Ionization and Scattering for Short Lived Potentials

A. Soffer * and M.I. Weinstein †

Dedicated to the Memory of Moshe Flato

Abstract

We consider perturbations of a model quantum system consisting of a single bound state and continuum radiation modes. In many problems involving the interaction of matter and radiation one is interested in the effect of time dependent perturbations. A time dependent perturbation will couple the bound and continuum modes causing "radiative transitions". Using techniques of time dependent resonance theory, developed in earlier work on resonances in linear and nonlinear Hamiltonian dispersive systems, we develop the scattering theory of short lived ($\mathcal{O}(t^{-1-\varepsilon})$) spatially localized perturbations. For weak pertubations, we compute (to second order) the ionization probability, the probability of transition from the bound state to the continuum states. These results can also be interpreted as a calculation, in the paraxial approximation, of the energy loss resulting from wave propagation in a waveguide in the presence of a localized defect.

1. Introduction

The interactions of matter with electromagnetic fields play a major role in many fields of physics, chemistry and engineering [2]. The study of laser interactions with atoms and electron microscopes, as well as the sudden approximation in Quantum Mechanics, lead us to consideration of the time-dependent Schrödinger equation with a perturbing potential which is short lived, that is, localized in time. A typical model of this type is: In the special case of

$$i\partial_t\phi(t,\mathbf{x}) \;=\; \left(-\Delta_{\mathbf{x}} + V(\mathbf{x})\right)\phi(t,\mathbf{x}) \;+\; \beta(t,\mathbf{x})\phi(t,\mathbf{x}), \tag{1.1}$$

where $V(\mathbf{x})$ is localized in $\mathbf{x} \in \mathbb{R}^d$ and β is localized in $\mathbf{x}$ and $t \in \mathbb{R}$.

Another important area of application is in the propagation of waves in optical waveguides (*e.g.* optical fibers) in the *paraxial approximation.* Let $\mathbf{x}$ denote the coordinates

*Department of Mathematics, Rutgers University, New Brunswick, NJ

†Mathematical Sciences Research, Bell Laboratories - Lucent Technologies, and Department of Mathematics, University of Michigan, Ann Arbor, MI

which are transverse to the guiding structure and let z denote the coordinate along the direction of propagation in the wave guide. Then, $\phi = \phi(z, \mathbf{x})$, the slowly varying envelope of the electric field, solves an equation of the form (1.1) where z plays the role of the time variable. The potential $V(\mathbf{x})$ models transverse variations in the refractive index and may have localized modes associated with it. The perturbation, $\beta(z, \mathbf{x})$ models perturbations in the refractive index which vary along the wave-guide. A "short-lived" (in z) potential corresponds to a local "defect" or local perturbation in the refractive index [7].

We consider a general class of problems including (1.1) in which the unperturbed Hamiltonian, H_0, has one bound state ψ_0, $\|\psi_0\| = 1$. The perturbation, $\beta(t)$, has the effect of coupling the bound state to continuum (dispersive) modes and leads to energy transfer. Our aim in this work is the analysis of the large time behaviour of such systems where $\beta(t)$ is symmetric *short lived* and localized; see (2.4). The quantities $P(t) \equiv |\langle \phi(t), \psi_0 \rangle|^2$ and $1 - P(t)$ can be interpreted, respectively, as the probability that the system is in the state ψ_0 and the ionization probability. In the context of optical waveguides, $P(t)$ is a measure of the amount of energy trapped by the defect and $1 - P(t)$ the amount of energy radiated away.

Our results are stated precisely in Theorems 2.1 and 2.2. Theorem 2.1 states that any solution of the initial value problem converges as $t \to \pm\infty$ to an asymptotic bound state $e^{-i\lambda_0 t} A(\pm\infty)\psi_0$ plus a purely dispersive part. The perturbation is not assumed to be small in Theorem 2.1. Theorem 2.2 gives a more detailed description of the solution as $t \to \infty$ and in particular gives, in the case of a small perturbation $\beta(t)$, a useful expression for the the asymptotic bound state component. Suppose the perturbed dynamical system is initialized in the state ψ_0. Theorem 2.2 implies

$$P(\infty) \; = \; e^{-2\Gamma} \; \left(1 \; + \; \mathcal{O}(\; \|h_\beta\|_{L^1}^2 \;) \right) \tag{1.2}$$

where $\Gamma > 0$ and $\|h_\beta\|_{L^1}$ is a time-integrated measure of the strength of the perturbation; see (2.4). In the special case of the Schrödinger equation (1.1)

$$\|h_\beta\|_{L^1} \; = \; \int_{\mathbb{R}} dt \; \left(\int_{\mathbb{R}^d} \; |\langle \; \mathbf{x} \rangle^\sigma \beta(t, \mathbf{x}) \langle \; \mathbf{x} \rangle^\sigma \; |^2 \; d\mathbf{x} \right)^{\frac{1}{2}}, \; \sigma > 0. \tag{1.3}$$

This gives an explicit and useful expression for the ionization probability for weak perturbations. Note that it's validity does not rely on pointwise smallness of $\beta(t)$. The quantity Γ is given by the formula:

$$\begin{aligned} \Gamma \; &= \; \|P_c \hat{\beta}(\lambda_0 - H_0)\psi_0\|^2 \\ &= \; \int \gamma(\mu) \; d\mu, \text{ where} \end{aligned} \tag{1.4}$$

$$\gamma(\mu) \; = \; \left| \; \mathcal{F}_{H_0}[\hat{\beta}(\mu)\psi_0](\lambda_0 - \mu) \; \right|^2. \tag{1.5}$$

where $\mathcal{F}_{H_0}$ denotes the (generalized) Fourier transform with respect the continuous spectral part of the operator H_0; see also (2.17). The integration in (1.4) is over the support of $\hat{\beta}(\mu)$. This formula is an analogue of the *Fermi golden rule* [1] appropriate for perturbations which are localized in t.

[1] See [10], [11] and references cited therein.

A T-periodic perturbation may be thought of as a sequence of $t-$ localized defects. After encountering N defects we have $P(NT) \sim \exp(-2N\Gamma_{\text{one defect}})$. Thus roughly, neglecting dispersive component of the solution, we have exponential decay as $|t|$ increases. The results of [11], [6] (see also the exactly solvable model considered in [9], [3]) show that this is valid for large but finite t. For example, in [11] it is proved for $\beta(t,x) = \cos(\mu_0 t)\beta(x)$ that $P(t) \sim \exp(-2\gamma(-\mu_0)|t|)$, for $|t| \sim \mathcal{O}(\beta^{-2})$. For a large class of perturbations with discrete time-frequency content (periodic, almost periodic in t), the very large time behavior ($t >> \mathcal{O}(\beta^{-2})$) is characterized by algebraic decay to zero with increasing t [11], [6].

The methods of this paper are a natural extension of the techniques of time dependent resonance theory developed in [10],[11],[12] to the case of time-localized perturbations. These methods can be extended to the case where H_0 has multiple bound states [6].

2. Preliminaries and results

We assume that the system under consideration is given by the following Hamiltonian

$$H(t) \equiv H_0 + \beta(t), \tag{2.1}$$

and satisfies the Schrödinger equation

$$i\frac{\partial \phi}{\partial t} = H(t)\phi. \tag{2.2}$$

We assume that H_0 is a self adjoint operator densely defined on a Hilbert space $\mathcal{H}$ and having exactly one bound state, with eigenvalue λ_0, and normalized eigenfunction ψ_0:

$$H_0\psi_0 \;=\; \lambda_0\psi_0,\ \ \|\psi_0\| = 1. \tag{2.3}$$

The projection $P_c = I - \langle\cdot,\psi_0\rangle\psi_0$ is onto the continuous part of H_0, We define "weights" w_+ and w_- satisfying:

$$\begin{array}{ll} (i) & w_+ \geq cI, c > 0, \\ (ii) & w_- \in \mathcal{L}(\mathcal{H}) \\ (iii) & w_-w_+\, P_c \;=\; P_c \;=\; P_c\, w_-w_+. \end{array}$$

We also assume $w_+\psi_0 \in \mathcal{H}$. This is the setting used in [11].

Example 2.1. *The example to keep in mind is $\mathcal{H} = L^2(\mathbb{R}^n)$, $H_0 = -\Delta + V(x)$, with $V(x)$ sufficiently regular and decaying rapidly as $|x| \to \infty$. Typical weights chosen in this case are: $w_\pm = \langle x\rangle^{\pm\sigma}$, where $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$ and σ is sufficiently large and positive. In this case, the estimates (2.6-2.7) hold; see, for example, [10].*

$\beta(t)$ is assumed to be a symmetric and bounded operator, such that

$$h_\beta(t) \;\equiv\; \|w_+\beta(t)w_+\|_{B(\mathcal{H})} \;\in\; L^1(\mathbb{R}; dt). \tag{2.4}$$

For simplicity we'll assume that $\beta(t)$ is supported for $t > 0$ and extend it as an even operator-valued function on $\mathbb{R}$. In the above example, one often has in applications a perturbation of the form $\beta(t,x) \equiv \sum_j \beta_j(t)V_j(x)$.

Remark and Warning: Introduce the Fourier transform of the operator β defined for any $v \in \mathcal{H}$:

$$\beta(t)\ v = \int_{\mathbb{R}} e^{i\mu t}\hat{\beta}(\mu)\ v\ d\mu,\ \hat{\beta}(\mu)v\ =\ (2\pi)^{-1}\int_{\mathbb{R}} e^{-i\mu t}\ \beta(t)\ v\ dt, \tag{2.5}$$

Using the spectral theorem one can then define $\hat{\beta}(A)$, where A is a self-adjoint operator; we shall use this below with $A = \lambda_0 - H_0$. In applications, as in the example above, one often has $\beta(t) = \beta(t,x)$ and therefore $\hat{\beta}(A) = \hat{\beta}(A,x)$ may be, for example, a pseudo-differential operator. In our general setting we shall simply write $\hat{\beta}(A)$ with the understanding that this is **not** simply a function of the operator A. Thus, in general $\hat{\beta}(H_0)\psi_0 \neq \hat{\beta}(\lambda_0)\psi_0$.

We also assume that H_0 satisfies the following:

Local energy decay: For some $r > 1$

$$\|w_- e^{-iH_0t}P_c(H_0)f\|_2 \leq c\langle t\rangle^{-r}\|w_+f\|_2. \tag{2.6}$$

Theorem 2.1 and most of Theorem 2.2 rely mainly on the estimate (2.6) alone. The assertion in Theorem 2.2 concerning the rate of approach of $P(t)$ to its asymptotic value requires an additional technical assumption. In particular, we shall use the following estimate which holds in the situation where f may not lie in the domain of w_+:
Weak local energy decay:

$$\int_{\mathbb{R}} \|w_- e^{-iH_0t}\ P_c(H_0)f\|_2^2\ dt\ \leq C\ \|f\|_2^2. \tag{2.7}$$

Remark 2.1. *The estimate (2.7) is applied with $f \equiv \hat{\beta}(\lambda_0 - H_0)\psi_0$. Consider the setting of Example 2.1, $H_0 = -\Delta + V(x)$. Let $\sigma = 2 + \theta$, $\theta > 0$, $|V(x)|\ \leq\ C\langle x\rangle^{-\sigma}$ and assume that zero is neither an eigenvalue nor a resonance of H_0. Then, (2.7) is a consequence of results in [1].*

Remark 2.2. *Improvements in decay, which are of an intermediate character between (2.7) and (2.6) can be obtained from appropriate assumptions on commutators of weights with $\hat{\beta}(\lambda_0 - H_0)$. For Example 2.1 by assuming that the commutators:*

$$\left[\langle x\rangle, \cdots \left[\langle x\rangle, \hat{\beta}(\lambda_0 - H_0)\right]\right],\ (j \leq k \text{ times}) \tag{2.8}$$

are bounded operators on $\mathcal{H}$, we can obtain improved decay:

$$\|\langle x\rangle^{-\sigma}e^{-iH_0t}\ P_c(H_0)f\|_2 \leq c\ \langle t\rangle^{-\min\{k,\frac{n}{2}\}}\|\langle x\rangle^k f\|_2. \tag{2.9}$$

The above boundedness assumption on commutators amounts to spatial localization of $\beta(t,x)$ and smoothness of $\hat{\beta}(\mu,x)$ in μ. These are ensured by the assumption:

$$\int_{\mathbb{R}} \langle t\rangle^k\ \|\langle x\rangle^\sigma\ \beta(t,\cdot)\|_{L^2}\ dt\ <\ \infty \tag{2.10}$$

Remark 2.3. *For the choice of r appearing in estimates (2.6), the integral*

$$I_r[h_\beta](t) \ \equiv \ \int_0^t \ \langle t-s\rangle^{-r} \ h_\beta(s) \ ds, \tag{2.11}$$

arises. Since $r > 1$, by Young's inequality

$$\begin{aligned} \|I_r[h_\beta]\|_{L^1} &\leq c\|h_\beta\|_{L^1}, \text{ and furthermore} \\ \|I_r[h_\beta]\|_{L^\infty} &\leq \|h_\beta\|_{L^1}. \end{aligned}$$

Under the above conditions it is known that $\phi(t)$ exists for all times t, for any given $\phi(x,0) \in \mathcal{D}(H_0)$, the domain of H_0.

Furthermore, since β is symmetric, we have conservation of energy:

$$\|\phi(t)\| = \|\phi(0)\|. \tag{2.12}$$

We will prove the following

Theorem 2.1. *Consider the initial value problem for (2.2) with data $\phi(0) \in D(\mathcal{H})$. The large time behavior of $\phi(t)$ is a linear combination of free wave and the bound state ψ_0 (orthogonal to $\phi(t)$ for all t). In particular, there exists a complex number $A(\infty)$ and a vector $\phi_\pm \in \mathcal{H}$ such that:*

$$\phi(t) \ - \ \left[e^{-i\lambda_0 t} A(\infty)\psi_0 \ + \ e^{-iH_0 t} P_c(H_0)\phi_+\right] \tag{2.13}$$

tends to zero in $\mathcal{H}$ as $t \to \pm\infty$.

The following result gives a more detailed picture of the asymptotic behavior and provides a useful expression for the small β (in $\|h_\beta\|_{L^1}$) behavior.

Theorem 2.2. *For $t \geq 0$, and $\|h_\beta\|_{L^1} \leq 1$ we have:*

$$\begin{aligned} \phi(t) \ &= \ e^{-i(\lambda_0 t + \int_0^\infty \ \langle \psi_0, \beta(s)\psi_0\rangle \ ds + \Lambda)} \ e^{-\Gamma} \\ &\quad \times \ \left(a_\infty \ + \ \mathcal{O}(\int_t^\infty \ h_\beta(s) \ ds)\right) \psi_0 \\ &\quad + \ \phi_d(t), \end{aligned} \tag{2.14}$$

where

$$a_\infty \ = \ a(0) \ + \ \mathcal{O}\left(\ \|\phi_d(0)\| \ \|h_\beta\|_{L^1} \ + \ \|\phi(0)\| \ \|h_\beta\|_{L^1}^2 \ \right) \tag{2.15}$$

and $\phi_d(t)$ is dispersive and satisfies the decay estimate:

$$\|w_-\phi_d(t)\| \leq c_1 \ \|w_+\phi_d(0)\| \ \langle t\rangle^{-r} \ + \ c_2 \ \|\phi(0)\| \ I_r[h_\beta](t). \tag{2.16}$$

Γ and Λ are given by:

$$\begin{aligned} \Gamma \ &= \ \pi^2 \int_{\mathbb{R}} d\mu \ \langle \ P_c\hat{\beta}(\mu)\psi_0, \delta(H_0 - (\lambda_0 - \mu)) \ P_c\hat{\beta}(\mu)\psi_0 \ \rangle \qquad (2.17) \\ &= \ \|P_c\hat{\beta}(\lambda_0 - H_0)\psi_0\|^2 \\ \Lambda \ &= \ \pi \ \text{P.V.} \int_{\mathbb{R}} d\mu \ \langle \ P_c\hat{\beta}(\mu)\psi_0, (H_0 - (\lambda_0 - \mu))^{-1} \ P_c\hat{\beta}(\mu)\psi_0 \ \rangle \qquad (2.18) \end{aligned}$$

Finally, for $P(t)$ (see the introduction) we have that there exists $p_\infty \geq 0$ such that

$$P(t) \ = \ e^{-2\Gamma} p_\infty \ + \ \mathcal{O}\left(\int_t^\infty \ h_\beta(s) \ \alpha_\beta(s) \ ds \ \right), \tag{2.19}$$

where

$$p_\infty \ = \ 1 \ + \ \mathcal{O}\left(\|h_\beta\|_{L^1}^2 \right) \ \text{and} \ \ \alpha_\beta(t) \ \in L^2(\mathbb{R}; dt). \tag{2.20}$$

3. Decomposition and proof of Theorem 2.1

Our first aim is to derive a system of equations, equivalent to (2.1), for the coupled evolution of the bound state and dispersive channels [10], [11], [12]. A natural basis to use is that of the unperturbed dynamics.

Let

$$\phi(t) \equiv a(t)\psi_0 + \phi_d(t), \ \langle \psi_0, \phi_d(t) \rangle = 0 \tag{3.1}$$

with $H_0\psi_0 = \lambda_0\psi_0$.

Then, from (2.1) we get

$$i\partial_t a(t)\psi_0 + i\partial_t \phi_d = H_0\phi_d + \lambda_0 a(t)\psi_0 + \beta a(t)\psi_0 + \beta\phi_d. \tag{3.2}$$

Taking the scalar product with ψ_0, we get

$$i\partial_t a(t) = \lambda_0 a(t) + \langle \psi_0, \beta\psi_0 \rangle a(t) + \langle \psi_0, \beta\phi_d \rangle \tag{3.3}$$

letting

$$a(t) \equiv e^{-i\lambda_0 t} A(t), \tag{3.4}$$

we have

$$i\partial_t A(t) = \langle \psi_0, \beta\psi_0 \rangle A(t) + e^{i\lambda_0 t} \langle \psi_0, \beta\phi_d \rangle \tag{3.5}$$

and by applying P_c to (3.3) we get

$$i\partial_t \phi_d = H_0\phi_d + P_c\beta\phi_d + e^{-i\lambda_0 t} A(t) P_c\beta\psi_0 \tag{3.6}$$

and $P_c\phi_d = \phi_d$.

Our aim is to prove that $A(\infty) \ \equiv \ \lim_{t\to\infty} A(t)$ exists.

Solving equation (3.6) we get

$$\begin{aligned} \phi_d(t) &= e^{-iH_0 t}\phi_d(0) - i\int_0^t e^{-iH_0(t-s)} P_c\beta(s)\phi_d(s) ds \\ &-i\int_0^t e^{-iH_0(t-s)} e^{-i\lambda_0 s} A(s) P_c\beta(s)\psi_0 \ ds \\ &\equiv \phi_0 + \phi_1 + \phi_2 \end{aligned} \tag{3.7}$$

We estimate $\|w_-\phi_d(t)\|$ by considering each term individually. First, $\phi_0(t) = e^{-iH_0 t}\phi_d(0)$ satisfies local decay since $\phi_d(0) = P_c\phi_d(0) = P_c\phi_0$. Thus,

$$\| \ w_-\phi_0(t) \ \| \ \leq \ c \ \|w_+\phi_d(0)\| \langle t \rangle^{-r}. \tag{3.8}$$

Estimating ϕ_1, we have

$$\begin{aligned}\| w_- \phi_1(t) \| &\leq \| w_- \int_0^t e^{-iH_0(t-s)} P_c\beta(s)\phi_d(s)ds\|_{L^2} \\ &\leq c\int_0^t \langle t-s\rangle^{-r} \|w_+\beta(s)\ w_+ \ \cdot w_-\phi_d(s)\|_{B(\mathcal{H})}ds \\ &\leq c\|\phi_d(0)\|\ I_r[h_\beta](t). \end{aligned} \tag{3.9}$$

We have also used that $\|\langle x\rangle^{-\sigma}\phi_d(s)\|_{L^2} \leq \|\phi_d(0)\|$ is bounded as a consequence of L^2 conservation and orthogonality:

$$\|\phi(t)\|^2 = |A(t)|^2 + \|\phi_d(t)\|^2 = \ \|\phi(0)\|^2.$$

Finally, we estimate the local decay norm of ϕ_2 in a similar way:

$$\| w_- \ \phi_2(t) \| \leq c\int_0^t \langle t-s\rangle^{-r}|A(s)|\ \|\ w_+ \ \beta(s)\psi_0\|ds$$

$$\leq \left(c\int_0^t \langle t-s\rangle^{-r} h_\beta(s)\ ds\right) \times \left(\sup_{a<s\leq t} |A(s)|\right) \tag{3.10}$$

$$\leq c\ \|\phi(0)\|\ \ I_r[h_\beta](t). \tag{3.11}$$

Therefore, we have local energy decay of ϕ_d:

$$\|w_-\phi_d(t)\| \ \leq\ c_1\ \|w_+\phi_d(0)\|\langle t\rangle^{-r}\ +\ c_2\ \|\phi(0)\|\ I_r[h_\beta](t). \tag{3.12}$$

To study the detailed asymptotic behavior of $\phi_d(t)$ we rewrite (3.7) as:

$$\begin{aligned} &\phi_d(t) \ - \ e^{-iH_0t}\phi_+ \\ = \ & e^{-iH_0t}\ \left(i\int_t^\infty e^{iH_0s}P_c\ \left[\beta(s)\phi_d(s)\ +\ e^{-i\lambda_0 s}A(s)\beta(s)\psi_0\ \right] ds\right),\end{aligned}$$

where

$$\phi_+ \ \equiv\ \phi_d(0)\ -\ i\int_0^\infty e^{iH_0s}P_c\beta(s)\phi_d(s)\ ds - i\int_0^\infty e^{iH_0s}\ e^{-i\lambda_0 s}A(s)P_c\beta(s)\psi_0\ ds \tag{3.13}$$

It follows that

$$\|\phi_d(t)\ -\ e^{-iH_0t}\phi_+\ \| \ =\ \mathcal{O}(\int_t^\infty h_\beta(s)\ ds). \tag{3.14}$$

If one additionally has asymptotic completeness for H_0 [8], then there exists $\bar{\phi}_+ \in \mathcal{H}$ such that

$$\|e^{-iH_0t}\phi_+\ -\ e^{i\Delta t}\bar{\phi}_+\| \to 0. \tag{3.15}$$

To see that $A(\infty) \equiv \lim_{t\uparrow\infty} A(t)$ exists, we solve (3.5) for $A(t)$ to get

$$\begin{aligned} A(t) \ &=\ e^{-i\int_0^t \langle\psi_0,\beta(s)\psi_0\rangle ds}A(0) \\ &+\ e^{-i\int_0^t \langle\psi_0,\beta(s)\psi_0\rangle ds}\left[\int_0^t e^{i\int_0^s \langle\psi_0,\beta(u)\psi_0\rangle} J(s)\ ds\right], \end{aligned} \tag{3.16}$$

where $J(s) = e^{i\lambda_0 s}\langle\psi_0, \beta(s)\phi_d(s)\rangle$

Since $\langle\psi_0, \beta(s)\psi_0\rangle$ is integrable in s, and $\phi_d(s)$ is bounded (even decaying), the existence of the limit $A(\infty)$ follows.

4. Proof of Theorem 2 - ionization probability

Our aim now, is to find $A(\infty)$ to second order in β, for β small in appropriate sense. However, the expressions we get can be used as the starting point to finding the ionization probability for the general case (β large).

From equations (3.5) and (3.7) we have

$$\partial_t A(t) = -i\langle\psi_0, \beta(t)\psi_0\rangle A(t) - ie^{i\lambda_0 t}\sum_{j=0}^{2}\langle\psi_0, \beta(t)\phi_j(t)\rangle. \tag{4.1}$$

We now prove:

Proposition 4.1.

$$\partial_t A(t) \ = \ -g(t)\ A(t) \ + \ R(t), \tag{4.2}$$

where

$$\begin{aligned} g(t) \quad &= \quad i\langle\ \psi_0, \beta(t)\psi_0\ \rangle \ + \ \pi\langle\ e^{-i(\lambda_0 - H_0)t}P_c\beta(t)\psi_0, P_c\hat{\beta}(\lambda_0 - H_0)\ \psi_0\ \rangle \\ &- \quad i\ \text{P.V.} \int d\mu\langle e^{-i\mu t}\beta(t)\psi_0\ ,\ (H_0 - (\lambda_0 - \mu))^{-1}\ P_c\hat{\beta}(\mu)\psi_0\rangle \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} R(t) \quad &= \quad -ie^{i\lambda_0 t}\ \sum_{j=0}^{1}\ \langle\psi_0, \beta(t)\phi_j(t)\rangle \\ &- \quad ie^{i\lambda_0 t}A(0)\langle\beta(t)\psi_0, e^{-iH_0 t}\mathcal{T}(0)\psi_0\ \rangle \\ &- \quad ie^{i\lambda_0 t}\langle\beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ e^{i(H_0-\lambda_0)s}\mathcal{T}(s)\dot{A}(s)ds\ \psi_0\ \rangle \end{aligned} \tag{4.4}$$

Proof: The proposition is proved by expanding the $j = 2$ term in the sum in (4.1).

$$\begin{aligned} \langle\psi_0, \beta(t)\phi_2(t)\rangle \ &= \ \langle\psi_0, \beta(t)(-i)\int_0^t e^{-iH_0(t-s)}e^{-i\lambda_0 s}P_c\beta(s)\psi_0\ ds\rangle \\ = \ &-i\langle\beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ ds\ e^{iH_0 s}e^{-i\lambda_0 s}A(s)P_c\int\ d\mu\ e^{i\mu s}\hat{\beta}(\mu)\psi_0\rangle \\ = \ &-i\langle\beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ ds\int\ d\mu\ e^{i(H_0-(\lambda_0-\mu))s}\ A(s)P_c\ \hat{\beta}(\mu)\ \psi_0\ \rangle. \end{aligned}$$

Now it could be that for some μ in the support of $\hat{\beta}$ that $\lambda_0 - \mu$ is in the continuous spectrum of H_0. For such μ there is a resonance and we now compute its effect. First, we introduce the operator

$$\mathcal{T}(t) \ = \ \lim_{\varepsilon\downarrow 0}\int\ e^{i\mu t}\ (H_0 - (\lambda_0 - \mu) - i\varepsilon)^{-1}\ P_c\hat{\beta}(\mu)\ d\mu \tag{4.5}$$

The resonance effects are calculated using integration by parts:

$$\begin{aligned} &-ie^{i\lambda_0 t}\ \langle\psi_0, \beta(t)\phi_2(t)\rangle \\ = \quad &-ie^{i\lambda_0 t}\ \lim_{\varepsilon\downarrow 0} -i\langle\beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ ds\int\ d\mu\ e^{i(H_0-(\lambda_0-\mu)-i\varepsilon)s}\ A(s)P_c\ \hat{\beta}(\mu)\ \psi_0\ \rangle \\ = \quad &ie^{i\lambda_0 t}\ \lim_{\varepsilon\downarrow 0}\langle\beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ ds\int\ d\mu\frac{d}{ds}e^{i(H_0-(\lambda_0-\mu)-i\varepsilon))s}\ (H_0 - (\lambda_0 - \mu) - i\varepsilon)^{-1}A(s)P_c\ \hat{\beta}(\mu)\ \psi_0\ \rangle \end{aligned}$$

$$
\begin{aligned}
&= \; ie^{i\lambda_0 t}\ \langle \beta(t)\psi_0, e^{-iH_0 t} \int d\mu\ e^{i(H_0-(\lambda_0-\mu))t}\ (H_0-(\lambda_0-\mu)-i0)^{-1}A(t)P_c\ \hat\beta(\mu)\ \psi_0\rangle \\
&- \; ie^{i\lambda_0 t}\ \langle \beta(t)\psi_0, e^{-iH_0 t}\ A(0)\int d\mu\ (H_0-(\lambda_0-\mu)-i0)^{-1}P_c\ \hat\beta(\mu)\ \psi_0\ \rangle \\
&- \; ie^{i\lambda_0 t}\ \langle \beta(t)\psi_0, e^{-iH_0 t} \int d\mu \int_0^t ds\ e^{i(H_0-(\lambda_0-\mu))s}(H_0-(\lambda_0-\mu)-i0)^{-1}\dot A(s)P_c\ \hat\beta(\mu)\ \psi_0\ \rangle \\
&= \; i\langle\ \beta(t)\psi_0, \mathcal{T}(t)\psi_0\ \rangle A(t) \\
&- \; ie^{i\lambda_0 t}A(0)\langle \beta(t)\psi_0, e^{-iH_0 t}\mathcal{T}(0)\psi_0\ \rangle \\
&- \; ie^{i\lambda_0 t}\langle \beta(t)\psi_0, e^{-iH_0 t}\int_0^t\ e^{i(H_0-\lambda_0)s}\mathcal{T}(s)\dot A(s)ds\ \psi_0\ \rangle \\
&\equiv \; S_1\ +\ S_2\ +\ S_3.
\end{aligned} \tag{4.6}
$$

Remark 4.1. *Operators of the type appearing in the integrand of $\mathcal{T}(t)$ arise in [10], [11], [6], where the time-frequency content of the perturbation is discrete. In this work* singular local energy decay *estimates were required. The main difference here is that the perturbation β contains a continuum of time-frequencies and therefore it is possible that for some μ in the support of $\hat\beta$, we have that $-\lambda_0+\mu$ is at a threshold energy of H_0 (e.g. endpoint of the continuous spectrum). Typically, for such energies weaker local energy decay estimates hold and in the above cited works on quantum resonances, and Hamiltonian systems parametrically forced by $t-$ periodic and almost periodic potentials, the discreteness of the frequency spectrum essentially required that resonances with the continuous spectrum be bounded away from thresholds. Here however the μ integral has a smoothing effect and we require no such hypothesis; see Proposition 4.2 below.*

The explicit evaluation of S_1 and estimation of S_2-S_3 will proceed using the identity:

$$
(H_0-(\lambda_0-\mu)-i0)^{-1}\ =\ \text{P.V.}\ (H_0-(\lambda_0-\mu))^{-1}\ +\ i\pi\delta(H_0-(\lambda_0-\mu)). \tag{4.7}
$$

and the the following:

Proposition 4.2. *Assume $w_+\chi\in\mathcal{H}$. Then,*

$$
\begin{aligned}
\mathcal{T}(t)\chi \;&=\; i\pi\ e^{i(\lambda_0-H_0)t}\ P_c\ \hat\beta(\lambda_0-H_0)\ \chi \\
&\quad -i\int_{\mathbb{R}}\ e^{i(\lambda_0-H_0)(t-\tau)}\ P_c\ \text{sgn}(t-\tau)\ \beta(\tau)\ d\tau\ \chi.
\end{aligned} \tag{4.8}
$$

Proof: By (4.7)

$$
\begin{aligned}
\mathcal{T}(t)\chi \;&=\; i\pi\ \int_{\mathbb{R}} e^{i\mu t}\delta(\mu-(\lambda_0-H_0))\hat\beta(\mu)\psi_0 d\mu \\
&+\; \text{P.V.}\ \int_{\mathbb{R}} e^{i\mu t}\hat\beta(\mu)(\mu-(\lambda_0-H_0))^{-1}\ d\mu \\
&=\; i\pi\ e^{i\lambda_0 t}e^{-iH_0 t}\hat\beta(\lambda_0-H_0)\psi_0\ +\ \text{H.T.}[\hat\beta(\cdot+t)](\lambda_0-H_0),
\end{aligned}
$$

where H.T.$[f]$ $(x)\equiv$ P.V. $\int f(y)/(x-y)\ dy$ is the Hilbert transform.

Evaluation of the latter term using the Plancherel identity and the fact that $\widehat{H[f]}[\xi]=i\text{sgn}(\xi)$ yields (4.8).

We use properties of the Fourier transform, (2.5), and the identity (4.7).

We now apply this proposition to the evaluation and estimation of of the terms (4.6). Consider first S_1. We have

$$\begin{aligned} S_1 &= -\pi\langle e^{-i(\lambda_0-H_0)t}\ P_c\beta(t)\psi_0, P_c\hat{\beta}(\lambda_0-H_0)\ \psi_0\ \rangle A(t) \\ &+\ \langle \beta(t)\psi_0, \mathrm{P.V.}\int d\mu\ e^{i\mu t}\ (H_0-(\lambda_0-\mu))^{-1}P_c\ \hat{\beta}(\mu)\ \psi_0\ \rangle A(t). \end{aligned} \tag{4.9}$$

The two terms in S_1 are incorporated in the definition of $g(t)$, displayed in (4.3), while the terms S_2 and S_3 are included in $R(t)$. This proves the Proposition 4.1.

We next estimate $R(t)$. Before stating the estimate we employ Proposition 4.2 to obtain the following useful expressions for S_2 and S_3 which we then estimate. Beginning with S_2, we have

$$\begin{aligned} S_2 &= -ie^{i\lambda_0 t}A(0)\langle \beta(t)\psi_0, e^{-iH_0t}\mathcal{T}(0)\psi_0\ \rangle \\ &= \pi A(0)\ \langle\ \beta(t)\psi_0, e^{i(\lambda_0-H_0)t}P_c\hat{\beta}(\lambda_0-H_0)\psi_0\ \rangle \\ &\quad +A(0)\ \langle\ \beta(t)\psi_0, \int_{\mathbb{R}}\ e^{i(\lambda_0-H_0)(t-\tau)}P_c\ \mathrm{sgn}(\tau)\beta(\tau)\ d\tau\ \psi_0\ \rangle. \end{aligned}$$

Therefore,

$$|S_2| \le\ c|A(0)|\ h_\beta(t)\ \left(\ \|w_-e^{-iH_0t}P_c\ \hat{\beta}(\lambda_0-H_0)\ \psi_0\|\ +\ \int_{\mathbb{R}}\ \langle t-\tau\rangle^{-r}\ h_\beta(\tau)\ d\tau\ \right). \tag{4.10}$$

For S_3 we have

$$\begin{aligned} S_3 &= \pi\ e^{i\lambda_0 t}\ (A(t)-A(0))\ \langle\beta(t)\psi_0, e^{-iH_0t}P_c\hat{\beta}(\lambda_0-H_0)\ \psi_0\ \rangle \\ &\quad -e^{i\lambda_0 t}\langle\beta(t)\psi_0, e^{-iH_0t}\int_0^t\ \dot{A}(s)\ ds\ \int_{\mathbb{R}}\ e^{-i\lambda_0(s-\tau)}e^{iH_0(s-\tau-t)}\ P_c\ \mathrm{sgn}(s-\tau)\ \beta(\tau)\ d\tau\ ds\ \psi_0\ \rangle \end{aligned} \tag{4.11}$$

Estimation of S_3 gives:

$$|S_3| \le\ c\ \|\phi(0)\|\ h_\beta(t)\ \left(\ \|w_-e^{-iH_0t}P_c\ \hat{\beta}(\lambda_0-H_0)\ \psi_0\|\ +\ h_\beta(t)\ \int_{\mathbb{R}}\ I_r[h_\beta](\omega)h_\beta(\omega-t)\ d\omega\ \right). \tag{4.12}$$

By weak local energy decay (2.7)

$$\alpha_\beta(t)\ \equiv\ \|w_-e^{-iH_0t}P_c\ \hat{\beta}(\lambda_0-H_0)\ \psi_0\| \in L^2(\mathbb{R};dt). \tag{4.13}$$

Note also that α_β also satisfies the following elementary uniform bound:

$$\begin{aligned} \alpha_\beta(t) &\le\ \|\hat{\beta}(\lambda_0-H_0)\ \psi_0\| \\ &\le\ \|\int e^{-i(\lambda_0-H_0)t}\beta(t)\psi_0\ dt\ \le\ \int \|\beta(t)\psi_0\|\ dt \\ &\le\ c\|h_\beta\|_{L^1}. \end{aligned}$$

These considerations imply the following

Proposition 4.3.

$$
\begin{aligned}
&|R(t)| \;\le\; C_1\|\phi_d(0)\|\;\langle t\rangle^{-r} I_r[h_\beta](t) \;+\; C_2\|\phi(0)\|\; h_\beta(t)\; I_r[h_\beta](t)\\
+\quad & C_3\|\phi(0)\|\; h_\beta(t)\\
&\times \left(\; \min\{\alpha_\beta(t), \|h_\beta\|_{L^1}\} \;+\; \int_{\rm I\!R} \langle t-\tau\rangle^{-r}\; h_\beta(\tau)\; d\tau \;+\; \int_{\rm I\!R} I_r[h_\beta](\omega)\; h_\beta(\omega - t)\; d\omega \;\right).
\end{aligned}
\tag{4.14}
$$

Using the above bound for $R(t)$, we now analyze the equation for $A(t)$. From (4.2) we have:

$$
A(t) \;=\; e^{-\int_0^t g(s)\; ds}\;\left(\; A(0) + \int_0^t e^{\int_0^\xi g(\tau)\; d\tau}\; R(\xi)\; d\xi\;\right), \tag{4.15}
$$

where $g(t)$ is given by (4.3). We now seek a simple relation between the initial bound state amplitude, $A(0)$, and the asymptotic bound state amplitude, $A(\infty)$ by reexpressing $A(t)$ in (4.15) as its asymptotic value plus a decaying perturbation.

$$
\begin{aligned}
A(t) \;&=\; A(\infty)\; + \rho(t), \text{ where}\\
A(\infty) \;&=\; e^{-\int_0^\infty g(s)\; ds}\left(\; A(0) \;+\; \int_0^\infty e^{\int_0^s g(s_1)\; ds_1} R(s)\; ds\;\right),
\end{aligned}
\tag{4.16}
$$

and $\rho(t)$ decays as $t\to\infty$ and is given explicitly by:

$$
\begin{aligned}
\rho(t) \;&=\; e^{-\int_0^\infty g(s) ds}\;\left(e^{\int_t^\infty g(s)\; ds} - 1\right)\;\left(\; A(0) \;+\; \int_0^\infty e^{\int_0^s g(s_1)\; ds_1} R(s)\; ds\right)\\
&-\; e^{-\int_0^\infty g(s)\; ds}\; \int_t^\infty e^{\int_0^s g(s_1)\; ds_1} R(s)\; ds.
\end{aligned}
$$

From (4.16) we see that it is necessary to evaluate the integral of g to obtain $A(\infty)$, and the ionization probability, $1-P(\infty)$. We do this using the expression for g in Proposition 4.1. The result is given in

Proposition 4.4.

$$
\begin{aligned}
\int_0^\infty g(s)\; ds \;&=\; \frac{i}{2}\int_{\rm I\!R} dt\; \langle \psi_0, \beta(t)\psi_0\rangle \;+\; \Gamma \;-i\Lambda, \text{ where} && (4.17)\\
\Gamma \;&=\; \pi^2 \int_{\rm I\!R} d\mu\; \langle\; P_c\hat\beta(\mu)\psi_0, \delta(H_0 - (\lambda_0-\mu))\; P_c\hat\beta(\mu)\psi_0\;\rangle && (4.18)\\
&=\; \pi^2\; \|\; P_c\hat\beta(\lambda_0 - H_0)\psi_0\;\|^2\\
\Lambda \;&=\; \pi\ \text{P.V.} \int_{\rm I\!R} d\mu\; \langle\; P_c\hat\beta(\mu)\psi_0, (H_0 - (\lambda_0-\mu))^{-1}\; P_c\hat\beta(\mu)\psi_0\;\rangle && (4.19)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A(\infty) \;&=\; e^{-\Gamma}\; e^{-i(\frac12\int_{\rm I\!R}\langle\psi_0,\beta(s)\psi_0\rangle\; ds+\Lambda)}\\
&\times\;\; \left[\; A(0) \;+\; \mathcal{O}\left(\|\phi(0)\|\; \|h_\beta\|_{L^1}^2\right)\;\right],\\
|A(\infty)| \;&=\; e^{-\Gamma}\left[|A(0)| \;+\; \mathcal{O}\left(\|\phi(0)\|\; \|h_\beta\|_{L^1}^2\;\right)\right].
\end{aligned}
\tag{4.20}
$$

Finally, the assertions concerning the rate of convergence of $\langle\phi(t),\psi_0\rangle$ to an asymptotic bound state and of $P(t)$ to its asymptotic value are obtained by estimation of the above expressions for $A(t)-A(\infty)$ and of $|A(t)|-|A(\infty)|$. These estimates follow from (4.14) and an estimatimation of $\exp(\int_t^\infty g(s)ds)-1$ and $\exp(\int_t^\infty \Re g(s)\ ds)-1$ using local energy and weak local energy decay estimates (2.6-2.7).

This concludes the proof of Theorem 2.2.

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