi_0), \tag{2.13}$$

$$\Lambda \equiv \left(W\psi_0,\ \text{P.V. } (H_0 - \omega)^{-1} W\psi_0\right), \text{ and} \tag{2.14}$$

$$\Gamma \equiv \pi\left(W\psi_0, \delta(H_0 - \omega)(I - P_0)W\psi_0\right). \tag{2.15}$$

We also have the estimates

$$\left\|\langle x\rangle^{-\sigma}\mathcal{R}(t)\right\|_2 \leq C|||W|||, \quad t \geq 0 \tag{2.16}$$

$$\left\|\langle x\rangle^{-\sigma}\mathcal{R}(t)\right\|_2 \leq C|||W|||^\varepsilon \langle t\rangle^{-r+1},$$
$$t \geq |||W|||^{-2(1+\delta)},\ \delta > 0,\ \varepsilon = \varepsilon(\delta) > 0. \tag{2.17}$$

Remark. Though phrased in the setting of the space $L^2(\mathbb{R}^n)$, our approach is quite general and our results hold with $L^2(\mathbb{R}^n)$ replaced by a Hilbert space, $\mathcal{H}$. In this general setting, the weight function, $\langle x\rangle$, is to be replaced by a "weighting operator", A, in the hypotheses (H), (W) and in the definition of the norm of W, $|||W|||$. Additionally, $P_c^\#$ can be taken to be a smoothed out spectral projection onto the subspace of $\mathcal{H}$ where the local decay estimate (H4) holds.

Given an eigenstate ψ_0 associated with an embedded eigenvalue, λ_0, of the unperturbed Hamiltonian, H_0, a quantity of physical interest is the *lifetime* of the state ψ_0 for the perturbed dynamics. To find the lifetime, consider the quantum expectation value that the system is in the resonant state, ψ_0,

$$(\psi_0, e^{-itH}\psi_0). \tag{2.18}$$

Note that

$$e^{-iHt}\psi_0 = e^{-iHt}g_\Delta(H)\psi_0 + e^{-iHt}\left(g_\Delta(H_0) - g_\Delta(H)\right)\psi_0. \tag{2.19}$$

Theorem 2.1 and the techniques used in the proofs of Propositions 3.1 and 3.2 yield the following result concerning the lifetime of the state ψ_0.

COROLLARY 2.1. *Let*

$$H_* = H - \mathrm{Re}\omega_* I\,. \tag{2.20}$$

Then, for any $T > 0$ there is a constant $C_T > 0$ such that for $0 \le t \le T|||W|||^{-2}$

$$\left|(\psi_0, e^{-iH_*t}\psi_0) - e^{-\Gamma t}\right| \le C_T|||W|||\,, \quad \text{as} \quad |||W||| \to 0\,. \tag{2.21}$$

3 Decomposition and Isolation of Resonant Terms

We begin with the following decomposition of the solution of (1.1):

$$e^{-iHt}\phi_0 = \phi(t) = a(t)\psi_0 + \tilde\phi(t) \tag{3.1}$$

$$\big(\psi_0, \tilde\phi(t)\big) = 0 \qquad -\infty < t < +\infty\,. \tag{3.2}$$

Substitution into (1.1) yields

$$i\partial_t\tilde\phi = H_0\tilde\phi + W\tilde\phi - (i\partial_t a - \lambda_0 a)\psi_0 + aW\psi_0\,. \tag{3.3}$$

Recall now that $I = P_0 + P_1 + P_c^\#$. Taking the inner product of (3.3) with ψ_0 gives the amplitude equation,

$$i\partial_t a = \big(\lambda_0 + (\psi_0, W\psi_0)\big)a + (\psi_0 W P_1\tilde\phi) + (\psi_0, W\phi_d)\,, \tag{3.4}$$

where,

$$\phi_d \equiv P_c^\#\tilde\phi\,. \tag{3.5}$$

The following equation for ϕ_d is obtained by applying $P_c^\#$ to equation (3.3):

$$i\partial_t\phi_d = H_0\phi_d + P_c^\# W(P_1\tilde\phi + \phi_d) + aP_c^\# W\psi_0\,. \tag{3.6}$$

Our goal is to derive a closed system for $\phi_d(t)$ and $a(t)$. To achieve this, we now propose to obtain an expression for $P_1\tilde\phi$, to be used in equations (3.4) and (3.6). Since $g_\Delta(H)\phi(\cdot,t) = \phi(\cdot,t)$, we find

$$\big(I - g_\Delta(H)\big)\phi = \big(I - g_\Delta(H)\big)[a\psi_0 + P_1\tilde\phi + P_c^\#\tilde\phi] = 0 \tag{3.7}$$

or

$$\big(I - g_\Delta(H)g_I(H_0)\big)P_1\tilde\phi = -\overline{g}_\Delta(H)[a\psi_0 + \phi_d]\,, \tag{3.8}$$

where $g_I(\lambda)$ is a smooth function, which is identically equal to one on the support of $P_1(\lambda)$, and which has support disjoint from Δ. Therefore,

$$P_1\tilde\phi = -B\overline{g}_\Delta(H)\,(a\psi_0 + \phi_d)\,, \tag{3.9}$$

where

$$B = \left(I - g_\Delta(H) g_I(H_0)\right)^{-1}.$$

This computation is justified by the following result which is proved in Appendix B.

PROPOSITION 3.1. *The operator $B = (I - g_\Delta(H) g_I(H_0))^{-1}$ is a bounded operator on $\mathcal{H}$*

From (3.9) we get

$$\begin{aligned} \phi(t) &= a(t)\psi_0 + \phi_d + P_1\tilde{\phi} \\ &= \tilde{g}_\Delta(H)\big(a(t)\psi_0 + \phi_d(t)\big), \end{aligned} \tag{3.10}$$

with

$$\tilde{g}_\Delta(H) \equiv I - B\overline{g}_\Delta(H) = B g_\Delta(H)\big(I - g_I(H_0)\big). \tag{3.11}$$

Although $\tilde{g}(H)$ is not really defined as a function of H, we indulge in this mild abuse of notation to emphasize its dependence on H. In fact, we shall prove that, in some sense, $\tilde{g}_\Delta(H) \sim g_\Delta(H) \sim g_\Delta(H_0)$.

Substitution of the above expression (3.9) for $P_1\tilde{\phi}$ into (3.6) gives

$$i\partial_t \phi_d = H_0\phi_d + aP_c^{\#} W \tilde{g}_\Delta(H)\psi_0 + P_c^{\#} W \tilde{g}_\Delta(H)\phi_d \tag{3.12}$$

and

$$\begin{aligned} i\partial_t a &= \left[\lambda_0 + (\psi_0, W\tilde{g}_\Delta(H)\psi_0)\right] a + \big(\psi_0, W\tilde{g}_\Delta(H)\phi_d\big) \\ &= \omega a + (\omega_1 - \omega)a + \big(\psi_0, W\tilde{g}_\Delta(H)\phi_d\big), \end{aligned} \tag{3.13}$$

where

$$\omega = \lambda_0 + (\psi_0, W\psi_0), \tag{3.14}$$

$$\omega_1 = \lambda_0 + \big(\psi_0, W\tilde{g}_\Delta(H)\psi_0\big). \tag{3.15}$$

The decay of $a(t)$ and ϕ_d is driven by a resonance. From equation (3.13), the second term on the right-hand side of (3.12) oscillates approximately like $e^{-i\lambda_0 t}$. Since λ_0 lies in the continuous spectrum of H_0, this term resonates with the continuous spectrum of H_0. To make explicit the effect of this resonance, we first write (3.12) as an equivalent integral equation.

$$\begin{aligned} \phi_d(t) &= e^{-iH_0 t}\phi_d(0) - i\int_0^t e^{-iH_0(t-s)} a(s) P_c^{\#} W\tilde{g}_\Delta(H)\psi_0 ds \\ &\qquad - i\int_0^t e^{-iH_0(t-s)} P_c^{\#} W \tilde{g}_\Delta(H)\phi_d ds \\ &\equiv \phi_0(t) + \phi_{res}(t) + \phi_1(t). \end{aligned} \tag{3.16}$$

Our next goal is to obtain the leading order behavior of $\phi_{res}(t)$. For $\epsilon > 0$ introduce the following regularization:

$$\phi_{res}^{\epsilon}(t) = -i\int_0^t e^{-iH_0(t-s)} a(s) e^{\epsilon s} P_c^{\#} W\tilde{g}_\Delta(H)\psi_0 ds. \tag{3.17}$$

Then, $\phi^\epsilon_{res}(t) \to \phi_{res}(t)$. To extract the dominant oscillatory part of $a(t)$, we let

$$A(t) = e^{i\omega t} a(t) \,. \tag{3.18}$$

We now expand $\phi^\epsilon_{res}(t)$ using integration by parts,

$$\begin{aligned}
\phi^\epsilon_{res}(t) &= -i \int_0^t e^{-iH_0 t} e^{i(H_0 - i\epsilon)s} a(s) P_c^\# W \tilde{g}_\Delta(H) \psi_0 ds \\
&= -i \int_0^t e^{-iH_0 t} e^{i(H_0 - i\epsilon - \omega)s} \big[e^{i\omega s} a(s)\big] P_c^\# W \tilde{g}_\Delta(H) \psi_0 ds \\
&\equiv -i e^{-iH_0 t} \int_0^t e^{i(H_0 - i\epsilon - \omega)s} A(s) P_c^\# W \tilde{g}_\Delta(H) \psi_0 ds \\
&= -e^{-iH_0 t} \ \big[(H_0 - \omega - i\epsilon)^{-1} e^{i(H_0 - i\epsilon - \omega)s} A(s) P_c^\# W \tilde{g}_\Delta(H) \psi_0 \big|_{s=0}^{s=t} \\
&\quad + e^{-iH_0 t} \int_0^t (H_0 - \omega - i\epsilon)^{-1} e^{i(H_0 - i\epsilon - \omega)s} \partial_t A(s) P_c^\# W \tilde{g}_\Delta(H) \psi_0 ds \,.
\end{aligned} \tag{3.19}$$

With a view toward taking $\epsilon \downarrow 0$ we first note that by hypothesis (H), since $|||W|||$ is assumed sufficiently small, we have that $\omega \in \Delta$. The limit is therefore singular, and we'll find a resonant, purely imaginary, contribution coming from the boundary term at $s = t$. Furthermore, to study the last term in (3.19) we will use the equation

$$\partial_t A = -i e^{i\omega t} \big(\psi_0, W \tilde{g}_\Delta(H) \phi_d\big) + i(\omega - \omega_1) A \,. \tag{3.20}$$

Now, taking $\epsilon \to 0$, we get in $L^2(\langle x \rangle^{-2\sigma}\, dx)$,

PROPOSITION 3.2. *The following expansion for $\phi_{res}(t)$ holds:*

$$\begin{aligned}
\phi_{res}(t) &= -a(t)(H_0 - \omega - i0)^{-1} P_c^\# W \tilde{g}_\Delta(H) \psi_0 \\
&\quad + a(0) e^{-iH_0 t} (H_0 - \omega - i0)^{-1} P_c^\# W \tilde{g}_\Delta(H) \psi_0 \\
&\quad - i \int_0^t e^{-iH_0(t-s)} (H_0 - \omega - i0)^{-1} P_c^\# W \tilde{g}_\Delta(H) \psi_0 \cdot \big(\psi_0, W \tilde{g}_\Delta(H) \phi_d(s)\big) ds \\
&\quad + i(\omega - \omega_1) \int_0^t e^{-iH_0(t-s)} (H_0 - \omega - i0)^{-1} P_c^\# W \tilde{g}_\Delta(H) \psi_0 \cdot a(s) ds \\
&\equiv -a(t)(H_0 - \omega - i0)^{-1} P_c^\# W \tilde{g}_\Delta(H) \psi_0 + \phi_2(t) + \phi_3(t) + \phi_4(t) \,.
\end{aligned} \tag{3.21}$$

REMARK. To see that the terms in (3.21) are well defined we refer to the proof of Proposition 2.1 in Appendix A. Localizing near and away from the energy ω

$$\begin{aligned}
(H - \omega - i0)^{-1} e^{-iH_0 t} P_c^\# &= (H - \omega - i0)^{-1} e^{-iH_0 t} P_c^\# g_\Delta \\
&\quad + (H - \omega - i0)^{-1} e^{-iH_0 t} \overline{g}_\Delta
\end{aligned}$$

$$\equiv T^t_{\Delta,0} + S^t_{\Delta,0} \,.$$

In Appendix A it is proved that, for $\epsilon \geq 0$,

$$T^t_{\Delta,\epsilon}, S^t_{\Delta,\epsilon} : L^2\big(\langle x\rangle^{2\sigma}dx\big) \mapsto L^2\big(\langle x\rangle^{-2\sigma}dx\big) \,, \qquad t \geq 0 \,.$$

Substitution of (3.16) and (3.21) into (3.13) yields the following equation for $a(t)$:

$$i\partial_t a(t) = \omega_* a(t) + \big(\psi_0, W\tilde{g}_\Delta(H)\{\phi_0(t) + \phi_1(t) + \phi_2(t) + \phi_3(t) + \phi_4(t)\}\big) \,. \tag{3.22}$$

Here,

$$\omega_* = \lambda_0 + \big(\psi_0, W\tilde{g}_\Delta(H)\psi_0\big) - \big(\psi_0, W\tilde{g}_\Delta(H)(H_0 - \omega - i0)^{-1}P_c^{\#}W\tilde{g}_\Delta(H)\psi_0\big) \,. \tag{3.23}$$

In order see the resonant decay we must first consider the behavior of the *complex* frequency ω_* for small $|||W|||$. The next proposition contains an expression for ω_* which depends explicitly on the "data" of the resonance problem, H_0 and W, plus a controllable error.

PROPOSITION 3.3.

$$\omega_* = \lambda_0 + (\psi_0, W\psi_0) - \Lambda - i\Gamma + E(W) \,, \tag{3.24}$$

where

$$\begin{aligned} \Gamma &= \pi\big(W\psi_0, \delta(H_0 - \omega)(I - P_0)W\psi_0\big) \,, \\ \Lambda &= \big(W\psi_0, \text{P.V. } (H_0 - \omega)^{-1}W\psi_0\big) \,, \\ E(W) &\leq C_1|||W|||^3 \,, \end{aligned} \tag{3.25}$$

where ω is given by (2.13).

The term, Γ, in (3.25) is the Fermi golden rule appearing in resonance hypothesis (W3) ($\Gamma \neq 0$).

The proof of Proposition 3.3 is a lengthy computation which we present in Appendix C.

We conclude this section with a summary of the coupled equations for $\phi_d(t)$ and $a(t)$.

PROPOSITION 3.4.

$$i\partial_t a = \omega_* a + \big(\psi_0, W\tilde{g}_\Delta(H)\{\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4\}\big) \,, \tag{3.26}$$

$$\phi_d(t) = e^{-iH_0t}\phi_d(0) - i\int_0^t e^{-iH_0(t-s)}a(s)P_c^{\#}W\tilde{g}_\Delta(H)\psi_0 ds - i\int_0^t e^{-iH_0(t-s)}P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(s)ds \,, \tag{3.27}$$

where

$$\phi_0(t) = e^{-iH_0 t} P_c^{\#} \phi_d(0) \tag{3.28}$$

$$\phi_1(t) = -i \int_0^t e^{-iH_0(t-s)} P_c^{\#} W \tilde{g}_\Delta(H) \phi_d(s) ds \tag{3.29}$$

$$\phi_2(t) = -a(0) e^{-iH_0 t} (H_0 - \omega - i0)^{-1} P_c^{\#} W \tilde{g}_\Delta(H) \psi_0 \tag{3.30}$$

$$\begin{aligned}\phi_3(t) = -i \int_0^t e^{-iH_0(t-s)} (H_0 - \omega - i0)^{-1} P_c^{\#} W \tilde{g}_\Delta(H) \psi_0 \\ \cdot \big(\psi_0, W\tilde{g}_\Delta(H)\phi_d(s)\big)\, ds\end{aligned} \tag{3.31}$$

$$\phi_4(t) = i(\omega - \omega_1) \int_0^t e^{-iH_0(t-s)} (H_0 - \omega - i0)^{-1} P_c^{\#} W \tilde{g}_\Delta(H) \psi_0 \cdot a(s) ds \tag{3.32}$$

To prove the main theorem we estimate $a(t)$ and $\phi_d(t)$ from (3.26)-(3.32). Note that since $\operatorname{Im}\omega_* \sim -\operatorname{Im}\Gamma$ is negative, it is evident that this resonant contribution has the effect of driving $a(t)$ to zero.

REMARK. Although we have the general result of Theorem 2.1, in a given example it may prove beneficial to analyze the system (3.26)-(3.32) directly in order to exploit special structure.

REMARK. Using the above expansion and definitions, we have

$$\begin{aligned}\phi(t) = \ & e^{-i\omega_* t} a(0)\psi_0 + e^{-iH_0 t}(I - P_0) g_\Delta \phi_0 \\ & + \left[\tilde{g}_\Delta(H) - g_\Delta(H_0)\right]\left[e^{-i\omega_* t} a(0)\psi_0 + e^{-iH_0 t} P_c^{\#} \phi_0\right] + \mathcal{R}(t)\,,\end{aligned} \tag{3.33}$$

where

$$\mathcal{R}(t) = \tilde{g}(H) \left[\sum_{j=0}^{4} R_j(t)\psi_0 + \phi_{res}(t) + \phi_1(t) \right]. \tag{3.34}$$

See (4.10) and (4.8) for the definition of R_j. The expansion in part (c) of Theorem 2.1 is obtained by estimates of the terms in (3.33) and (3.34). These estimates are carried out in sections 4 and 5.

In the next two sections we estimate the solution over various time scales.

4 Local Decay of Solutions

In this section we begin our analysis of the large time behavior of solutions. To prove local decay, we introduce the norms

$$[a](T) \equiv \sup_{0 \le s \le T} \langle s \rangle^\alpha |a(s)| \quad \text{and} \quad [\phi_d]_{LD}(T) \equiv \sup_{0 \le s \le T} \langle s \rangle^\alpha \big\| \langle x \rangle^{-\sigma} \phi_d(s) \big\|_2\,,$$

for which we seek to obtain upper bounds that are uniform in $T \in \mathbb{R}$. Because of terms like $\phi_j(t)$, $j = 2, 3, 4$ (see Proposition 3.4) and the singular local decay estimate of Proposition 2.1, it is natural study these norms with $\alpha = r - 1$. In this section, it turns out that we require the restriction on α, $1 < \alpha < 3/2$. Thus, throughout this section we shall assume the constraints

$$\alpha \equiv r - 1\,, \quad 1 < \alpha < 3/2\,.$$

In section 5 we relax the upper bound on α.

REMARK. In the estimates immediately below and in subsequent sections we shall require bounds on the following quantities like $\|\langle x\rangle^a W \tilde{g}_\Delta(H) \langle x\rangle^b\|$ with $a, b \in \{0, \sigma\}$. That all these can be controlled in terms of the norm $|||W|||$ is ensured by the following proposition, which is proved in Appendix B.

PROPOSITION 4.1. *For* $a, b \in \{0, \sigma\}$,

$$\big\|\langle x\rangle^a W \tilde{g}_\Delta(H) \langle x\rangle^b\big\| \le C_{a,b} |||W|||\,. \tag{4.1}$$

We begin by estimating the local decay norm of ϕ_d.

Local decay estimates for $\phi_d(t)$. From equation (3.27)

$$\begin{aligned}
\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\|_2 &\le \big\|\langle x\rangle^{-\sigma} e^{-iH_0 t}\phi_d(0)\big\|_2 \\
&\quad + \int_0^t \big|a(s)\big|\big\|\langle x\rangle^{-\sigma} e^{-iH_0(t-s)} P_c^{\#} W \tilde{g}_\Delta(H)\psi_0\big\|_2 ds \\
&\quad + \int_0^t \big\|\langle x\rangle^{-\sigma} e^{-iH_0(t-s)} P_c^{\#} W \tilde{g}_\Delta(H)\phi_d(s)\big\|_2 ds \\
&\le C\langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|_2 + C\big\|\langle x\rangle^{\sigma} W \tilde{g}_\Delta(H)\psi_0\big\|_2 \int_0^t \langle t-s\rangle^{-r}\big|a(s)\big| ds \\
&\quad + \big\|\langle x\rangle^{\sigma} W \tilde{g}_\Delta(H)\langle x\rangle^{\sigma}\big\| \int_0^t \langle t-s\rangle^{-r}\big\|\langle x\rangle^{-\sigma}\phi_d(s)\big\|_2 ds\,.
\end{aligned} \tag{4.2}$$

This implies, for $0 \le t \le T$,

$$\begin{aligned}
&\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\|_2 \le C\langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|_2 \\
&+ C\langle t\rangle^{-\alpha}\big(\|\langle x\rangle^{\sigma} W \tilde{g}_\Delta(H)\psi_0\|_2 [a](T) + \big\|\langle x\rangle^{\sigma} W \tilde{g}_\Delta(H)\langle x\rangle^{\sigma}\big\| [\phi_d]_{LD}(T)\big) \\
&\quad \le C_1\langle t\rangle^{-r}\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|_2 + C_2|||W|||\langle t\rangle^{-\alpha}\big([a](T) + [\phi_d]_{LD}(T)\big)\,.
\end{aligned} \tag{4.3}$$

It follows that

$$[\phi_d]_{LD}(T) \le C_1\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|_2 + C_2|||W|||\big([a](T) + [\phi_d]_{LD}(T)\big)\,, \tag{4.4}$$

and therefore

$$\big(1 - C_2|||W|||\big)[\phi_d]_{LD}(T) \le C_1\big\|\langle x\rangle^{\sigma}\phi_d(0)\big\|_2 + C_2|||W|||[a](T)\,. \tag{4.5}$$

An additional simple consequence of (4.2) and the orthogonality of the decomposition (3.10), is

$$\left\|\langle x\rangle^{-\sigma}\phi_d(t)\right\|_2 \leq C\langle t\rangle^{-r}\left\|\langle x\rangle^{\sigma}\phi_d(0)\right\|_2 + C|||W|||\|\phi_0\|_2 \,. \tag{4.6}$$

Estimation of $a(t)$. We estimate $a(t)$ using equation (3.26). This equation has the form

$$i\partial_t a = \omega_* a + \sum_{j=0}^{4} F_j \,, \tag{4.7}$$

where

$$F_j(t) \equiv \left(\psi_0, W\tilde{g}_\Delta(H)\phi_j\right) . \tag{4.8}$$

Therefore,

$$a(t) = e^{-i\omega_* t}a(0) + \sum_{j=0}^{4} R_j(t) \,, \tag{4.9}$$

where

$$R_j(t) = -i\int_0^t e^{-i\omega_*(t-s)}F_j(s)ds \,. \tag{4.10}$$

We next estimate each R_j. In the course of carrying out the analysis we shall frequently apply the following:

LEMMA 4.1. *Let Γ, α and β denote real numbers such that $\Gamma > 0$ and $\beta > 1$. Define*

$$I_{\alpha,\beta}(t) = \langle t\rangle^{\alpha}\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-\beta}ds \,. \tag{4.11}$$

Then,

(i)
$$I_{\alpha,\beta}(t) \leq C(\langle t\rangle^{\alpha}e^{-\frac{1}{2}\Gamma t} + \langle t\rangle^{\alpha-\beta}\Gamma^{-1}) \,. \tag{4.12}$$

(ii) *If $\alpha \leq \beta$, we have*

$$\sup_{t\geq 0} I_{\alpha,\beta}(t) \leq C(\Gamma^{-\alpha} + \Gamma^{-1}) \,. \tag{4.13}$$

To prove this lemma, note that

$$\begin{aligned} I_{\alpha,\beta}(t) &= \langle t\rangle^{\alpha}\left(\int_0^{t/2} + \int_{t/2}^t\right)\{\cdots\}ds \\ &\leq \langle t\rangle^{\alpha}e^{-\frac{1}{2}\Gamma t}\int_0^{t/2}\langle s\rangle^{-\beta}ds + C\langle t\rangle^{\alpha-\beta}\int_{t/2}^t e^{-\Gamma(t-s)}ds \,. \end{aligned}$$

Part (i) follows by explicitly carrying out the integrals, using that $\beta > 1$, and part (ii) follows by noting that the supremum over $t \geq 0$ of the expression obtained in (i).

Estimation of $R_0(t)$.

$$R_0(t) = -i\int_0^t e^{-i\omega_*(t-s)}\big(\psi_0, W\tilde{g}_\Delta(H)e^{-iH_0 s}P_c^{\#}\phi_d(0)\big)ds\,. \tag{4.14}$$

Estimation of the integrand gives

$$\begin{aligned}\big|(\psi_0, W\tilde{g}_\Delta(H)e^{-iH_0 s}\phi_d(0))\big| &= \big|(\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma}e^{-iH_0 s}\phi_d(0))\big|\\ &\le \big\|\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0\big\|_2\big\|\langle x\rangle^{-\sigma}e^{-iH_0 s}\phi_d(0)\big\|_2\\ &\le C\big\|\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0\big\|_2\big\|\langle x\rangle^\sigma\phi_d(0)\big\|_2\langle s\rangle^{-r}\\ &\le C|||W|||\big\|\langle x\rangle^\sigma\phi_d(0)\big\|_2\langle s\rangle^{-r}\,.\end{aligned} \tag{4.15}$$

Use of (4.15) in (4.14) yields

$$\big|R_0(t)\big| \le C|||W|||\big\|\langle x\rangle^\sigma\phi_d(0)\big\|_2\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-r}ds\,. \tag{4.16}$$

Multiplication of (4.16) by $\langle t\rangle^\alpha$, use of Lemma 4.1 and the lower bound for Γ, (2.8), yields the bound

$$\langle t\rangle^\alpha\big|R_0(t)\big| \le C|||W|||^{1-2\alpha}\big\|\langle x\rangle^\sigma\phi_d(0)\big\|_2\,,\quad t\ge 0\,. \tag{4.17}$$

It also follows from (4.16), since $r>1$, that

$$\big|R_0(t)\big| \le C|||W|||\big\|\langle x\rangle^\sigma\phi_d(0)\big\|_2\,. \tag{4.18}$$

Estimation of $R_1(t)$. We must bound the expression

$$R_1(t) = -\int_0^t e^{-i\omega_*(t-s)}\left(\psi_0, W\tilde{g}_\Delta(H)\int_0^s e^{-iH_0(s-\tau)}P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)d\tau\right)ds. \tag{4.19}$$

This can be rewritten as

$$\begin{aligned}R_1(t) = \int_0^t e^{-i\omega_*(t-s)}ds\Big(&\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0,\\ &\int_0^s\langle x\rangle^{-\sigma}e^{-iH_0(s-\tau)}P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)d\tau\Big)\,.\end{aligned} \tag{4.20}$$

which satisfies the bound

$$\begin{aligned}\big|R_1(t)\big| \le C\big\|\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0\big\|_2&\int_0^t e^{-\Gamma(t-s)}ds\\ &\cdot\int_0^s\big\|\langle x\rangle^{-\sigma}e^{-iH_0(s-\tau)}P_c^{\#}W\tilde{g}_\Delta(H)\phi_d(\tau)\big\|d\tau\,.\end{aligned} \tag{4.21}$$

Use of the assumed local decay estimate (H4) gives that $R_1(t)$ is bounded by

$$\begin{aligned}C\big\|\langle x\rangle^\sigma\tilde{g}_\Delta(H)W\psi_0\big\|_2\big\|\langle x\rangle^\sigma W\tilde{g}_\Delta(H)\langle x\rangle^\sigma\big\|&\int_0^t e^{-\Gamma(t-s)}ds\\ &\cdot\int_0^s\langle s-\tau\rangle^{-r}\big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|_2\,d\tau\,,\end{aligned} \tag{4.22}$$

and therefore

$$\left|R_1(t)\right| \le C|||W|||^2 \int_0^t e^{-\Gamma(t-s)}ds \int_0^s \langle s-\tau\rangle^{-r}\langle\tau\rangle^{-\alpha}d\tau[\phi_d]_{LD}(T)\,. \tag{4.23}$$

Using Lemma 4.1, we have

$$\langle t\rangle^{\alpha}\left|R_1(t)\right| \le C\big(1+|||W|||^{2-2\alpha}\big)[\phi_d]_{LD}(T)\,. \tag{4.24}$$

Furthermore, use of (4.6) in (4.22) gives

$$\left|R_1(t)\right| \le C|||W|||\big\|\langle x\rangle^{\sigma}\phi_0\big\|_2\,, \quad t\ge 0\,. \tag{4.25}$$

Estimation of $R_2(t)$.

$$R_2(t) = ia(0)\int_0^t e^{-i\omega_*(t-s)} \cdot \big(\psi_0, W\tilde g_\Delta(H)e^{-iH_0 s}(H_0-\omega-i0)^{-1}P_c^{\#}W\tilde g_\Delta(H)\psi_0\big)\,ds\,. \tag{4.26}$$

Therefore, by Proposition 2.1,

$$\left|R_2(t)\right| \le C\left|a(0)\right||||W|||^2\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-r+1}ds\,. \tag{4.27}$$

A first simple consequence, since $r>2$, is that

$$\left|R_2(t)\right| \le C\left|a(0)\right||||W|||^2\,. \tag{4.28}$$

Next, multiplication of (4.27) by $\langle t\rangle^{\alpha}$, taking supremum over the interval $0\le t\le T$ and applying Lemma 4.1 yields the bound

$$\langle t\rangle^{\alpha}\left|R_2(t)\right| \le C_1\left|a(0)\right|\big(1+|||W|||^{2-2\alpha}\big)\,, \tag{4.29}$$

Estimation of $R_3(t)$. We begin by recalling

$$R_3(t) = -i\int_0^t e^{-i\omega_*(t-s)}F_3(s)ds\,. \tag{4.30}$$

Therefore,

$$\left|R_3(t)\right| \le C\int_0^t e^{-\Gamma(t-s)}\left|F_3(s)\right|ds\,. \tag{4.31}$$

$F_3(t) = (\psi_0, W\tilde g_\Delta(H)\phi_3(t))$ is given explicitly by the expression

$$\begin{aligned} &-i\int_0^s d\tau\big(\psi_0, W\tilde g_\Delta(H)e^{-iH_0(s-\tau)}(H_0-\omega-i0)^{-1}P_c^{\#}W\tilde g_\Delta(H)\psi_0\big) \\ &\qquad\qquad \times\big(\psi_0, W\tilde g_\Delta(H)\phi_d(\tau)\big) \\ &= -i\int_0^s d\tau\big(\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma}e^{-iH_0(s-\tau)}(H_0-\omega-i0)^{-1}P_c^{\#}\langle x\rangle^{-\sigma} \\ &\qquad \cdot\langle x\rangle^{\sigma}W\tilde g_\Delta(H)\psi_0\big)\times\big(\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma}\phi_d(\tau)\big) \end{aligned} \tag{4.32}$$

Estimation of $F_3(t)$ yields the bound

$$\big|F_3(t)\big| \le C_W \int_0^s \big\|\langle x\rangle^{-\sigma} e^{-iH_0(t-\tau)}(H_0-\omega-i0)^{-1}P_c^{\#}\langle x\rangle^{-\sigma}\big\| \cdot \big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|_2 \, d\tau\,, \quad (4.33)$$

where

$$C_W = \big\|\langle x\rangle^{\sigma}\tilde g_\Delta(H)W\psi_0\big\|_2^2 \cdot \big\|\langle x\rangle^{\sigma}W\tilde g_\Delta(H)\psi_0\big\|_2 \le C|||W|||^3\,. \quad (4.34)$$

By Proposition 2.1 and (4.33)-(4.34),

$$\big|F_3(s)\big| \le C|||W|||^3 \int_0^s \langle s-\tau\rangle^{-r+1}\big\|\langle x\rangle^{-\sigma}\phi_d(\tau)\big\|_2 d\tau\,. \quad (4.35)$$

If we bound $||\langle x\rangle^{-\sigma}\phi_d(\tau)||_2$ simply by $||\phi_0||_2$ we obtain, from (4.35) and (4.31),

$$\big|R_3(t)\big| \le C|||W|||\,\|\phi_0\|_2\,. \quad (4.36)$$

On the other hand, bounding $||\langle x\rangle^{-\sigma}\phi_d(\tau)||_2$ by $[\phi_d]_{LD}(T)\,\langle\tau\rangle^{-\alpha}$ ($\alpha = r-1$) in (4.35) we obtain

$$\big|F_3(s)\big| \le C|||W|||^3\langle s\rangle^{-\alpha}[\phi_d]_{LD}(T)\,. \quad (4.37)$$

Finally, using (4.37) in (4.31) and applying Lemma 4.1 we have

$$\langle t\rangle^{\alpha}\big|R_3(t)\big| \le C\big(|||W|||+|||W|||^{3-2\alpha}\big)[\phi_d]_{LD}(T)\,. \quad (4.38)$$

Estimation of $R_4(t)$.

$$\begin{aligned} R_4(t) &= -i\int_0^t e^{-i\omega_*(t-s)}\ \big(\psi_0, W\tilde g_\Delta(H)\phi_4(s)\big)\,ds \\ &= (\omega-\omega_1)\int_0^t e^{-i\omega_*(t-s)}\Big(\tilde g_\Delta(H)W\psi_0, \\ &\qquad \int_0^s a(\tau)e^{-iH_0(s-\tau)}P_c^{\#}W\tilde g_\Delta(H)\psi_0\Big)\,d\tau \end{aligned}$$

By Proposition 2.1,

$$\begin{aligned} \big|R_4(t)\big| &\le |\omega-\omega_1|\,|||W|||^2\int_0^t e^{-\Gamma(t-s)}ds\int_0^s\langle s-\tau\rangle^{-r+1}\big|a(\tau)\big|d\tau \\ &\le |\omega-\omega_1|\,|||W|||^2\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-\alpha}ds[a](t)\,. \end{aligned}$$

We now estimate the $|\omega-\omega_1|$. By (3.14)-(3.15),

$$\omega_1-\omega = \big(\psi_0, W\tilde g_\Delta(H)\psi_0\big) - (\psi_0, W\psi_0) \equiv \beta\,. \quad (4.39)$$

An explicit expression, (9.6), is obtained for β in Appendix C,

$$\beta = -\big(W\psi_0, B\bar g_\Delta(H)(H-\lambda_0)^{-1}W\psi_0\big)\,. \quad (4.40)$$

From Theorem 11.1 of Appendix E and an argument along the lines of the proof of (4.1) we have $|\beta| \leq C|||W|||^2$. Therefore, using Lemma 4.1, we find

$$\langle t\rangle^\alpha \big|R_4(t)\big| \leq C|||W|||^{4-2\alpha}[a](T)\,. \tag{4.41}$$

If $\alpha < 3/2$, then

$$\langle t\rangle^\alpha \big|R_4(t)\big| \leq C|||W|||[a](T)\,. \tag{4.42}$$

Closing the estimates and completion of the proof. We can now combine the upper bounds (4.17), (4.24), (4.29), (4.38) and (4.41) for the $R_j(t), 0 \leq j \leq 4$ to obtain, via (4.9), the following upper bound for $a(t)$ provided $|||W||| < 1/2$:

$$[a](T) \leq c_1\big|a(0)\big|\,|||W|||^{-2\alpha} + c_2|||W|||^{1-2\alpha}\big|\langle x\rangle^\sigma \phi_0\big\|_2 \\ + c_3\big(1 + |||W|||^{2-2\alpha}\big)[\phi_d]_{LD}(T)\,.$$

Substitution of this bound into (4.4) gives the following bound for ϕ_d:

$$[\phi_d]_{LD}(T) \leq C_0\big(1 + |||W|||^{2-2\alpha}\big)\big\|\langle x\rangle^\sigma \phi_d(0)\big\|_2 + c_1|||W|||^{1-2\alpha}\big|a(0)\big| \\ + C_3\big(|||W||| + |||W|||^{3-2\alpha}\big)[\phi_d]_{LD}(T)\,. \tag{4.44}$$

Use of (4.44) as a bound for the last term in (4.43) yields a bound for $[a](T)$,

$$[a](T) \leq c_1\big|a(0)\big|\,|||W|||^{-2\alpha} + c_2|||W|||^{1-2\alpha}\big\|\langle x\rangle^\sigma\ \phi_d(0)\big\|_2\,. \tag{4.45}$$

Finally, for $|||W|||$ sufficiently small and $\alpha < 3/2$ we have

$$[\phi_d]_{LD}(T) \leq C\big(1 + |||W|||^{2-2\alpha}\big)\big\|\langle x\rangle^\sigma \phi_d(0)\big\|_2 + C|||W|||^{1-2\alpha}\big|a(0)\big|\,. \tag{4.46}$$

Taking $T \to \infty$ we conclude the decay of $\phi(t)$, with initial data ϕ_0 in the range of $P_\Delta(H)$, with rate $\langle t\rangle^{-\alpha}$, $0 < \alpha < 3/2$. It follows [RSim] that the interval Δ consists of absolutely continuous spectrum of H, as asserted in parts (a) and (b) of Theorem 2.1.

5 Local Decay of Solutions for Large r

In the preceding subsection we proved the decay of solutions, $\phi(t, x)$, in the local decay sense, with a slow rate of decay $\langle t\rangle^{-\alpha}$ with $1 < \alpha < 3/2$; $\alpha = r - 1$. A consequence of this result is that, in the interval Δ, the spectrum of H is absolutely continuous. Now if Δ contains no thresholds of H, we expect decay as $t \to \infty$ at a rate which is faster than any polynomial. (For example, this is what one has for constant coefficient dispersive equations for energy intervals containing no points of stationary phase.) In this section we show that this result holds in the sense of (2.17) in Theorem 2.1. This requires some adaptation of the methods of section 4. We shall indicate here only the required modifications to the argument of the previous section.

(1) The origin of the restriction $\alpha < 3/2$ can be traced to our application of part (ii) of Lemma 4.1. In particular, in obtaining (4.13) we use that

$$\sup_{t\geq 0}\langle t\rangle^{\alpha} e^{-\Gamma t} = \mathcal{O}(\Gamma^{-\alpha})\,. \tag{5.1}$$

It follows that certain coefficients are found to be large for $|||W|||$ small, an obstruction to closing the system of estimates for $[a]$ and $[\phi_d]_{LD}$, unless $\alpha < 3/2$. This is remedied by taking the supremum in (5.1) over t in the interval $[\Gamma^{-1-\delta}, T]$, where $\delta > 0$.

Lemma 5.1. *Let $M \equiv \Gamma^{-1-\delta} \sim |||W|||^{-2(1+\delta)}$; see* (W3). *There exists $\theta_* > 0$ such that if $|||W||| < \theta_*$ and $t \geq M$, then*

(a) $\langle t\rangle^{r-1}\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-r}ds \leq C\Gamma^{\delta} \sim |||W|||^{2\delta}$.

(b) $\langle t\rangle^{r-1}\int_0^t e^{-\Gamma(t-s)}\langle s\rangle^{-\alpha}ds \leq C\Gamma^{-1}$.

(2) Assume $r > 2$ ($\alpha > 1$). The analysis of section 4 yields a coupled system of integral inequalities for the functions $a(t)$ and

$$L(t) \equiv \left\|\langle x\rangle^{-\sigma}\phi_d(t)\right\|_2\,. \tag{5.2}$$

The precise form of these inequalities can be seen as follows. Let

$$I_r\{L\}(t) = \int_0^t \langle t-s\rangle^{-r}L(s)ds\,. \tag{5.3}$$

Then, by (4.2), (4.9) and the estimates for $R_j(t)$, $j = 0,1,2,3,4$, the inequalities for $L(t)$ and $a(t)$ take the form

$$\begin{aligned} L(t) &\leq C_0\langle t\rangle^{-r} + C_1|||W|||I_r\{|a|\}(t) + C_2|||W|||I_r\{L\}(t) \\ |a(t)| &\leq A_0e^{-\Gamma t} + A_1|||W|||I_r\{e^{-\Gamma s}\}(t) + A_2|||W|||^2 I_{r-1}\{e^{-\Gamma s}\}(t) \\ &\quad + A_3|||W|||^2\int_0^t e^{-\Gamma(t-s)}I_r\{L\}(s)ds \\ &\quad + A_4|||W|||^3\int_0^t e^{-\Gamma(t-s)}I_{r-1}\{L\}(s)ds \\ &\quad + A_5|||W|||^4\int_0^t e^{-\Gamma(t-s)}I_{r-1}\{|a|\}(s)ds\,, \end{aligned} \tag{5.4}$$

where the C_j and A_j denote positive constants.

(3) The procedure is first to consider the functions $L(t)$ and $a(t)$ on a large but finite time interval, $0 \leq t \leq \Gamma^{-1-\delta} \equiv M$, where δ is positive and suitably chosen. An explicit bound for $L(t)$ and $a(t)$ can be found by

iteration of the inequalities (5.4). For this, we use the following estimate of $I_r\{e^{-\Gamma s}\}$, which is proved using integration by parts,

$$I_r\{e^{-\Gamma s}\} \leq c_0 e^{-\Gamma t} + \sum_{k=1}^{r-2} c_k \ \Gamma^{k-1}\langle t\rangle^{-r+k-1} + c_{r-1}\Gamma^{r-2-\rho}\langle t\rangle^{-1}\,, \tag{5.5}$$

where $\rho > 0$ is arbitrary.

(4) To show decay for arbitrary, in particular, large $\alpha = r-1$, and the estimates of $\mathcal{R}(t)$ of Theorem 2.1, we introduce the norms

$$[a]^{\Gamma}(T) \equiv \sup_{M\leq t\leq T} \langle t\rangle^{\alpha}\big|a(t)\big| \tag{5.6}$$

and

$$[\phi_d]^{\Gamma}_{LD}(T) \equiv \sup_{M\leq t\leq T} \langle t\rangle^{\alpha}\big\|\langle x\rangle^{-\sigma}\phi_d(t)\big\|\,. \tag{5.7}$$

We now reexpress the system (5.4) for $L(t)$ and $a(t)$ by breaking the time integrals in (5.4) into a part over the interval $[0,M]$ and a part over the interval $[M,t]$. Using the estimate of part (3) above, the integrals over $[0,M]$ are estimated to be of order $|||W|||^{\varepsilon}\langle t\rangle^{-r+1}$ for some $\varepsilon = \varepsilon(\delta)$.

In this way, the resulting system for $L(t)$ and $a(t)$ is now reduced to one which can be studied using Lemma 5.1 and the approach of section 4. Using this approach estimates for the norms (5.6) and (5.7), and consequently of $\mathcal{R}(t)$ can be obtained.

6 Examples and Applications

In this section we sketch examples and applications of Theorem 2.1. Most of these examples have been previously studied, under more stringent hypotheses on H_0 and W, *e.g.* some type of analyticity: dilation analyticity for the Helium atom, translation analyticity for the Stark Hamiltonian; see [CyFKS] and references cited therein. Theorem 2.1 enables us to relax this requirement and gives the detailed time-behavior of solutions near the resonant energy at all time-scales. Example 5 concerns the instability of an eigenvalue embedded at a threshold, a result which we believe is new and not tractable by techniques of dilation analyticity.

We begin with the remark that in the examples below, one can often replace the operator, $-\Delta$ by $H_1 \equiv \omega(p)$, where $p = -i\nabla$. The necessary hypothesis on local decay, (H4), is reduced to its verification for $H_1 + V$. By the general discussion of local decay estimates of Appendix D (see also [Si3]), we have

Theorem 6.1. *The operator $H = H_1 + V$ satisfies the required local decay estimates of (H4) under the following hypotheses:*

Hypotheses on $\omega(p)$:

(i) *$\omega(p)$ is real valued and $\omega(p) \to \infty$ as $|p| \to \infty$.*
(ii) *$\omega(p)$ is C^m function, $m \geq 4$.*
(iii) *$\nabla_p \omega = 0$ on at most finitely many points, in any compact domain.*

Hypotheses on $V(x)$: $V(x)$ is real valued and such that

(V1) *$V(x)$, $x \cdot \nabla V$, $(x \cdot \nabla)^2 V$, $(x \cdot \nabla)^3 V$ are all $g(H_1)$ bounded for $g \in C_0^\infty$.*
(V2) *$|V(x)| = 0(\langle x \rangle^{-\epsilon})$, $\epsilon > 0$, $|x| \to \infty$.*
(V3) *$\chi_R V$, $\chi_R (x \cdot \nabla)^m V$, m=1, 2, 3 are $g(H_1)$-compact, for $\chi_R \equiv \chi_{[R,\infty)}(|x|)$ with some $R > 0$.*

The proof of this result follows from the procedure outlined in Appendix D where we use the hypotheses on $\omega(p)$ and V and the choice for the operator A is

$$A = \tfrac{1}{2}(x \cdot \nabla_p \omega + \nabla_p \omega \cdot x)\,.$$

Remark. Due to lack of assumptions on analyticity of $\omega(p)$ or $V(x)$ one cannot simply apply the technique of analytic deformation used in other approaches.

Example 1: *Dispersive Hamiltonian.* With the above assumptions on $\omega(p)$ and $V(x)$, Theorem 2.1 applies directly to the operator $H_0 \equiv \omega(p) + V(x)$.

Example 2: *Direct Sum.* Let

$$H_0 = \begin{pmatrix} -\Delta_x & 0 \\ 0 & -\Delta_x + q(x) \end{pmatrix}$$

acting on $\mathbb{C}^2 \otimes L^2(\mathbb{R}^n)$, where $q(x)$ is a well behaved potential having some positive discrete eigenvalues. An example of this type is considered in [W].

Consider, for example, the case where $q(x) = P(x)$, is a polynomial which is bounded below. In this case, the spectrum of $-\Delta_x + P(x)$ is discrete and consists of an infinite set of eigenvalues $\lambda_1 < \lambda_2 \cdots$ with corresponding eigenfunctions $\psi_1, \psi_2, \ldots$. The spectrum of H is then

$$\big\{\text{eigenvalues of } -\Delta_x + P(x)\big\} \cup [0, \infty)$$

and therefore H_0 has nonnegative eigenvalues embedded in its continuous spectrum.

Let

$$W = \begin{pmatrix} 0 & W(x) \\ W(x) & 0 \end{pmatrix}$$

with W satisfying conditions (W).

Theorem 6.2. *For H_0 and W as above, if for some strictly positive simple eigenvalue $\lambda > 0$ the resonance condition (Fermi golden rule) (2.7) holds, then in an interval Δ around λ, the spectrum of H is absolutely continuous and the other conclusions of Theorem 2.1 hold. Furthermore, if $n > 4$, Theorem 2.1 holds even when $\lambda = 0$ is an eigenvalue.*

Proof. In this case local decay must be proved for $-\Delta_x$, with $r > 2$. This is well known. What is more, if the spatial dimension is larger than four, $n > 4$, then $\lambda = 0$ is also allowed, since in this case we use

$$\begin{aligned}\left\|\langle x\rangle^{-\frac{n}{2}-\epsilon}e^{i\Delta_x t}\psi\right\|_2 &\le \left\|\langle x\rangle^{-\frac{n}{2}-\epsilon}\right\|_2\|e^{i\Delta_x t}\psi\|_\infty\\ &\le Ct^{-n/2}\|\psi\|_1\,.\end{aligned}$$

Hence, for $\psi \in \mathcal{D}(\langle x\rangle^{(n/2)+\epsilon})$, and $n > 4$ we have the necessary decay,

$$\left\|\langle x\rangle^{-\frac{n}{2}-\epsilon}e^{i\Delta_x t}\langle x\rangle^{-\frac{n}{2}-\epsilon}\right\| \le Ct^{-n/2}$$

with $r = n/2 > 2$.

Example 3: *Tensor Products.* Let $H_0 = \mathbf{1}\otimes h_1 + h_2 \otimes \mathbf{1}$ act on $L^2(\mathbb{R}^n_{x_1}) \otimes L^2(\mathbb{R}^n_{x_2})$, where

$$h_1 = -\Delta_{x_1} \quad \text{and} \quad h_2 = -\Delta_{x_2} + q(x_2)\,. \tag{6.1}$$

Then,

$$\sigma(H_0) = \{\lambda : \lambda = \lambda_1 + \lambda_2\,,\ \lambda_1 \in \sigma(-\Delta_{x_1}) \text{ and } \lambda_2 \in \sigma(-\Delta_{x_2} + q(x_2))\}\,. \tag{6.2}$$

Let $W(x_1, x_2)$ act on $L^2 \otimes L^2$, satisfying ($\mathbf{W}$), with $\langle x\rangle^2 \equiv 1+|x_1|^2+|x_2|^2$. Then we have

Theorem 6.3. *The embedded eigenvalues of H_0 are unstable and Theorem 2.1 holds.*

Example 4: *Helium Type Hamiltonians* [RSim]. Consider H_0 as in Example 3 with

$$h_1 = -\Delta_{x_1} - |x_1|^{-1}\,, \quad h_2 = -\Delta_{x_2} - |x_2|^{-1}\,. \tag{6.3}$$

Also, let W be of the form

$$W(x_1, x_2) = W(x_1 - x_2)\,.$$

In this case the weight $\langle x\rangle^2 = 1 + |x_1|^2 + |x_2|^2$. We now discuss the hypothesis (W).

H_0 has infinitely many negative eigenvalues embedded in the continuous spectrum [CyFKS]. If Δ is a subinterval of the negative real line containing exactly one negative eigenvalue, E, then g_Δ is a sum of terms of the form

$$g^c_{\Delta - E}(h_1) \otimes \mathbf{P} \quad \text{and} \quad \mathbf{P} \otimes g^c_{\Delta - E}(h_2)\,. \tag{6.4}$$

Here, $g^c_{\Delta-E}(h_j)$ is a spectral projection onto the continuous spectral part associated with an interval $\Delta - E$, the translate of Δ by $-E$, and $\mathbf{P}$ denotes a (negative) bound state projection. Thus, g_Δ localizes either the x_1 or the x_2 variable, and so while $\langle x\rangle^{2\sigma}W$ is not bounded we do have that

$$\langle x\rangle^{2\sigma} W g_\Delta(H_0)$$

is bounded provided, for example, W is short range.

In the case, where W is long range, *i.e.*

$$W(x_1 - x_2) = \mathcal{O}\big(\langle x_1 - x_2\rangle^{-1}\big) \tag{6.5}$$

we first prove a *minimal velocity bound* and then use it to get local decay.

Going back to (3.27) we estimate

$$\left\| F\Big(\tfrac{|x|}{t} \leq \eta\Big)\phi_d(t)\right\|_2 \tag{6.6}$$

using the known propagation and minimal velocity estimates for H_0 [SiSo]. The problematic term, which is the last term on the right-hand side of (3.27) is then bounded by

$$c_1 \int_0^t \langle t-s\rangle^{-1-\varepsilon} \big\|\langle x\rangle^{\frac{1}{2}+\delta} W\tilde{g}_\Delta(H)\big\|_2 \left\|F\Big(\tfrac{|x|}{s} \leq \eta\Big)\phi_d(s)\right\|_2 ds$$
$$+ c_2 \int_0^t \langle t-s\rangle^{-1-\varepsilon} \left\|\langle x\rangle^{\frac{1}{2}+\delta} W\tilde{g}_\Delta(H) F\Big(\tfrac{|x|}{s} \geq \eta\Big)\right\| \big\|\phi_d(s)\big\|_2 ds\,.$$

Since

$$\left\|\langle x\rangle^{\frac{1}{2}+\delta} W\tilde{g}_\Delta(H) F\Big(\tfrac{|x|}{s} \geq \eta\Big)\right\| \leq c_3 |||W||| \langle s\rangle^{-\frac{1}{2}+\delta}\,, \tag{6.7}$$

we can close the inequalities and obtain

$$\langle t\rangle^{\frac{1}{2}-\delta} \left\|F\Big(\tfrac{|x|}{t} \leq \eta\Big)\phi_d(t)\right\| \leq \ c_0 + c_1 \sup_{0\leq s\leq t} \left|\langle s\rangle^{\frac{1}{2}-\delta} a(s)\right| \ . \tag{6.8}$$

The above estimate, together with the estimates for $a(t)$ lead to local decay with a rate $\langle t\rangle^{-(1/2)+\delta}$. This rate is not sufficient to preclude singular continuous spectrum. However, the Mourre estimate holds in this interval for $H_0 + W$ which implies local decay and absence of singular continuous spectrum; see Theorem 10.1 and Theorem 10.2 in Appendix D.

EXAMPLE 5: *Threshold Eigenvalues.* Let $H_0 = -\Delta + V(x)$ in $L^2(\mathbb{R}^n)$, $n > 4$. Assume $V(x)$ is smooth and rapidly decaying for simplicity. Then, under certain conditions on the spectrum of H_0, and the behavior of its resolvent at zero energy, one can prove local decay and L^∞ decay with a rate $r > 2$; see [JeK], [JoSoSog].

In such cases, it follows by Theorem 2.1 that a threshold eigenvalue at $\lambda = 0$, if it exists, is unstable with respect to small and generic perturbations, W.

EXAMPLE 6: *Stark Effect - an atom in a uniform electric field.* The Stark Hamiltonian is given by

$$H = -\Delta + V(x) + \vec{E} \cdot x \tag{6.9}$$

acting on $L^2(\mathbb{R}^n)$. If $V(x)$ is real valued and not too singular, then for $\vec{E} \neq 0$ the continuous spectrum of H is $(-\infty, \infty)$. To see this apply Theorem 9.1 with $A \equiv \vec{E} \cdot p$, $p = -i\nabla$. Thus, if H has an eigenvalue, it is necessarily embedded in the continuous spectrum.

Our results can be used to show that any embedded eigenvalue is generically unstable (*i.e.* provided the Fermi golden rule resonance condition (W3) holds) and perturbs to a resonance.

To see this, one can proceed by a decoupling argument; see [CHi]. This reduces the problem to a direct sum of Hamiltonians, as in Example 2, with Hamiltonians of the form

$$\begin{aligned} H_1 &= -\Delta + \vec{E} \cdot x + \tilde{V}(x, -i\nabla)\,, \\ H_2 &= -\Delta + V_b(x)\,, \\ W &= \tilde{W}(x, -i\nabla)\,. \end{aligned}$$

The strategy is then to use the techniques of Appendix B to verify hypotheses (W) and the techniques of Appendix D to prove the necessary local decay estimates in (H) for the the operator $H_0 = \mathrm{diag}(H_1, H_2)$.

EXAMPLE 7: *The Radiation Problem.* The radiation problem is the fundamental problem which motivated work on quantum resonances. See the work of Weisskopf and Wigner [WeiWi], following Dirac [Di] and Landau [L]. We present here a very brief description of the problem and the relation to our methods. For a more detailed discussion of the formulation see [BFroSi].

The free Hamiltonian, H_0, is the direct sum operator acting on $\mathcal{H}_a \oplus \mathcal{H}_{photon}$. Here, $\mathcal{H}_a$ is the Hilbert space associated with an atom or molecule. $\mathcal{H}_{photon}$ is the Fock space of free photons. H_0 is then the Hamiltonian of a decoupled particle and free photon system

$$H_0 = H_a \otimes I + I \otimes H_{photon}\,. \tag{6.10}$$

The next step is to introduce the interaction term W that couples the photon-radiation field to the atom. In quantum electrodynamics, this coupling is given by the standard minimal coupling, but in general it is sufficient to consider a simple approximation *e.g.* the dipole approximation [AlE]. The goal is to show that all eigenvalues of the original atom, except the ground state, are destabilized by the coupling and become resonances.

This is the phenomenon of spontaneous emission. One is also interested in the computation of the *lifetime* and the transition probabilities.

A simplified Hamiltonian which incorporates the essential mathematical features of the radiation problem is

$$H = H_a \otimes I + I \otimes H_{photon} + \lambda W , \tag{6.11}$$

where $H_a = -\Delta + V(x)$ acting on $L^2(\mathbb{R}^n)$ describes the atom, and the coupling is given by

$$W = \int \left(\omega(\vec{k})\right)^{-1/2} \left(g(\vec{k}) e^{i\vec{k}x} a_{\vec{k}} + \overline{g}(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} a^{\dagger}_{\vec{k}}\right) d\vec{k} . \tag{6.12}$$

The Hamiltonian associated with the photon field is given by

$$H_{photon} = \int \omega(\vec{k}) a^{\dagger}_{\vec{k}} a_{\vec{k}} d\vec{k} , \tag{6.13}$$

the second quantization of multiplication in Fourier space by $\omega(\vec{k})$ in $L^2(\mathbb{R}^n)$. Hence, H_{photon} acts on the Fock space of bosons,

$$\mathcal{F} = \oplus_{m=1}^{\infty} \otimes_{sym}^{m} L^2(\mathbb{R}^3) , \tag{6.14}$$

where $\otimes_{sym}^m$ denotes the m-fold symmetric tensor product of $L^2(\mathbb{R}^3)$. The operator $a^{\dagger}_{\vec{k}}$ is the creation operator on $\mathcal{F}$ and $a_{\vec{k}}$, its adjoint.

For realistic photons, we must have $g \equiv 1$ and $\omega(\vec{k}) \sim |\vec{k}|$ for $\vec{k}$ near zero. However, to make mathematical sense of the above Hamiltonian we need to introduce the *ultraviolet cutoff*;

$$g = 0 \quad \text{for} \quad |\vec{k}| \gg 1 . \tag{6.15}$$

When the coupling constant λ is zero, and so $H = H_0$, it is fairly easy to verify our conditions (H) for H_0, even in the massless photon case, $\omega(\vec{k}) = |\vec{k}|$ [BFroSi],[BFSS]. The conditions (W) however fail when $\omega(k) = |\vec{k}|$ since in this case the interaction λW is not localized. On the other hand, in the massive case $\left(\omega(\vec{k}) = \sqrt{m^2 + |\vec{k}|^2},\ m \neq 0\right)$, the interaction is localized for quite general $g(\vec{k})$; see [Ge]. In this case, our conditions (W) can be verified and therefore the results of Theorem 2.1 can be applied.

7 Appendix A: Proof of Local Decay Proposition 2.1

Our aim is to prove local decay estimates for $e^{-iH_0t}(H_0 - \Lambda - i0)^{-1} P_c^{\#}$ using the given local decay estimates for $e^{-iH_0t} P_c^{\#}$, where $\Lambda \in \mathbf{T}^{\#}$. The proof is split into two parts: analysis near Λ and analysis away from Λ.

Let Δ be a small interval about Λ and g_Δ denote a smoothed out characteristic function of Δ and $\overline{g}_\Delta = 1 - g_\Delta$. We write

$$\begin{aligned} e^{-iH_0t}(H_0 - \Lambda - i0)^{-1}P_c^{\#} &= e^{-iH_0t}(H_0 - \Lambda - i0)^{-1}P_c^{\#}(g_\Delta + \overline{g}_\Delta) \\ &\equiv T_\Delta^t + S_\Delta^t \,. \end{aligned}$$

We first estimate the operator T_Δ^t. Let $\varepsilon > 0$ and set

$$T_{\Delta,\varepsilon}^t = e^{-i(H_0-\Lambda-i\varepsilon)t}(H_0 - \Lambda - i\varepsilon)^{-1}P_c^{\#} g_\Delta \,.$$

Then, by (H4),

$$T_{\Delta,\varepsilon}^t = i\int_t^\infty e^{-i(H_0-\Lambda-i\varepsilon)s}P_c^{\#} g_\Delta \, ds \,.$$

Let $\langle x\rangle^\sigma h \in L^2$. Then,

$$\begin{aligned} \big\|\langle x\rangle^{-\sigma}T_{\Delta,\varepsilon}^t h\big\|_2 &\le \int_t^\infty \big\|\langle x\rangle^{-\sigma}e^{-i(H_0-\Lambda-i\varepsilon)s}\,P_c^{\#} g_\Delta h\big\|_2 ds \\ &\le \int_t^\infty e^{-\varepsilon s}\big\|\langle x\rangle^{-\sigma}e^{-iH_0 s}P_c^{\#} g_\Delta h\big\|_2 ds \\ &\le \int_t^\infty e^{-\varepsilon s}\langle s\rangle^{-r}\big\|\langle x\rangle^{\sigma} h\big\|_2 ds \\ &\le C\langle t\rangle^{1-r}\big\|\langle x\rangle^{\sigma} h\big\|_2 \,. \end{aligned}$$

Therefore, taking $\varepsilon \downarrow 0$, we get

$$\big\|\langle x\rangle^{-\sigma}T_\Delta^t h\big\|_2 \le C\langle t\rangle^{1-r}\big\|\langle x\rangle^{\sigma} h\big\|_2 \,.$$

To estimate S_Δ^t, we exploit that the energy is localized away from Λ, and so the resolvent $(H_0 - \Lambda)^{-1}$ is bounded,

$$\begin{aligned} \langle x\rangle^{-\sigma}S_\Delta^t\langle x\rangle^{-\sigma} &= \langle x\rangle^{-\sigma}e^{-iH_0t}(H_0 - \Lambda - i0)^{-1}P_c^{\#}\overline{g}_\Delta\langle x\rangle^{-\sigma} \\ &= \langle x\rangle^{-\sigma}e^{-iH_0t}P_c^{\#}\langle x\rangle^{-\sigma}\cdot\langle x\rangle^{\sigma}(H_0 - \Lambda - i0)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma} \,. \end{aligned} \tag{7.1}$$

For the operator norm we then have the bound

$$\begin{aligned} \big\|\langle x\rangle^{-\sigma}S_\Delta^t\langle x\rangle^{-\sigma}\big\| \le{}& \big\|\langle x\rangle^{-\sigma}e^{-iH_0t}P_c^{\#}\langle x\rangle^{-\sigma}\big\| \\ &\cdot\big\|\langle x\rangle^{\sigma}(H_0 - \Lambda - i0)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma}\big\| \,. \end{aligned} \tag{7.2}$$

We bound the first factor in (7.2) using the assumed local decay estimate (H4). The second factor is controlled as follows. Note that

$$\begin{aligned} \langle x\rangle^{\sigma}(H_0 - \Lambda - i0)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma} &= \langle x\rangle^{\sigma}(H_0 + c)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma} \\ &+ (\Lambda + c)\langle x\rangle^{\sigma}(H_0 + c)^{-1}\langle x\rangle^{-\sigma}\cdot\langle x\rangle^{\sigma}(H_0 - \Lambda - i0)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma} \,. \end{aligned} \tag{7.3}$$

Taking operator norms and using hypothesis **(H5)** and Theorem 11.2 of Appendix E we obtain the following bound on the second factor in (7.2)

$$\big(1 - |\Lambda + c|\|\langle x\rangle^{\sigma}(H_0 + c)^{-1}\langle x\rangle^{-\sigma}\|\big)\big\|\langle x\rangle^{\sigma}(H_0 - \Lambda - i0)^{-1}\overline{g}_\Delta\langle x\rangle^{-\sigma}\big\| \le$$

$$\left\|\langle x\rangle^\sigma (H_0+c)^{-1}\overline{g}_\Delta \langle x\rangle^{-\sigma}\right\| \leq \left(1+\|\langle x\rangle^\sigma g_\Delta \langle x\rangle^{-\sigma}\|\right)\left\|\langle x\rangle^\sigma (H_0+c)^{-1}\langle x\rangle^{-\sigma}\right\|.$$

This completes the proof.

8 Appendix B: Operator Norm Estimates Involving $g_\Delta(H)$

In this section we prove Propositions 3.1 and 4.1. These propositions require some operator calculus.

Let $\hat{h}(\lambda)$ denote the Fourier transform of the function g, with the normalization,

$$\hat{h}(\mu) = (2\pi)^{-1}\int e^{i\mu\lambda} h(\mu)d\mu\,.$$

Proof of Proposition 3.1. Recall that λ_0 denotes an embedded eigenvalue of the unperturbed operator, H_0, g_Δ is a smoothed out characteristic function of the interval Δ, and I is an open set which contains the support of P_1 and is disjoint from Δ.

We need to show that

$$B = \left(I - g_\Delta(H) g_I(H_0)\right)^{-1} \tag{8.1}$$

is bounded and we do this by showing that $\|g_\Delta(H)g_I(H_0)\|$ has small norm. We use techniques of [SiSo].

Let $\tilde{\Delta}$ be an interval which contains and is slightly larger than Δ. Then

$$\begin{aligned} g_\Delta(H)g_I(H_0) &= g_\Delta(H)\big(I - g_{\tilde{\Delta}}(H_0)\big)g_I(H_0)\\ &= g_\Delta(H)g_{\Delta'}(H_0)g_I(H_0)\\ &= g_\Delta(H)\big(g_{\Delta'}(H_0) - g_{\Delta'}(H)\big)g_I(H_0)\,, \end{aligned} \tag{8.2}$$

where Δ and Δ' are disjoint.

We now obtain, an expression for the above difference, which is easily estimated. Using the Fourier transform we have that

$$g_{\Delta'}(H_0) - g_{\Delta'}(H) = \int (e^{i\mu H_0} - e^{i\mu H})\hat{g}_{\Delta'}(\mu)d\mu\,. \tag{8.3}$$

Furthermore,

$$\begin{aligned} e^{i\mu H_0} - e^{i\mu H} &= (I - e^{i\mu H}e^{-i\mu H_0})e^{i\mu H_0}\\ &= -\int_0^\mu \tfrac{d}{ds} e^{isH}e^{-isH_0}\,ds\, e^{i\mu H_0}\\ &= -\int_0^\mu e^{isH} i(H-H_0)e^{-isH_0}\,ds\, e^{i\mu H_0}\\ &= -i\int_0^\mu e^{isH} W e^{-isH_0}\,ds\, e^{i\mu H_0}\,. \end{aligned} \tag{8.4}$$

Substitution of (8.4) into (8.3) yields

$$g_{\Delta'}(H_0) - g_{\Delta'}(H) = -i\int \hat{g}_{\Delta'}(\mu)e^{i\mu H}d\mu \int_0^\mu e^{-isH}We^{isH_0}ds\,. \tag{8.5}$$

We now apply the operator $g_\Delta(H)$ to the expression in (8.5) and estimate

$$\begin{aligned}\big\|g_\Delta(H)(g_{\Delta'}(H_0) - g_{\Delta'}(H)\big\| &\le \int \big|\hat{g}_{\Delta'}(\mu)\big| \int_0^\mu \big\|g_\Delta(H)W\big\|ds\,d\mu \\ &\le \int \big|\hat{g}_{\Delta'}(\mu)\big||\mu|d\mu\big\|g_\Delta(H)W\big\| \\ &\le C|\Delta|^{-1}\big\|g_\Delta(H)W\big\| \le C|\Delta|^{-1}|||W|||\,. \end{aligned}\tag{8.6}$$

Therefore,

$$\big\|g_\Delta(H)g_I(H_0)\big\| \le C|\Delta|^{-1}|||W|||\,.$$

and $(I - g_\Delta(H)g_I(H_0))^{-1}$ is bounded provided $|\Delta|^{-1}|||W||| < \theta$ is sufficiently small; see (W4).

Proof of Proposition 4.1. We estimate the norm of the operator

$$\mathcal{G} = \langle x\rangle^\sigma W\tilde{g}_\Delta(H)\langle x\rangle^\sigma \tag{8.7}$$

in terms of $|||W|||$, defined in (W2).

Recall that by (3.11)

$$\tilde{g}_\Delta(H) = g_\Delta(H)\big(I - g_\Delta(H)g_I(H_0)\big)^{-1}\overline{g}_I(H_0)\,. \tag{8.8}$$

Using (8.8) we express $\mathcal{G}$ as the product of operators

$$\begin{aligned}&\langle x\rangle^\sigma W\tilde{g}_\Delta(H)\langle x\rangle^\sigma = \mathcal{G}_1\cdot\mathcal{G}_2\cdot\mathcal{G}_3 \\ &\equiv \langle x\rangle^\sigma Wg_\Delta(H)\langle x\rangle^\sigma\cdot\langle x\rangle^{-\sigma}\big[I - g_\Delta(H)g_I(H_0)\big]^{-1}\langle x\rangle^\sigma\cdot\langle x\rangle^{-\sigma}\overline{g}_I(H_0)\langle x\rangle^\sigma\,.\end{aligned}\tag{8.9}$$

Therefore it suffices to obtain upper bounds for $\|\mathcal{G}_j\|$, $j = 1,2,3$. We shall use some general operator calculus estimates of Appendix E, especially Theorem 11.1.

Bound on $\mathcal{G}_3$: This follows from Theorem 11.1 of Appendix E, with $A = H_0$ and $\varphi = \overline{g}_I$, a function which is smooth and rapidly decaying at infinity.

Bound on $\mathcal{G}_2$: By our hypotheses and the proof of Proposition 3.1, $\|g_\Delta(H)g_I(H_0)\|$ is small and

$$\big(I - g_\Delta(H)g_I(H_0)\big)^{-1} = \sum_{n=0}^\infty \big(g_\Delta(H)g_I(H_0)\big)^n \tag{8.10}$$

converges in the norm. We need to show this in the weighted norms. For this, we will show that the norm of $g_\Delta(H)g_I(H_0)$ is small in the weighted norm, *i.e.*

$$\langle x\rangle^\sigma g_\Delta(H) g_I(H_0)\langle x\rangle^{-\sigma} = \mathcal{O}\big(|||W|||\big)\,. \tag{8.11}$$

Since the supports of g_Δ and g_I are disjoint

$$\begin{aligned}\langle x\rangle^\sigma g_\Delta(H) g_I(H_0)\langle x\rangle^{-\sigma} &= \langle x\rangle^\sigma\big(g_\Delta(H) - g_\Delta(H_0)\big) g_I(H_0)\langle x\rangle^{-\sigma}\\ &= \langle x\rangle^\sigma\big(g_\Delta(H) - g_\Delta(H_0)\big)\langle x\rangle^{-\sigma}\cdot\langle x\rangle^\sigma g_I(H_0)\langle x\rangle^{-\sigma}\,.\end{aligned}$$

By parts (a) and (b), respectively, of Theorem 11.1 both

$$\begin{gathered}\big\|\langle x\rangle^\sigma g_I(H_0)\langle x\rangle^{-\sigma}\big\| < \infty \text{ and}\\ \langle x\rangle^\sigma\big(g_\Delta(H) - g_\Delta(H_0)\big)\langle x\rangle^{-\sigma} = \mathcal{O}\big(|||W|||\big)\,. \end{gathered}\tag{8.12}$$

Bound on $\mathcal{G}_1$: Expanding about the unperturbed operator, H_0, we have

$$\begin{aligned}\mathcal{G}_1 &= \langle x\rangle^\sigma W g_\Delta(H)\langle x\rangle^\sigma\\ &= \langle x\rangle^\sigma W(H+c)^{-1}\langle x\rangle^\sigma\cdot\langle x\rangle^{-\sigma}(H+c) g_\Delta(H_0)\langle x\rangle^\sigma\,.\end{aligned}\tag{8.13}$$

Taking norms, we get

$$\|\mathcal{G}_1\| \le \big\|\langle x\rangle^\sigma W(H+c)^{-1}\langle x\rangle^\sigma\big\|\cdot\big\|\langle x\rangle^{-\sigma}(H+c) g_\Delta(H_0)\langle x\rangle^\sigma\big\| \tag{8.14}$$

Consider the first factor in (8.14). We show that it is of order $|||W|||$ as $|||W||| \to 0$. Note that

$$\begin{aligned}\langle x\rangle^\sigma W(H+c)^{-1}\langle x\rangle^\sigma = \langle x\rangle^\sigma W(H_0+c)^{-1}\langle x\rangle^\sigma - \langle x\rangle^\sigma W(H_0+c)^{-1}\langle x\rangle^{-\sigma}&\\ \cdot\langle x\rangle^\sigma W(H+c)^{-1}\langle x\rangle^\sigma\,.&\end{aligned}$$

Taking norms we obtain

$$\begin{aligned}\big(1 - \|\langle x\rangle^\sigma W(H_0+c)^{-1}\langle x\rangle^{-\sigma}\|\big)\big\|\langle x\rangle^\sigma W(H+c)^{-1}\langle x\rangle^\sigma\big\|&\\ \le \big\|\langle x\rangle^\sigma W(H_0+c)^{-1}\langle x\rangle^\sigma\big\|\,.&\end{aligned}\tag{8.15}$$

Therefore, if $|||W||| < 1/2$ the first factor of (8.14) is bounded by $2|||W|||$. The second factor of (8.14) is bounded by Theorem 11.1.

Finally, we note that the above bounds on $\mathcal{G}_j$ complete the proof of Proposition 4.1.

9 Appendix C: Expansion of the Complex Frequency, ω_*

In this section we prove Proposition 3.3, in which an expansion of the complex frequency, ω_*, is presented. In particular, our goal will be to obtain an expansion of ω_* which is explicit to second order in the perturbation, W, with an error term of order $|||W|||^3$.

Recall that

$$\omega_* = \lambda_0 + \omega_A - \omega_B\,, \tag{9.1}$$

where

$$\omega_A = \big(\psi_0, W\tilde{g}_\Delta(H)\psi_0\big) = \omega_1 - \lambda_0 \quad (\text{see } (3.15))\,, \tag{9.2}$$

and

$$\omega_B = \big(W\psi_0, \tilde{g}_\Delta(H)R_{H_0}(\omega + i0)P_c^{\#}W\tilde{g}_\Delta(H)\psi_0\big)\,. \tag{9.3}$$

Expansion of ω_A:

$$\begin{aligned}\omega_A &\equiv \big(\psi_0, W\tilde{g}_\Delta(H)\psi_0\big)\\ &= (\psi_0, W\psi_0) + \big(W\psi_0, B[g_\Delta(H) - g_\Delta(H_0)]\psi_0\big)\\ &\equiv (\psi_0, W\psi_0) + \beta\,.\end{aligned} \tag{9.4}$$

In what follows, we shall frequently use the notation $(H-\lambda)^{-1}$ and $\frac{1}{H-\lambda}$ interchangeably.

Proposition 9.1.

$$\big[g_\Delta(H) - g_\Delta(H_0)\big]\psi_0 = -\bar{g}_\Delta(H-\lambda_0)^{-1}W\psi_0 \tag{9.5}$$

Proof. Noting that $H - H_0 = W$, we have the expansion formula

$$\begin{aligned}g_\Delta(H) - g_\Delta(H_0) &= \int \hat{g}_\Delta(\lambda)(e^{i\lambda H} - e^{i\lambda H_0})d\lambda\\ &= \int \hat{g}_\Delta(\lambda)e^{i\lambda H}(1 - e^{-i\lambda H}e^{i\lambda H_0})d\lambda\\ &= i\int \hat{g}_\Delta(\lambda)e^{i\lambda H}\int_0^\lambda e^{-isH}We^{isH_0}\,ds\,d\lambda\,.\end{aligned}$$

We next apply this expansion to ψ_0, where $H_0\psi_0 = \lambda_0\psi_0$ and obtain

$$\begin{aligned}\big(g_\Delta(H) - g_\Delta(H_0)\big)\psi_0 &= i\int \hat{g}_\Delta(\lambda)e^{i\lambda H}\int_0^\lambda e^{-isH}We^{is\lambda_0}\psi_0 ds\,d\lambda\\ &= i\int \hat{g}_\Delta(\lambda)e^{i\lambda H}\int_0^\lambda e^{-isH+is\lambda_0}W\psi_0 ds\,d\lambda\\ &= i\int \hat{g}_\Delta(\lambda)e^{i\lambda H}\frac{e^{-i\lambda H+i\lambda\lambda_0} - 1}{-iH+i\lambda_0}W\psi_0 d\lambda\\ &= -\int \hat{g}_\Delta\frac{e^{i\lambda\lambda_0}}{H-\lambda_0}d\lambda\,W\psi_0 + \int \hat{g}_\Delta(\lambda)\frac{e^{i\lambda H}}{H-\lambda_0}W\psi_0 d\lambda\\ &= -g_\Delta(\lambda_0)\frac{1}{H-\lambda_0}W\psi_0 + g_\Delta(H)\frac{1}{H-\lambda_0}W\psi_0\\ &= -\big(1 - g_\Delta(H)\big)\frac{1}{H-\lambda_0}W\psi_0\,, \quad \big(g_\Delta(\lambda_0) = 1\big)\\ &\equiv -\bar{g}_\Delta(H)(H-\lambda_0)^{-1}W\psi_0\,.\end{aligned}$$

This completes the proof of the proposition.

Substitution of (9.5) into the above expression for β yields

$$\beta = -\big(W\psi_0, B\bar{g}_\Delta(H)(H-\lambda_0)^{-1}W\psi_0\big)\,. \tag{9.6}$$

Let $h(\lambda)$ be a function which is equal to one on the support of $\bar{g}_\Delta$ and is zero outside a small neighborhood of the support of $\bar{g}_\Delta$. Therefore, $(H_0-\lambda_0)^{-1}h(H_0)$ is bounded. A computation yields

PROPOSITION 9.2.

$$\begin{aligned}
\omega_A &= (\psi_0, W\psi_0) + \beta \\
&= (\psi_0, W\psi_0) - \big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\lambda_0)^{-1}\, W\psi_0\big) \qquad (9.7)\\
&\quad - \big(W\psi_0, \bar{g}_\Delta(H)(H-\lambda_0)^{-1}(h(H)-I)W\psi_0\big)\\
&\quad - \big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\lambda_0)^{-1}[h(H)-h(H_0)]W\psi_0\big)\\
&\quad + \big(W\psi_0, [g_\Delta(H)-g_\Delta(H_0)]h(H)(H-\lambda_0)^{-1}W\psi_0\big)\\
&\quad + \big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\lambda_0)^{-1}W(H-\lambda_0)^{-1}h(H)W\psi_0\big)\\
&\quad - \big(W\psi_0, Bg_\Delta(H)g_I(H_0)\bar{g}_\Delta(H)(H-\lambda_0)^{-1}W\psi_0\big)\,. \qquad (9.8)
\end{aligned}$$

Note also that the second term in (9.7) can be expressed as

$$\begin{aligned}
\big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\lambda_0)^{-1}W\psi_0\big) &= \big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\omega)^{-1}W\psi_0\big)\\
&\quad - (W\psi_0,\psi_0)\cdot\big(W\psi_0, \bar{g}_\Delta(H_0)(H_0-\lambda_0)^{-1}(H_0-\omega)^{-1}W\psi_0\big) \qquad (9.9)
\end{aligned}$$

Expansion of ω_B. Let $R_H(\lambda) \equiv (H-\lambda)^{-1}$. Recall that ω_B is given by the expression

$$\omega_B = \big(W\psi_0, \tilde{g}_\Delta(H)R_{H_0}(\omega+i0)P_c^{\#}W\tilde{g}_\Delta(H)\psi_0\big)\,,$$

and $\tilde{g}_\Delta(H) = Bg_\Delta(H)(I-P_1)$. We find after some computation

PROPOSITION 9.3.

$$\begin{aligned}
\omega_B &= \big(W\psi_0, g_\Delta(H_0)R_{H_0}(\omega+i0)P_c^{\#}W\psi_0\big)\\
&\quad + \big(W\psi_0, [B-I]g_\Delta(H_0)R_{H_0}(\omega+i0)P_c^{\#}WBg_\Delta(H)\psi_0\big)\\
&\quad + \big(W\psi_0, [g_\Delta(H)-g_\Delta(H_0)]R_{H_0}(\omega+i0)P_c^{\#}W\tilde{g}_\Delta(H)\psi_0\big)\\
&\quad + \big(W\psi_0, g_\Delta(H_0)R_{H_0}(\omega+i0)P_c^{\#}WB[g_\Delta(H)-g_\Delta(H_0)]\psi_0\big)\\
&\quad + \big(W\psi_0, (B-I)[g_\Delta(H)-g_\Delta(H_0)]\overline{g}_I(H_0)R_{H_0}(\omega+i0)P_c^{\#}WBg_\Delta(H)\psi_0\big)\,.
\end{aligned} \tag{9.10}$$

Here, we have used that $B\psi_0=\psi_0$. More generally, $(B-I)g_\Delta(H_0) = Bg_\Delta(H)g_I(H_0)g_\Delta(H_0) = 0$, and therefore the second term in (9.10) is zero.

It follows from (9.4), (9.9) and (9.10) that

$$\omega_* = \lambda_0 + \omega_A - \omega_B$$

$$\equiv \lambda_0 + (\psi_0, W\psi_0) - (\Lambda + i\Gamma) + \sum_{j=1}^{9} E_j\,, \tag{9.11}$$

where

$$\begin{aligned}\Lambda + i\Gamma = &\big(W\psi_0, \bar g_\Delta(H_0)(H_0-\omega)^{-1}W\psi_0\big)\\ &+ \big(W\psi_0, g_\Delta(H_0)(H_0-\omega-i0)^{-1}P_c^\# W\psi_0\big)\,.\end{aligned} \tag{9.12}$$

and

$$\begin{aligned}
E_1 &= (W\psi_0,\psi_0)\cdot\big(W\psi_0, \bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}(H_0-\omega)^{-1}W\psi_0\big)\\
E_2 &= \big(W\psi_0, (H-\lambda_0)^{-1}\big(h(H)-I\big)\bar g_\Delta(H)W\psi_0\big)\\
E_3 &= -\big(W\psi_0, \bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}[h(H)-h(H_0)]W\psi_0\big)\\
E_4 &= \big(W\psi_0, [g_\Delta(H)-g_\Delta(H_0)]h(H)(H-\lambda_0)^{-1}W\psi_0\big)\\
E_5 &= \big(W\psi_0, \bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}W(H-\lambda_0)^{-1}h(H)W\psi_0\big)\\
E_6 &= \big(W\psi_0, Bg_\Delta(H)g_I(H_0)[g_\Delta(H_0)\\
&\qquad - g_\Delta(H)]\bar g_I(H_0)R_{H_0}(\omega+i0)P_c^\# WBg_\Delta(H)\psi_0\big)\\
E_7 &= \big(W\psi_0, g_\Delta(H_0)R_{H_0}(\omega+i0)P_c^\# WB[g_\Delta(H_0)-g_\Delta(H)]\psi_0\big)\\
E_8 &= \big(W\psi_0, [g_\Delta(H_0)-g_\Delta(H)]R_{H_0}(\omega+i0)P_c^\# WBg_\Delta(H)\psi_0\big)\\
E_9 &= \big(W\psi_0, B[g_\Delta(H_0)-g_\Delta(H)]g_I(H_0)\bar g_\Delta(H)(H-\lambda_0)^{-1}W\psi_0\big)
\end{aligned}$$

We now claim that the terms E_j, $j=1,\dots,9$, are all of order $|||W|||^3$. Consider first $E_1 = E_1^a\cdot E_1^b$. Estimation of the first factor gives

$$|E_1^a| \le C|||W|||\,, \tag{9.13}$$

by Proposition 4.1.

Estimation of the second factor gives

$$\begin{aligned}
|E_1^b| &= \big|(Wg_\Delta(H_0)\psi_0, \bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}(H_0-\omega)^{-1}Wg_\Delta(H_0)\psi_0)\big|\\
&\le \big\|\langle x\rangle^\sigma Wg_\Delta(H_0)\psi_0\big\|_2^2\big\|\langle x\rangle^{-\sigma}\bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}(H_0-\omega)^{-1}\langle x\rangle^{-\sigma}\big\|\\
&\le C|||W|||^2\big\|\langle x\rangle^{-\sigma}\bar g_\Delta(H_0)(H_0-\lambda_0)^{-1}(H_0-\omega)^{-1}\langle x\rangle^{-\sigma}\big\| \le C|||W|||^2\,,
\end{aligned}$$

by Theorem 11.1. Therefore, $|E_1|\le C|||W|||^3$.

The term E_2 is zero; $(h-1)\bar g_\Delta \equiv 0$ since $h\equiv 1$ on the support of $\bar g_\Delta$.

The term E_5 can be treated by the same type of estimates as E_1. The remaining terms are E_j, $j=2,3,4,6,7,8,9$. Each of these expressions has two explicit occurrences of the perturbation, W, as well as a difference of operators: $g_\Delta(H)-g_\Delta(H_0)$ or $h(H)-h(H_0)$. By (8.12), these differences are $\mathcal{O}(|||W|||)$, so we expect each of these terms to be $\mathcal{O}(|||W|||^3)$. We carry this argument out for the term E_7. The other terms are similarly estimated.

Consider E_7. Let $\tilde{\Delta}$ be an interval properly containing Δ so restricted to the interval Δ, $g_{\tilde{\Delta}} \equiv 1$ and $g_\Delta = g_\Delta g_{\tilde{\Delta}}$. Then,

$$\begin{aligned}|E_7| &= \big|(\langle x\rangle^\sigma g_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma} g_{\tilde{\Delta}}(H)R_0(\omega+i0)P_c^{\#}\langle x\rangle^{-\sigma} \\ &\qquad \cdot \langle x\rangle^\sigma WB[g_\Delta(H_0)-g_\Delta(H)]\psi_0)\big| \\ &\le \big|(\langle x\rangle^\sigma g_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma} g_{\tilde{\Delta}}(H_0)R_0(\omega+i0)P_c^{\#}\langle x\rangle^{-\sigma} \\ &\qquad \cdot \langle x\rangle^\sigma WB[g_\Delta(H_0)-g_\Delta(H)]\psi_0)\big| \\ &+ \big|(\langle x\rangle^\sigma g_\Delta(H)W\psi_0, \langle x\rangle^{-\sigma}[g_{\tilde{\Delta}}(H)-g_{\tilde{\Delta}}(H_0)]\langle x\rangle^\sigma\langle x\rangle^{-\sigma}R_0(\omega+i0)P_c^{\#}\langle x\rangle^{-\sigma} \\ &\qquad \cdot \langle x\rangle^\sigma WB[g_\Delta(H_0)-g_\Delta(H)]\psi_0)\big|. \end{aligned} \tag{9.14}$$

Using Proposition 4.1, Theorem 11.1 and (8.12) we have that

$$|E_7| \le C|||W|||^3 \big\|\langle x\rangle^{-\sigma}(H_0-\omega-i0)^{-1}P_c^{\#}\langle x\rangle^{-\sigma}\big\|. \tag{9.15}$$

That the term $\|\langle x\rangle^{-\sigma}(H_0-\omega-i0)^{-1}\langle x\rangle^{-\sigma}\|$ is finite is a consequence of Proposition 2.1 with $t=0$. Thus we have the following proposition from which Proposition 3.4 follows.

Proposition 9.4.

$$\begin{aligned}&(1)\quad \Lambda + i\Gamma = \big(W\psi_0, \text{P.V.}(H_0-\omega)^{-1}W\psi_0\big) \\ &\qquad\qquad + i\pi\big(W\psi_0, \delta(H_0-\omega)(I-P_0)W\psi_0\big)\,, \\ &(2)\quad |E_j| \le C|||W|||^3, \qquad 1\le j\le 9\,, \\ &(3)\quad \omega_* = \lambda_0 + (\psi_0, W\psi_0) - \Lambda - i\Gamma + E(W)\,,\ \text{where } |E(W)| \le C|||W|||^3. \end{aligned} \tag{9.16}$$

It remains to verify part (1). This follows from an application of the well-known distributional identity

$$(x \mp i0)^{-1} \equiv \lim_{\varepsilon\to0^+}(x\mp i\varepsilon)^{-1} = \text{P.V. } x^{-1} \ \pm\ i\pi\delta(x) \tag{9.17}$$

to the second term in equation (9.12) and the identity $g_\Delta(H_0)P_c^{\#} = I - P_0$.

10 Appendix D: General Approach to Local Decay Estimates

Hypothesis (H4) for our main theorem is one requiring that our unperturbed operator, H_0, satisfy a suitable local decay estimate, (2.3). In this section we give an outline to a very general approach to obtaining such estimates based on a technique originating in the work of Mourre [Mou]; see also [PeSiSim]. In the following general discussion we shall let H denote self-adjoint operator on a Hilbert space, $\mathcal{H}$, keeping in mind that our application is to the unperturbed operator H_0. Let $E \in \sigma(H)$, and assume that an operator A can be found such that A is self-adjoint and $\mathcal{D}(A) \cap \mathcal{H}$ is dense in $\mathcal{H}$. Let Δ denote an open interval with compact closure. We shall use the notation

$$\mathrm{ad}_A^n(H) = \left[\cdots [H, A], A, \cdots A \right], \tag{10.1}$$

for the n-fold commutator.

Assume the two conditions

(M1) The operators

$$g_\Delta(H)\mathrm{ad}_A^n(H)g_\Delta(H), \quad 1 \le n \le N \tag{10.2}$$

can all be extended to a bounded operator on $\mathcal{H}$.

(M2) *Mourre estimate*:

$$g_\Delta(H)i[H, A]g_\Delta(H) \ge \theta g_\Delta(H)^2 + K \tag{10.3}$$

for some $\theta > 0$ and compact operator, K.

Theorem 10.1 (Mourre; see [CyFKS, Theorem 4.9]). *Assume conditions (M1)-(M2), with $N = 2$. Then, in the interval Δ, H can only have absolutely continuous spectrum with finitely many eigenvalues of finite multiplicity. Moreover, the operator*

$$\langle A\rangle^{-1} g_\Delta(H)(H - z)^{-1}\langle A\rangle^{-1} \tag{10.4}$$

is uniformly bounded in z, as an operator on $\mathcal{H}$. If $K = 0$, then there are no eigenvalues in the interval Δ.

Theorem 10.2 (Sigal–Soffer; see [SiSo],[GeSi], [HuSi]). *Assume conditions (M1)-(M2) with $N \ge 2$ and $K = 0$. Then, for all $\varepsilon > 0$*

$$\left\| F\left(\tfrac{|A|}{t} < \theta\right) e^{-iHt} g_\Delta(H)\psi \right\|_2 \le C\langle t\rangle^{-\frac{N}{2}+\varepsilon} \left\| |A|^{\frac{N}{2}}\psi \right\|_2, \tag{10.5}$$

and therefore

$$\left\| \langle A\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\|_2 \le C\langle t\rangle^{-\sigma} \left\| |A|^{N/2}\psi \right\|_2, \tag{10.6}$$

for $\sigma < N/2$. Here, F is a smoothed out characteristic function, and $F(|A|/t < \theta)$ is defined by the spectral theorem.

Let Δ_1 denote an open interval containing the closure of Δ.

Corollary 10.1. *Assume that $\langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma}$ is bounded. Then, in the above theorems we can replace the weight $\langle A\rangle^{-\sigma}$ by $\langle x\rangle^{-\sigma}$.*

The strategy for using the above results to prove local decay estimates like that in (H4) is as follows. Then

$$\begin{aligned}\left\| \langle x\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\|_2 &= \left\| \langle x\rangle^{-\sigma} g_{\Delta_1}(H) e^{-iHt} g_\Delta(H)\psi \right\|_2 \\ &= \left\| \langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma} \cdot \langle A\rangle^{-\sigma} e^{-iHt} g_\Delta(H)\psi \right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\langle x\rangle^{-\sigma} g_{\Delta_1}(H)\langle A\rangle^{\sigma}\right\| \cdot \left\|\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \\
&\leq C\left\|\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\| \\
&\leq C_1 \left\|F\left(\tfrac{|A|}{t} < \theta\right)\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\|_2 \\
&\quad + C_2 \left\|F\left(\tfrac{|A|}{t} \geq \theta\right)\langle A\rangle^{-\sigma} e^{-iHt} g_{\Delta}(H)\psi\right\|_2 . \qquad (10.7)
\end{aligned}
$$

Theorem 10.2 is used to obtain the decay of the first term on the right-hand side of (10.7), while we can replace $|A|$ by θt in the second term.

REMARK. Here we return to our comment in the introduction on the relation between our assumption (H4) (local decay for e^{-iH_0t}) and the hypothesis of dilation analyticity, used in previous works. Dilation analyticity or its generalization, analytic deformation, is the requirement that the map,

$$d(\theta) : \theta \mapsto (\, e^{i\theta A} H_0 e^{-i\theta A} f, f)\,, \qquad (10.8)$$

has analytic continuation to a strip, for f in a dense subset of $\mathcal{H}$. Since the n^{th} derivative of $d(\theta)$ at $\theta = 0$ is $(\mathrm{ad}_A^n(H_0)f, f)$, by the above local decay result, the assumption (H4) is the requirement that the mapping, $d(\theta)$ be of class C^3.

11 Appendix E: Weighted Norm Estimates for Functions of Operators

In Appendices A, B and C we frequently require facts and estimates of functions of a self-adjoint operator. In this section we give some basic definitions and provide the statements and proofs of such estimates. We shall refer to certain known results and our basic references are [RSim] and [AmMoG].

Let A denote a self-adjoint operator with domain $\mathcal{D}$ which is dense in a Hilbert space $\mathcal{H}$. Then we have that for any bounded continuous complex-valued function, $\varphi \in L^1(\mathbb{R})$,

$$\varphi(A) = \text{weak} - \lim_{\varepsilon \downarrow 0} \pi^{-1} \int \varphi(\lambda) \Im R_A(\lambda + i\varepsilon)^{-1} d\lambda\,, \qquad (11.1)$$

where $R_A(\lambda) = (A - \lambda)^{-1}$ denotes the resolvent of A. Here and throughout this section all regions of integration are assumed to be over $\mathbb{R}$ unless explicitly stated otherwise.

Theorem 11.1. *Let $\tilde{A}$ and $\tilde{B}$ denote bounded self-adjoint operators, and let Γ be a contour in the complex plane, not passing through the origin, surrounding $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$ and lying in the strip $|\Im\zeta| < 1$.*

(a) *Let* $\psi : \mathbb{R} \to \mathbb{C}$ *be a* $W^{2,1}$ *function. Suppose*

$$\eta_{\tilde{A}} \equiv \left\| \langle x\rangle^\sigma \tilde{A} \langle x\rangle^{-\sigma} \right\| < \tfrac{1}{2} \min \left[\mathrm{distance}(\Gamma, 0), 1\right] . \tag{11.2}$$

Then, there exists a positive number $C_1 = C_1(\|\psi\|_{W^{2,1}}, \eta_{\tilde{A}})$ *such that*

$$\left\| \langle x\rangle^\sigma \psi(\tilde{A})\ \langle x\rangle^{-\sigma} \right\| < C_1 . \tag{11.3}$$

(b) *Let* ψ *be as in part* (a). *Assume that* $\tilde{A}$ *and* $\tilde{B}$ *both satisfy condition* (11.2). *Then, there is a constant* $C_2 = C_2(\|\psi\|_{W^{2,1}}, \eta_{\tilde{A}}, \eta_{\tilde{B}})$ *such that*

$$\left\| \langle x\rangle^\sigma [\psi(\tilde{A}) - \psi(\tilde{B})] \langle x\rangle^{-\sigma} \right\| \le C_2 \left\| \langle x\rangle^\sigma (\tilde{A} - \tilde{B}) \langle x\rangle^{-\sigma} \right\| . \tag{11.4}$$

The following result shows that the case of unbounded self-adjoint operators is reducible to Theorem 11.1.

Theorem 11.2. *Suppose that Theorem 11.1 holds, and let* Γ *and* φ *be as in Theorem 11.1. Furthermore, assume that* $x^2\varphi''(x)$ *and* $x\varphi'(x)$ *are* L^1 *functions. Let* A *and* B *be densely defined self-adjoint operators for which* $(A+c)^{-1}$ *and* $(B+c)^{-1}$ *are bounded for some real number* c *and satisfy the estimate* (11.2). *Then,*

$$\left\| \langle x\rangle^\sigma \varphi(A) \langle x\rangle^{-\sigma} \right\| < C_1(\psi) . \tag{11.5}$$

and

$$\left\| \langle x\rangle^\sigma [\varphi(A) - \varphi(B)] \langle x\rangle^{-\sigma} \right\| \le C_2(\psi) \left\| \langle x\rangle^\sigma (A - B) \langle x\rangle^{-\sigma} \right\| , \tag{11.6}$$

where the constants $C_1(\psi)$ *and* $C_2(\psi)$ *are as in Theorem 11.1, with* $\psi(x) \equiv \varphi(x^{-1} - c)$.

Proof. Let $\tilde{A} = (A+c)^{-1}$ and note that $\varphi(A) = \varphi(\tilde{A}^{-1} - c) = \psi(\tilde{A})$. It suffices to show that $\psi(x) = \varphi(x^{-1} - c)$ satisfies the hypotheses of Theorem 11.1. It is simple to check that $\partial_x^j \psi(x) \in L^1$, for $j = 0, 1, 2$. This proves Theorem 11.2.

We now embark on the proof of Theorem 11.1. A key tool is an expansion formula for $\varphi(A)$; see Proposition 6.1.4 on page 239 of [AmMoG].

Theorem 11.3. *Let* A *be a densely defined self-adjoint operator and* φ *be as in the statement of Theorem 11.1. Then,*

$$\begin{aligned}\varphi(A) = \tfrac{1}{\pi} \int \varphi(\lambda) \Im R_A(\lambda + i) d\lambda + \tfrac{1}{\pi} \int \varphi'(\lambda) \Im i R_A(\lambda + i) d\lambda \\ + \tfrac{1}{\pi} \int_0^1 \tau\, d\tau \int \varphi''(\lambda) \Im i^2 R_A(\lambda + i\tau) d\lambda \\ \equiv \varphi_1 + \varphi_2 + \varphi_3 .\end{aligned} \tag{11.7}$$

where all integrals exist in the norm of the space of bounded operators on $\mathcal{H}$.

To prove Theorem 11.1 we first obtain a simple expression for the third summand in (11.7) by interchanging order of integration. We begin with a calculation of the τ-integral

$$\begin{aligned}\int_0^1 \tau\, d\tau \Im\big(i^2\ R_A(\lambda+i\tau)\big) &= -\int_0^1 \tau \tfrac{1}{2i}\big(R_A(\lambda+i\tau)-R_A(\lambda-i\tau)\big)\\ &= -\int_0^1 d\tau\, \tau^2\big[(A-\lambda)^2+\tau^2\big]^{-1}\\ &= f(A;\lambda)-1\,,\end{aligned}$$

where

$$f(z;\lambda)=(z-\lambda)\int_0^{\frac{1}{z-\lambda}}(1+\mu^2)^{-1}d\mu\,. \tag{11.8}$$

For each λ in the support of φ, the function $f(z;\lambda)$ is analytic in the strip $|\Im z|<1$; this corresponds to choice an appropriate branch of the function $z\mapsto (z-\lambda)\arctan(z-\lambda)^{-1}$. By (11.7)

$$\varphi_3(A)=\tfrac{1}{\pi}\int \varphi''(\lambda)f(A;\lambda)d\lambda\,. \tag{11.9}$$

The strategy is as follows:

First, we observe that $\langle x\rangle^\sigma\varphi_j(\tilde A)\langle x\rangle^{-\sigma}$ is bounded for $j=1,2$. This is true because $\varphi,\varphi'\in L^1$ and (11.2) can be used to bound the weighted norm of the resolvent by a convergent geometric series. Therefore, it remains to bound the operator $\varphi_3(\tilde A)$, where φ_3 is given explicitly (11.9).

LEMMA 11.1. *Let $\tilde A$ and $\tilde B$ denote bounded self-adjoint operators and $f(\zeta)$ be a function which is defined and analytic in a neighborhood of $\sigma(\tilde A)\cup\sigma(\tilde B)$. Let Γ be a smooth contour in the domain of analyticity of f, surrounding $\sigma(\tilde A)\cup\sigma(\tilde B)$, not passing through the origin and such that the estimate,*

$$\big\|\langle x\rangle^\sigma M\langle x\rangle^{-\sigma}\big\|\le \tfrac12\min_{\zeta\in\Gamma}|\zeta|\,, \tag{11.10}$$

holds with $M=\tilde A$ and $M=\tilde B$. Then, there exist positive constants C_1 and C_2 such that

$$\big\|\langle x\rangle^\sigma f(\tilde A)\langle x\rangle^{-\sigma}\big\|\le C_1 \tag{11.11}$$

$$\big\|\langle x\rangle^\sigma[f(\tilde A)-f(\tilde B)]\langle x\rangle^{-\sigma}\big\|\le C_2\big\|\langle x\rangle^\sigma(\tilde A-\tilde B)\langle x\rangle^{-\sigma}\big\| \tag{11.12}$$

Proof. By the Cauchy integral formula we have

$$f(\tilde A)=(2\pi i)^{-1}\int_\Gamma f(\zeta)(\tilde A-\zeta I)^{-1}d\zeta\,. \tag{11.13}$$

Part (a) follows by use of (11.10) to expand the resolvent in a geometric series and by termwise estimation in the weighted norm.

Part (b) follows by the same method; by (11.13) applied to $\tilde{B}$ and computation of the difference, we get

$$\begin{aligned} f(\tilde{A}) - f(\tilde{B}) &= (2\pi i)^{-1} \int_\Gamma f(\zeta)\big[(\tilde{A} - \zeta I)^{-1} - (\tilde{B} - \zeta I)^{-1}\big]\, d\zeta \\ &= (2\pi i)^{-1} \int_\Gamma f(\zeta)\big[(\tilde{A} - \zeta I)^{-1}(\tilde{A} - \tilde{B})(\tilde{B} - \zeta I)^{-1}\big]\, d\zeta\,. \end{aligned}$$

Estimation in the weighted space yields (11.12). This completes the proof of the lemma.

To complete the proofs of Theorems 11.1 and Theorem 11.2, we need to estimate the operator $\langle x\rangle^\sigma \varphi_3(\tilde{A})\langle x\rangle^{-\sigma}$, where $\tilde{A}$ is the bounded self-adjoint operator defined by $\tilde{A} = (A+c)^{-1}$. We accomplish this by applying the previous lemma to the function $f(\zeta;\lambda)$ defined in (11.8), where λ is in the support of φ. The function $f(\zeta;\lambda)$ is analytic in the strip $|\Im\zeta| < 1$, and Γ is, by hypothesis, a contour in its domain of analyticity, surrounding $\sigma(\tilde{A})$ (respectively, $\sigma(\tilde{A}) \cup \sigma(\tilde{B})$,) and so that (11.10) holds. Then, by Lemma 11.1 we have that $f(\tilde{A};\lambda)$ and $f(\tilde{B};\lambda)$ satisfy (11.11) and (11.12). Finally, using the representation formula for φ_3, (11.9), we have

$$\begin{gathered} \big\|\langle x\rangle^\sigma \varphi_3(\tilde{A})\langle x\rangle^{-\sigma}\big\| \le C_1\|\varphi''\|_{L^1}\,, \\ \big\|\langle x\rangle^\sigma [\varphi_3(\tilde{A}) - \varphi_3(\tilde{B})]\langle x\rangle^{-\sigma}\big\| \le C_2\|\varphi''\|_{L^1}\big\|\langle x\rangle^\sigma[\tilde{A} - \tilde{B}]\langle x\rangle^{-\sigma}\big\|\,. \end{gathered} \tag{11.14}$$

This completes the proof.

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