A Reaction Dispersion System and Raman Interactions

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Abstract

We consider the problem of amplification of an optical signal wave with an optical pump wave when both are propagating in the fundamental mode of a single mode optical waveguide. We introduce a system of Ginzburg Landau type and study the radiation loss due to the nonlinear interaction between the signal and the pump waves. The linear dynamics are dispersive, while nonlinearity governs the transfer of energy from the pump wave to the signal wave. The strength of the effect is shown to depend on a dimensionless parameter, which is given by the ratio of the diffraction length and amplification length. If this parameter is small then the radiation loss is small. This result is established by (i) verifying the absence of resonant terms that can potentially drive the growth of radiative components and (ii) then by estimating the oscillatory (non-resonant) terms by proving the relevant PDE *á priori* estimates. These estimates require appropriate bounds on the solutions of the PDE, whose only conserved integral is the L^2 norm. However, the special structure of the nonlinear term, dictated by the physics of the Raman effect, implies a weak space-time bound involving the signal and pump intensities. This bound and L^2 conservation are used together with Strichartz (space-time) estimates for the Schrödinger equation to obtain control of stronger classical norms of the signal and pump fields.

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1 Introduction

In this paper we study a system of nonlinear and dispersive partial differential equations, where nonlinearity is of "reaction" type, *i.e.* in the absence of dispersion it induces pure energy exchange between the fields. Such systems are reminiscent of reaction-diffusion systems; here the diffusive mechanism is replaced by dispersion. While, the latter has been widely studied, the former has received very little attention.

Our reaction-dispersion system arises naturally in mathematical modeling of the stimulated Raman process, but will also arise in other systems (perhaps in somewhat modified form), where two dispersive waves interact nonlinearly, while other nonlinear effects (such as self-phase modulation) and diffusion are negligible. This system can be also considered as a special case of CGL (complex Ginzburg-Landau system), which has been studied in a different parameter regime [4]. We will often refer to the systems (1) and (2) as the Raman model, due to their relation to the motivating physical context.

The Raman effect is one where light of one frequency, ω_s ("signal"), is amplified by light of a down shifted frequency, ω_p ("pump"). Taking the energy transfer characteristics of the Raman process into account as well as diffraction leads to the system of nonlinear evolution equations, discussed in greater detail in appendix A:

$$\begin{aligned} i\partial_t u_s &= -\Delta\ u_s\ +\ i\epsilon|u_p|^2 u_s \\ i\partial_t u_p &= -\Delta\ u_p\ -\ i\epsilon|u_s|^2 u_p, \end{aligned} \tag{1}$$

In a waveguide setting, which is of importance in optical communications, the equations take the form:

$$\begin{aligned} i\partial_t u_s &= H\ u_s\ +\ i\epsilon|u_p|^2 u_s \\ i\partial_t u_p &= H u_p\ -\ i\epsilon|u_s|^2 u_p \end{aligned} \tag{2}$$

where

$$H = -\Delta + V$$

Here, $u_s = u_s(x,t)$ and $u_p = u_p(x,t)$ denote, respectively, the signal and pump complex electric field envelopes. Systems (1) and (2) are valid, assuming the *paraxial approximation.* Δ denotes the Laplace operator with the respect to x ($x \in \mathcal{R}^1$ or $x \in \mathcal{R}^2$). The longitudinal coordinate (z), a *time-like* variable with which propagation distance is measured, is denoted by t. In the waveguide setting, the "potential" $V(x)$ is determined by the transverse refractive index profile of the waveguide. The parameter ϵ measures the size of the nonlinear effects relative to the linear effects (*e.g.* diffraction, dispersion). The particular application to optical communications is discussed in appendix A.

Our study of the systems (2) and (1) is motivated by a fundamental issue arising in the modeling of the Raman interaction in a waveguide setting. In optical communication applications, the weak signal field whose envelope encodes bits of information, is amplified by the strong pump field. This process takes place in an optical fiber waveguide, with one transverse localized state or "guided mode" and radiation modes. Raman amplification of the signal is based on the intended net transfer of energy from the pump to the signal. Physicists have found that good agreement with experiment is achieved by an ODE model, in which one neglects the effect of nonlinear coupling of bound states to radiation modes:

$$\begin{aligned} \partial_t I_s &= \epsilon\ g_s I_s\ I_p \\ \partial_t I_p &= -\epsilon\ g_p I_s\ I_p, \\ I_s\ \sim |u_s|^2,\ \ I_p &\sim\ |u_p|^2, \end{aligned} \tag{3}$$

where $g_{s,p}$ are coefficients depending on the properties of the fundamental modes, *e.g.* on frequency and the so-called effective area. A satisfactory explanation for this has been lacking; see, for example, [5]. This motivated us to consider the full Raman energy transfer problem in the context of the model (2). We have found an explanation for the above statement about energy transfer using ideas and methods of resonance and averaging. In particular, in Theorem 3.1 we establish that if the initial field energy, which is not small, is in the guided mode, then this property persists with negligible error on the length scale of physical interest, $\mathcal{O}(\epsilon^{-1})$, and therefore the model (3) applies. Note that on the time interval of order ϵ^{-1}, the radiation of size ϵ can grow to become of order $\mathcal{O}(1)$ (since the rate of radiation change is of order $\mathcal{O}(\epsilon)$).

The proof of Theorem 3.1 requires a good understanding of the large time dynamics of the flow defined by (2). Thus we consider the question of global existence of the initial value problem for such systems and have derived results of independent interest. The standard approach to controlling the large time dynamics is to first prove local in time existence of the solutions to the initial value problem in a "natural" Banach space. A "natural" space is often one in which the physically relevant conserved integrals are defined. Global existence in time then follows from an appropriate *á priori* bound on the norm of this Banach space. If this norm remains bounded in terms of conserved integrals of the flow, global existence holds.

The ideas we use to prove global existence apply to both systems (1) and (2). Equations (1) and (2) have the L^2 (energy) conservation law:

$$\mathcal{P}[u_s(t), u_p(t)]\ \equiv\ \int_{-\infty}^{+\infty} (|u_s|^2 + |u_p|^2)dx\ =\ \mathcal{P}[u_s(0), u_p(0)]\ \equiv\ \mathcal{P}_0. \tag{4}$$

Unfortunately, L^2 is a very weak a space in which to control the nonlinear flow. Unlike the nonlinear Schrödinger equation, a Hamiltonian system, equations (2) and (1) do not have a second conserved integral (Hamiltonian), which controls $\|\nabla u_{s,p}\|_{L^2}$, and from which sufficient *á priori* control follows for global existence to hold.

We find that the key to a global existence theory is the following space-time integral *á priori* bound, which is consequence of the form of the nonlinear Raman interaction terms:

$$\int_0^T dt \int |u_s|^2 |u_p|^2 \, dx \leq \frac{1}{2}\mathcal{P}_0. \tag{5}$$

Outline of the paper:

The paper is structured as follows. We first consider the system (2) in one space and one time dimension. In section 2 we prove global well-posedness for the solution of the initial value problem. The key to this result are certain *á priori* estimates, whose point of departure is the L^2 conservation law (4) and the space-time *á priori* bound (5). This space-time estimate implies that (2) may be viewed as an inhomogeneous system of equations for u_s and u_p, with a source term, which is bounded in a space-time norm. Strichartz estimates [10] are then used to bound u_s and u_p individually in space-time norms and then finally in classical Sobolev norms. In section 3 the energy transfer from the bound mode of a single mode waveguide to radiation modes is studied by estimating non-resonant oscillating terms. Section 4 contains a theory of well-posedness in the case of two transverse dimensions. Finally, there are two appendices: one with a detailed discussion of the motivating application to optical communications and a second in which we prove a normal result, based on the special symmetries of the system, which is of independent interest.

2 Existence theory on $\mathcal{R}^1$

In this section we prove that the system (2) has a unique global solution in an appropriate (physical) function space. In subsection 2.1 we provide the required operator estimates we shall require. In subsection 2.2 we derive certain *á priori* bounds which are satisfied by solutions of (2). In subsection 2.3 existence in a "weak" space is proved. In section 2.4 it is shown how to extend these results to H^s.

2.1 Space time estimates for the propagator

We first introduce the fundamental solution of the Schrödinger equation.
(1) The solution of the initial value problem

$$i\partial_t u = -\partial_x^2 u, \; u(0,x) = f \tag{6}$$

is denoted by $U_0(t)f$. $U_0(t)$ is called the free propagator.

(2) The solution of the initial value problem

$$i\partial_t u = \left(-\partial_x^2 \ + \ V(x) \ \right) u \ = \ H \ u, \ u(0,x) = f \tag{7}$$

is denoted by $U(t)f$.

The operator, H, may have spectrum consisting of continuous and discrete parts, with associated spectral projections P_c and $P_b = I - P_c$. Intuitively, on the range of P_c we expect $U(t)$ to behave dispersively in a manner similar to $U_0(t)$. We use dispersive estimates which involve space-time integrals, often referred to as estimates of Strichartz type; see [10] and [11]. The proofs of the space time estimates for the free Schrödinger equation in the form we use are due to Ginibre & Velo [14] and, in inhomogeneous case, to Yajima [12] and Cazenave & Weissler [13]. For complete proofs see, for example, Theorems 3.3-3.4 of [6].

We now introduce the function spaces and the notation of an *admissible pair* in terms of which the space-time estimates are expressed.

(3) For an real interval I and a Banach space X, we denote by $L^p(I,X)$ the Banach space of functions $u : I \to X$ for which $\int_I \|u(t)\|_X dt$ is finite.

(4) A pair of real numbers (q,r) is called *admissible* (for dimension $n = 1$) if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{r}, \ r \in [2,\infty]. \tag{8}$$

We now state Strichartz type estimates for $U(t)$ for the initial value problem (7) and the inhomogeneous problem:

$$i\partial_t u \ = \ H \ u \ + \ P_c g. \tag{9}$$

Theorem 2.1 *Assume that the potential V satisfies*

$$\int_{-\infty}^{+\infty} |V(x)|(1+|x|)^{5/2} dx < \infty. \tag{10}$$

Let (q,r) be an admissible pair. For any $f \in L^2$, $U_0(t)f$ and $U(t)P_c f$ are of class $L^q(R,L^r)$ and satisfy the estimates

$$\begin{aligned} \|U_0(\cdot)f\|_{L^q(R,L^r)} &\leq C \ \|f\|_{L^2}, \\ \|U(\cdot)P_c f\|_{L^q(R,L^r)} &\leq C \ \|f\|_{L^2}, \end{aligned} \tag{11}$$

where C depends only on q

Theorem 2.2 *Let V satisfy (10) and let (γ,ρ) be an admissible pair $(2/\gamma = 1/2 - 1/\rho)$, $f \in L^{\gamma'}([0,T],L^{\rho'})$, where (γ',ρ') is conjugate to (γ,ρ). Then for any admissible pair (q,r) $(2/q =$*

$1/2 - 1/r$)

$$\left\| \int_0^t U_0(t-\tau) f(\tau) d\tau \right\|_{L^q([0,T],L^r)} \leq C \|f\|_{L^{\gamma'}([0,T],L^{\rho'})},$$
$$\left\| \int_0^t U(t-\tau) P_c f(\tau) d\tau \right\|_{L^q([0,T],L^r)} \leq C \|f\|_{L^{\gamma'}([0,T],L^{\rho'})}, \tag{12}$$

where C depends only on q, γ.

The proofs rely on the $L^p - L^{p'}$ estimates for the free Schroedinger equation

$$\|U_0(t) f\|_{L^p} \leq (\ 4\pi|t|\ )^{-\frac{1}{2}+\frac{1}{p}} \ \|f\|_{L^{p'}}. \tag{13}$$

In the case of a Schrödinger equation with a potential in one space dimension the analogous estimate for $U(t)\mathbf{P_c}$ was established by Weder [7]. Adapting the proofs in [6], for the free propagator $U_0(t)$, and using Weder's estimate, one obtains the Strichartz type estimates for Schrödinger equation with a potential.

The following Corollary, easily derived from the previous estimates by a change of variables, concerns the dependence of space-time estimates on a parameter, which arises when we rescale the equations (2).

Corollary 2.1 *Consider a one parameter family of Schrödinger initial value problems:*

$$i\partial_t u = \beta H u,$$
$$u(x,0) = f(x) \tag{14}$$

where $\beta \in [\beta_0, \infty)$ with $\beta_0 > 0$. Assume that the potential, $V(x)$, satisfies (10). Then, the conclusions of Theorem 2.1 and Theorem 2.2 hold with $\beta-$ dependent constants. In particular, if $U_\beta(t)f = U(\beta t)f$ denotes the solution of the initial value problem (14), then

$$\|U_\beta(\cdot) P_c f\|_{L^q(R,L^r)} \leq C_1(\beta, q)\ \|f\|_{L^2}, \tag{15}$$
$$\left\| \int_0^t U(t-\tau) P_c f(\tau) d\tau \right\|_{L^q([0,T],L^r)} \leq C_2(\beta, q, \gamma')\ \|f\|_{L^{\gamma'}([0,T],L^{\rho'})}, \tag{16}$$

where

$$C_1(\beta, q) = C_{11}(q) \beta^{-\frac{1}{q}}, \quad C_2(\beta, q, \gamma') = C_{22}(q, \gamma') \beta^{-1-\frac{1}{q}+\frac{1}{\gamma'}}. \tag{17}$$

2.2 Á priori space-time estimates

Essential in the proof that the system (2) defines the solution globally in time and that the solution does not develop singularities are *á priori* estimates which we now derive. For convenience, we rescale the time $\epsilon t = t_{\rm new}$, so the uniform bound on the interval $t_{\rm new} \in [0,T]$ will correspond to the interval $[0, T/\epsilon]$ in old time. We will continue to use t as the time variable

$$\begin{aligned} \partial_t u_s + i\beta\ H u_s &= |u_p|^2 u_s && (18)\\ \partial_t u_p + i\beta\ H u_p &= -|u_s|^2 u_p, && (19) \end{aligned}$$

where $\beta := \epsilon^{-1}$ is a dispersion/diffraction parameter and $\beta \in [\beta_0, \infty]$.

Multiplication of (18) by $\overline{u_s}$, taking the real part of the resulting equation and integrating over all gives:

$$\frac{d}{dt}\int\ |u_s|^2\ dx\ =\ 2\int\ |u_s|^2\ |u_p|^2\ dx. \qquad (20)$$

Similarly, multiplication of (19) by $\overline{u_p}$ yields

$$\frac{d}{dt}\int\ |u_p|^2\ dx\ =\ -2\int\ |u_s|^2\ |u_p|^2\ dx. \qquad (21)$$

Equations (20) and (21) expressed the gain of signal energy at the expense of pump energy and the depletion of pump energy at the expense of signal energy; see (2) on page 3.

Addition of (20) and (21) yields the conservation law:

$$\frac{d}{dt}\int\ |u_s|^2 + |u_p|^2\ dx = 0 \qquad (22)$$

or

$$\int\ |u_s|^2 + |u_p|^2\ dx\ =\ \int\ |u_s(0)|^2 + |u_p(0)|^2\ dx\ \equiv\ \mathcal{P}_0 \qquad (23)$$

An important step in our analysis is to use the *energy dissipation identity* (21). Integration of (21) over time interval $[0,T]$ yields:

$$2\int_0^T\int\ |u_s(x,t)|^2\ |u_p(x,t)|^2\ dx\ dt\ =\ \int\ |u_p(x,0)|^2\ dx\ -\ \int\ |u_p(x,t)|^2\ dx. \qquad (24)$$

A simple consequence of (24) is the *space-time bound*:

Proposition 2.1 *(***Á priori space- time estimate***)*

$$\int_0^T\int |u_s|^2|u_p|^2 dxdt \le \frac{1}{2}\mathcal{P}_0. \qquad (25)$$

Equation (19) and (18) can both be viewed as inhomogeneous Schrödinger equations of the form:

$$i\partial_t\ U\ =\ H\ U\ +\ g \qquad (26)$$

with a source term g given by

$$g\ =\ |u_p|^2 u_s\ \text{ or }\ g\ =\ -|u_s|^2 u_p \qquad (27)$$

We next show that the *á priori* estimate (25) implies bounds on the source terms (27) which are suitable for application of the inhomogeneous Strichartz estimate of Theorem 2.2.

Proposition 2.2 *Let g denote either term in (27). Then, for any $T>0$ and any $\kappa\in[0,2]$:*

$$\int_0^T \left|\int |g|dx\right|^{\kappa} dt \le 2^{-\frac{\kappa}{2}}\ \mathcal{P}_0^{\kappa}\ T^{\frac{2-\kappa}{2}}, \tag{28}$$

In particular, for $\kappa\in[1,2]$, $g\in L^{\kappa}([0,T],L^1)$, and

$$\|g\|_{L^{\kappa}([0,T],L^1)}\ \le 2^{-\frac{1}{2}}\mathcal{P}_0\ T^{\frac{2-\kappa}{2k}} \tag{29}$$

To prove Proposition 2.2, let $g=|u_s|^2|u_p$. The proof for $g=|u_p|^2|u_s$ is analogous. By the Cauchy-Schwarz inequality

$$\left|\int |u_s|^2|u_p|dx\right| \le \left(\int |u_s|^2dx\right)^{\frac{1}{2}}\left(\int |u_s|^2|u_p|^2dx\right)^{\frac{1}{2}} \tag{30}$$

Squaring this inequality and integrating the result over the time interval $[0,T]$ yields, after using that the L^2 norm of u_s is bounded by $\mathcal{P}_0$, that

$$\int_0^T \left|\int |u_s|^2|u_p|dx\right|^2 dt \le \frac{1}{2}\mathcal{P}_0^2. \tag{31}$$

This handles the case $\kappa=2$. For $\kappa=0$ the trivial bound of T holds. The result follows by interpolation.

Using the *á priori* bounds of Proposition 2.2 we can now estimate the solution in $L^q([0,T],L^r)$ spaces, for admissible (q,r).

Theorem 2.3 (Á priori bounds in $L^q([0,T],L^r)$) *Let (q,r) be an admissible pair; see (8). Then, any solution (u_s,u_p) of the system (18)-(19) for $0\le t\le T$ satisfies the bounds:*

$$\begin{aligned} ||u_s||_{L^q([0,T],L^r)} &\le C\mathcal{P}_0^{\frac{1}{2}}(T^{\frac{1}{4}}+1)\\ ||u_p||_{L^q([0,T],L^r)} &\le C\mathcal{P}_0^{\frac{1}{2}}(T^{\frac{1}{4}}+1), \end{aligned} \tag{32}$$

where C depends only on q,γ, and β_0.

Proof of Theorem 2.3: We estimate $||u_s||_{L^q([0,T],L^r)}$. The corresponding estimate for u_p is similar. Equation (18) can be rewritten as an equivalent integral equation:

$$u_s(t) = U_\beta(t)u_s(0)\ +\ \int_0^t U_\beta(t-\tau)\ |u_p(\tau)|^2u_s(\tau)d\tau. \tag{33}$$

Estimation of (33) using Corollary 2.1 gives

$$\begin{aligned} \|u_s\|_{L^q([0,T],L^r)} &\le C\left\|U_\beta(t)u_s(0)\ +\ \int_0^t U_\beta(t-\tau)\ |u_p(\tau)|^2u_s(\tau)d\tau\right\|_{L^q([0,T],L^r)}\\ &\le C_1\|u_s(0)\|_{L^2}\ +\ C_2\|u_p^2u_s\|_{L^{\gamma'}([0,T],L^{\rho'})} \end{aligned} \tag{34}$$

Now Proposition 2.2 implies a bound on $\|u_p^2 u_s\|_{L^{\gamma'}([0,T],L^{\rho'})}$, where $\rho' = 1$ and $\gamma' \in [0,2]$. Note that the exponents γ and ρ, dual to γ' and $\rho' = 1$, form an admissible pair provided $\gamma = 4/3$ and and $\gamma' = 4/3$.

Setting $\gamma' = 4/3$ and $\rho' = 1$ in (34) and applying Proposition 2.2 with $\kappa = \gamma' = 4/3$ implies

$$\begin{aligned} \|u_s\|_{L^q([0,T],L^r)} &\leq C_1\|u_s(0)\|_{L^2} + C_2\|u_p^2 u_s\|_{L^{\frac{4}{3}}([0,T],L^1)} \\ &\leq C_1\mathcal{P}_0^{\frac{1}{2}} + 2^{-\frac{1}{2}}C_2\mathcal{P}_0 T^{\frac{1}{4}} \end{aligned} \tag{35}$$

This completes the proof of Theorem 2.3.

2.3 Existence in $L^8(\mathcal{R}, L^4) \cap L^\infty(\mathcal{R}, L^2)$

In this subsection we prove existence of solutions in a function space $\mathcal{X}(T)$ define by

$$\mathcal{X}(T) = L^8([0,T], L^4) \cap L^\infty([0,T], L^2). \tag{36}$$

with the norm:

$$\|u\|_{\mathcal{X}(T)} = \|u\|_{L^8([0,T],L^4)} + \|u\|_{L^\infty([0,T],L^2)} \tag{37}$$

$$= \|u\|_{8,4} + \|u\|_{\infty,2}, \tag{38}$$

the latter being written when there is no ambiguity. For the two dimensional field (u_s, u_p), we naturally define the norm

$$\|u_s, u_p\|_{\mathcal{X}(T)} = \|u_s\|_{\mathcal{X}(T)} + \|u_p\|_{\mathcal{X}(T)} = \|u_s\|_{8,4} + \|u_s\|_{\infty,2} + \|u_p\|_{8,4} + \|u_p\|_{\infty,2}. \tag{39}$$

Since Theorem 2.3 gives *á priori* control of solutions in $L^q([0,T], L^r)$ spaces for any admissible pairs (q, r) it is natural to obtain a local existence theorem in a space, where the maximal time of existence depends only on $L^q([0,T], L^r)$ bounds. Then, global existence follows from Theorem 2.3; see the discussion below.

Define the mapping

$$(u_s, u_p) \mapsto A_\beta(u_s, u_p) \equiv \Big(A_\beta^{(s)}(u_s, u_p), A_\beta^{(p)}(u_s, u_p) \Big) \tag{40}$$

where

$$A_\beta^{(s)}(u_s, u_p) = U_\beta(t)u_s(0) + \int_0^t U_\beta(t-\tau)\ |u_p(\tau)|^2 u_s(\tau)d\tau \tag{41}$$

$$A_\beta^{(p)}(u_s, u_p) = U_\beta(t)u_p(0) - \int_0^t U_\beta(t-\tau)\ |u_s(\tau)|^2 u_p(\tau)d\tau \tag{42}$$

Then, the above the evolution equation has the equivalent formulation as a fixed point problem: For initial data $(u_s(0), u_p(0) \in L^2$, find $(u_s, u_p) \in \mathcal{X}(T)$ for some $T > 0$ such that

$$(u_s, u_p) = A_\beta(u_s, u_p) \tag{43}$$

Our local existence theorem is the following:

Theorem 2.4 (Local Existence)
(1) Given initial data $(u_s(0), u_p(0)) \in L^2$, there exists a $T > 0$ and a unique solution $(u_s, u_p) \in \mathcal{X}(T)$ of (43).
(2) Let $T_{\max} > 0$ denote the maximal time of existence. Either $T_{\max} = \infty$ (global existence in time) or

$$T_{\max} < \infty \quad \text{and} \quad \limsup_{t \to T_{\max}} \|(u_s, u_p)\|_{\mathcal{X}(T)} = \infty \tag{44}$$

Using the local existence theory of Theorem 2.4 and the *á priori* bounds of Theorem 2.3, we have $T_{\max} = \infty$. Therefore,

Theorem 2.5 (Global existence) *For any initial data $(u_s(0), u_p(0))$ in L^2, equation (43) has a unique global solution of class $L^8(\mathbb{R}_+, L^4) \cap L^\infty(\mathbb{R}_+, L^2)$.*

We need only prove the local existence Theorem 2.4. The proof follows from the next two propositions in which we establish that for L^2 initial conditions and T sufficiently small :
(i) the transformation A_β maps a specified ball, $\mathcal{B}(T)$ in $\mathcal{X}(T)$ into itself and that
(ii) A_β is a contraction mapping on $\mathcal{B}(T)$.

Proposition 2.3 *Let $(u_s(0), u_p(0))$ be in L^2. Define the ball in $\mathcal{X}(T)$*

$$\mathcal{B}(T) \;=\; \{\; (u_s, u_p) \in \mathcal{X}(T) \;:\; \|(u_s, u_p)\|_{\mathcal{X}(T)} \le 4C(||u_s(0)||_2 + ||u_p(0)||_2) \;\} \tag{45}$$

There exists $T_0 > 0$ such that for any $T < T_0$, the ball is mapped into itself, i.e. $A_\beta(\mathcal{B}(T)) \subset \mathcal{B}(T)$ for any $\beta \in [\beta_0, \infty]$, with T_0 depending on β_0.

Proof of Proposition 2.3: We estimate the action of $A_\beta^{(s)}$. The estimation for $A_\beta^{(p)}$ is similar. By Corollary 2.1

$$\begin{aligned} \|[A_\beta^{(s)}(u_s, u_p)\|_{q,r} &\le \|U_\beta(t) u_s(0)||_{q,r} + \| \int_0^t U_\beta(t-\tau)\; |u_p|^2 u_s \;\|_{q,r} \\ &\le C_1 \|u_s^0\|_2 \;+\; C_2 \|u_p^2 u_s\|_{\gamma', \rho'} \end{aligned}$$

Estimation of the cubic term proceeds as follows. By the Cauchy-Schwarz inequality

$$
\begin{aligned}
\|u_p^2 u_s\|_{\gamma',\rho'} &= \left[\int_0^T \left(\int |u_p^2 u_s|^{\rho'} dx\right)^{\gamma'/\rho'} dt\right]^{1/\gamma'} \\
&\leq \left[\int_0^T \left(\int |u_p|^{4\rho'} dx\right)^{\gamma'/\rho'} dt\right]^{1/2\gamma'} \left[\int_0^T \left(\int |u_s|^{2\rho'} dx\right)^{\gamma'/\rho'} dt\right]^{1/2\gamma'}
\end{aligned}
$$

Set $\rho' = 1$ and therefore [1] $\gamma' = 4/3$. Then the last expression becomes

$$
= \left[\int_0^T \left(\int |u_p|^4 dx\right)^{4/3} dt\right]^{3/8} \left[\int_0^T \left(\int |u_s|^2 dx\right)^{4/3} dt\right]^{3/8} \leq \tag{46}
$$

and by Hölder's inequality, applied to each factor, we have the bound

$$
\begin{aligned}
&\left[\left[\int_0^T \left(\int |u_p|^4 dx\right)^2 dt\right]^{(2/3)\cdot(3/8)} \left[\int_0^T 1^3 dt\right]^{(1/3)}\right]^{3/8} \left[\sup_t ||u_s(t)||_2^{2\cdot(4/3)} T\right]^{3/8} \\
&\leq \|u_p\|_{8,4}^2 ||u_s||_{\infty,2} T^{1/2}.
\end{aligned}
$$

Adding up all the terms, we obtain

$$
\begin{aligned}
\left\|A_\beta^{(s)}(u_s,u_p)\right\|_{\mathcal{X}(T)} &\leq 2C\left(\|u_s(0)\|_2 + \|u_p\|_{8,4}^2 \|u_s\|_{\infty,2}\, T^{\frac{1}{2}}\right) \\
\left\|\, A_\beta^{(p)}(u_s,u_p)\,\right\|_{\mathcal{X}(T)} &\leq 2C\left(\|u_p(0)\|_2 + \|u_s\|_{8,4}^2 \|u_p\|_{\infty,2} T^{\frac{1}{2}}\right).
\end{aligned}
$$

Finally, combining the last two terms, we have

$$
||A_\beta(u_s,u_p)||_{\mathcal{X}(T)} \leq 4C\left(||u_s(0)||_2 + ||u_p(0)||_2 + 2||(u_s,u_p)||_{\mathcal{X}(T)}^3\, T^{\frac{1}{2}}\right). \tag{47}
$$

Assume now that $||(u_s,u_p)||_{\mathcal{X}(T)} \leq 8C(||u_s(0)||_2 + ||u_p(0)||_2)$ and that T is sufficiently small, then A_β maps $\mathcal{B}(T)$ into itself. This completes the proof of Proposition 2.3.

Proposition 2.4 *For $T < T_1 \leq T_0$ sufficiently small, the transformation, A_β, is a contraction on $\mathcal{B}(T)$. That is,*

$$
\|A_\beta(u_s,u_p) - A_\beta(v_s,v_p)\|_{\mathcal{X}(T)} \leq q\ \|(u_s - v_s, u_p - v_p)\|_{\mathcal{X}(T)}, \tag{48}
$$

where $0 < q < 1$.

Proof of Proposition 2.4: Consider the first component of the map. By Corollary 2.1

$$
\begin{aligned}
\left\|A_\beta^{(s)}(u_s,u_p) - A_\beta^{(s)}(v_s,v_p)\right\|_{q,r} &\leq C_2 \left\|\, |u_p|^2 u_s - |v_p|^2 v_s\, \right\|_{\frac{4}{3},1} \\
&\leq \left\|\, u_p^2(u_s - v_s)\, \right\|_{\frac{4}{3},1} + \|u_p v_s(u_p - v_p)\|_{\frac{4}{3},1} + \|v_p v_s(u_p - v_p)\|_{\frac{4}{3},1}.
\end{aligned}
$$

[1] Indeed, since $1/\rho' + 1/\rho = 1, 1/\gamma + 1/\gamma' = 1$, and $2/\gamma = 1/2 - 1/\rho$, we have $\rho = \infty$, $\gamma = 4$, and $\gamma' = 4/3$.

These terms are all estimated in a similar manner. We focus on the second term. First, by the Cauchy-Schwarz inequality

$$
\begin{aligned}
\|u_p v_s(u_p - v_p)\|_{\frac{4}{3},1} &\leq \left[\int_0^T \left(\int |u_p|^2 |v_s|^2 dx\right)^{4/3} dt\right]^{3/8} \left[\int_0^T \left(\int |u_p - v_p|^2 dx\right)^{4/3} dt\right]^{3/8} \\
&\leq \left[\int_0^T \left(\int |u_p|^2 |v_s|^2 dx\right)^{4/3} dt\right]^{3/8} T^{3/8} \, ||u_p - v_p||_{\infty,2}
\end{aligned}
$$

Another application of the Cauchy-Schwarz inequality to the spatial integral in the first factor in the previous expression and then Hölder's inequality to the time integral, gives

$$
\begin{aligned}
&\left[\int_0^T \left(\int |u_p|^4 dx\right)^{4/3} dt\right]^{3/16} \left[\int_0^T \left(\int |v_s|^4 dx\right)^{4/3} dt\right]^{3/16} T^{3/8} ||u_p - v_p||_{\infty,2} \\
\leq\ & \|u_p\|_{8,4} \, \|v_s\|_{8,4} \, T^{\frac{2}{16}} \, T^{\frac{3}{8}} \, \|u_p - v_p\|_{\infty,2} \ = \ \|u_p\|_{8,4} \, \|v_s\|_{8,4} \, T^{1/2} \, ||u_p - v_p||_{\infty,2} \\
\leq\ & ||u_p||_{8,4} ||v_s||_{8,4} T^{1/2} ||(u_s - v_s, u_p - v_p)||_{\mathcal{X}(T)}.
\end{aligned}
$$

Adding the estimates for

$$
\left\| A_\beta^{(s)}(u_s, u_p) - A_\beta^{(s)}(v_s, v_p) \right\|_{\mathcal{X}(T)} \quad \text{and} \quad \left\| A_\beta^{(p)}(u_s, u_p) - A_\beta^{(p)}(v_s, v_p) \right\|_{\mathcal{X}(T)} \tag{49}
$$

and choosing, if necessary, $T_1 < T_0$, we obtain the contraction estimate This completes the proof.

2.4 Existence in $H^1(\mathcal{R})$

We consider the existence theory in H^1. In this section we prove

Theorem 2.6 *Let $(u_s(0), u_p(0))$ be in H^1 and let the potential V satisfy (10), then there exists a unique global solution for the system (2) in $L^\infty(\mathcal{R}, H^1)$.*

We first observe that our proof of local existence, via the contraction mapping principle, extends to the space

$$
\mathcal{X}_1(T) \equiv C([0,T], H^1) \cap \mathcal{X}(T). \tag{50}
$$

In particular, one need only prove that A_β maps a ball to a ball in this smaller space and is a contraction mapping there, for $T < T_2$, where $T_2 \leq T_1 \leq T_0$. This can be proven by applying $(H+i)^{\frac{1}{2}}$ to the equations, using equivalence of norms: $||(H+i)^{\frac{1}{2}} \cdot ||_{L^2}$ and $|| \cdot ||_{H^1}$ and carrying out the standard energy estimates. We use $(H+i)$ rather than $I - \Delta$ because functions of H commute with H and thus we avoid differentiation of the potential $V(x)$. Otherwise, we would require bounds on norms of $\partial_x V$.

If $T^*_{\max}$ denotes the maximal time of existence for the solution in $\mathcal{X}_1(T)$, then in view of the *á priori* estimates in $\mathcal{X}(T)$, global existence ($T^*_{\max} = \infty$) will follow from *á priori* bounds on (u_s, u_p) in H^1.

Applying to the system (18-19) operator $(HP_c + I)^{1/2}$, we obtain the system of PDE's':

$$\frac{\partial}{\partial t}\int \left(|(HP_c+I)^{1/2}u_s|^2 + |(HP_c+I)^{1/2}u_s|^2\right) dx \leq \tag{51}$$

$$\int \left(\overline{(HP_c+I)^{1/2}u_s}(HP_c+I)^{1/2}(|u_p|^2u_s) - \overline{(HP_c+I)^{1/2}u_p}(HP_c+I)^{1/2}(|u_s|^2u_p)\right) dx \tag{52}$$

Proposition 2.5 *Assume that V satisfies condition (10). Then the operator $(HP_c + I)^{1/2}(I - \Delta)^{-1/2}$ and its inverse are bounded in L^2, i.e. for any $f \in L^2$ both $(HP_c + I)^{1/2}(I - \Delta)^{1/2}f$ and $(I - \Delta)^{1/2}(HP_c + I)^{-1/2}f$ are bounded in L^2 and*

$$||(HP_c+I)^{1/2}(I-\Delta)^{-1/2}f||_{L^2} \leq C||f||_{L^2}$$

$$||(I-\Delta)^{1/2}(HP_c+I)^{-1/2}f||_{L^2} \leq C||f||_{L^2}$$

Proof: This proposition states that the $||\cdot||_{L^2}$ norm and $||(HP_c+I)^{1/2}\cdot||_{L^2}$ are equivalent. Then our strategy will be similar to the proof in the potential free case (as if $(HP_c + I)^{1/2}$ were ∂_x).

We prove the proposition using Weder's result on the continuity of wave operators [8]. Under the conditions stated above Weder proves that there exists wave operator Ω such that

$$\Omega(I-\Delta)\Omega^* = (H\mathbf{P_c} + I),$$

where Ω is a bounded continuous operator on H^1. Then, taking the square root, we obtain

$$\Omega(I-\Delta)^{-1/2}\Omega^* = (H\mathbf{P_c}+I)^{-1/2}, \tag{53}$$

which implies the result.

Corollary 2.2 *Let $f \in H^1$ then*

$$||(HP_c+I)^{1/2}f||_2 \leq C||(I-\Delta)^{1/2}f||_2$$
$$||(I-\Delta)^{1/2}f||_2 \leq ||(HP_c+I)^{1/2}f||_2.$$

First, using the above proposition, we estimate

$$\left|\int \left(\overline{(HP_c+I)^{1/2}u_s}(HP_c+I)^{1/2}(|u_p|^2u_s)\right) dx\right| \leq ||(HP_c+I)^{1/2}u_s||_2 \cdot ||(HP_c+I)^{1/2}(|u_p|^2u_s)||_2 =$$
$$||(HP_c+I)^{1/2}u_s||_2 \cdot ||(HP_c+I)^{1/2}(I-\Delta)^{-1/2}(I-\Delta)^{1/2}(|u_p|^2u_s)||_2 \leq$$
$$||(HP_c+I)^{1/2}u_s||_2 \cdot ||(HP_c+I)^{1/2}(I-\Delta)^{-1/2}||_{\mathcal{B}(L^2,L^2)} \cdot ||(I-\Delta)^{1/2}(|u_p|^2u_s)||_2.$$

Using Leibnitz formula for the fraction, see [9]

$$||(I-\Delta)^{1/2}(fg)||_2 \leq ||f||_\infty \cdot ||(I-\Delta)^{1/2}g||_2 + ||(I-\Delta)^{1/2}f||_2 \cdot ||g||_\infty,$$

we obtain

$$||(I-\Delta)^{1/2}(|u_p|^2u_s)||_2 \leq ||u_p||_\infty^2||(I-\Delta)^{1/2}u_s||_2 + 2||u_p||_\infty||u_s||_\infty||(I-\Delta)^{1/2}u_p||_2.$$

Combining this estimate, we obtain

$$\left|\int \left(\overline{(HP_c+I)^{1/2}u_s}(HP_c+I)^{1/2}(|u_p|^2u_s)\right)dx\right| \leq$$

$$C||(HP_c+I)^{1/2}u_s||_2 \cdot (||u_p||_\infty^2 + ||u_s||_\infty^2)(||(I-\Delta)^{1/2}u_p||_2 + ||(I-\Delta)^{1/2}u_s||_2) \leq$$
$$C\left(||(HP_c+I)^{1/2}u_s||_2^2 + ||(HP_c+I)^{1/2}u_p||_2^2\right) \cdot \left(||u_p||_\infty^2 + ||u_s||_\infty^2\right).$$

Finally adding the s and p components of the differential inequalities, we obtain:

$$\partial_t(||(HP_c+I)^{1/2}u_s||_2^2+||(HP_c+I)^{1/2}u_p||_2^2) \leq (||u_p||_\infty^2+||u_s||_\infty^2)\cdot(||(HP_c+I)^{1/2}u_s||_2^2+||(HP_c+I)^{1/2}u_p||_2^2)$$

which implies that

$$||(HP_c+I)^{1/2}u_s||_2^2 + ||(HP_c+I)^{1/2}u_p||_2^2 \leq$$
$$\exp\left(C\int_0^T \left[||u_s||_{L^\infty}^2 + ||u_p||_{L^\infty}^2\right]\right)\left(||(HP_c+I)^{1/2}u_s(0)||_2^2 + ||(HP_c+I)^{1/2}u_p(0)||_2^2\right)|.$$

Applying Hölder's inequality to the time integral in the exponent we have

$$\begin{aligned}
\int_0^T \left[||u_s||_{L^\infty}^2 + ||u_p||_{L^\infty}^2\right]dt &\leq C||u_s||_{L^4([0,T],L^\infty)}^2T^{\frac{1}{2}} + C||u_p||_{L^4([0,T],L^\infty)}^2T^{\frac{1}{2}} \\
&= C\left(||u_s||_{L^4([0,T],L^\infty)}^2 + ||u_p||_{L^4([0,T],L^\infty)}^2\right)T^{\frac{1}{2}} \\
&\leq C\ \mathcal{P}_0T^{\frac{1}{2}}\ T^{\frac{1}{2}}\ =\ C\ \mathcal{P}_0\ T.
\end{aligned}$$

The last inequality follows from the *á priori* space-time estimate of Theorem 2.3 and the fact that $(4,\infty)$ is an admissible pair. We, thus, establish the boundedness of (u_s,u_p) in $||(HP_c+I)(*)||_2$ norm

$$||(HP_c+I)^{1/2}u_s||_2^2+||(HP_c+I)^{1/2}u_p||_2^2 \leq K_1e^{K_2\mathcal{P}_0T}\left(||(HP_c+I)^{1/2}u_s(0)||_2^2 + ||(HP_c+I)^{1/2}u_p(0)||_2^2\right)$$

Therefore, using the equivalence of norms, see Corollary 2.2, we obtain

$$\|u_s(t)\|_{H^1} + \|u_p(t)\|_{H^1} \leq \tilde{K}_1e^{K_2\mathcal{P}_0T}\ \left(\|u_s(0)\|_{H^1} + \|u_p(0)\|_{H^1}\right). \tag{54}$$

for some $K_1, K_2 > 0$. This completes the proof of global existence in H^1.

3 Energy transfer from the guided mode to radiation modes

In this section, we prove that, over time scales of interest ($t \leq \mathcal{O}(\epsilon^{-1})$), radiation terms remain small during the amplification process and the finite dimensional model (3) is a valid approximation. In the discussion of this section we return to the time-scale, where nonlinear terms are of order ϵ:

$$\begin{aligned} i\partial_t u_s - Hu_s &= i\epsilon|u_p|^2 u_s && (55)\\ i\partial_t u_p - Hu_p &= -i\epsilon|u_s|^2 u_p. && (56)\end{aligned}$$

For this system, we are going to show that radiation is indeed bounded by $C\epsilon$ on a time scale of order $1/\epsilon$.

To proceed, we first orthogonally decompose a solution of (55- 56) into its bound state and continuous spectral (radiative) parts:

$$\begin{aligned} u_s(x,t) &= a_s(t)\phi(x) + U_s(x,t) && (57)\\ u_p(x,t) &= a_p(t)\phi(x) + U_p(x,t). && (58)\end{aligned}$$

We prove

Theorem 3.1 *Let $(u_s(0), u_p(0)) \in H^1$ and $\mathbf{P_c} u_s(0) = \mathbf{P_c} u_p(0) = 0$. Assume that $0 < \epsilon < \epsilon_0 < \infty$, V has no zero resonance and it satisfies (10) then for any $T > 0$ there exists $C(T, \epsilon_0)$ so that*

$$\max\{||U_s(t)||_{H^1}, ||U_p(t)||_{H^1}\} \leq C\epsilon \tag{59}$$

on the interval $t \in [0, T/\epsilon]$.

We begin the proof with the following proposition, which follows from the H^1 bounds on $u_{s,p}$, (54).

Proposition 3.1 *Let $0 < \epsilon < \epsilon_0 < \infty$ Then for any $T > 0$ there exists $C(T, \epsilon_0)$ such that*

$$||U_s||_{H^1}, ||U_p||_{H^1}, |a_s|, |a_p| \leq C \tag{60}$$

on the interval $t \in [0, T/\epsilon]$.

Substitution of (57-58) into (55-56) and projection onto ϕ and the range of $\mathbf{P_c}$ gives

$$\begin{aligned}
i\partial_t a_s - \lambda a_s &= i\epsilon\left[|a_p|^2 a_s\langle\phi^3|\phi\rangle + \overline{a_p}a_s\langle\phi^3|U_p\rangle + a_p a_s\langle\phi^3|\overline{U_p}\rangle + |a_p|^2\langle\phi^3|U_s\rangle + ... + \langle|U_p|^2 U_s|\phi\rangle\right] && (61)\\
i\partial_t a_p - \lambda a_p &= -i\epsilon\left[|a_s|^2 a_p\langle\phi^3|\phi\rangle + \overline{a_s}a_p\langle\phi^3|U_s\rangle + a_p a_s\langle\phi^3|\overline{U_s}\rangle + |a_s|^2\langle\phi^3|U_p\rangle + ... + \langle|U_s|^2 U_p|\phi\rangle\right] && (62)\\
i\partial_t U_s - HU_s &= i\epsilon\left[a_s|a_p|^2\mathbf{P_c}\phi^3 + \overline{a_p}a_s\mathbf{P_c}\phi^2 U_p + a_p a_s\mathbf{P_c}\phi^2\overline{U_p} + |a_p|^2\mathbf{P_c}\phi^2 U_s + ... + \mathbf{P_c}|U_p|^2 U_s\right] && (63)\\
i\partial_t U_p - HU_p &= -i\epsilon\left[a_p|a_s|^2\mathbf{P_c}\phi^3 + \overline{a_s}a_p\mathbf{P_c}\phi^2 U_s + a_p a_s\mathbf{P_c}\phi^2\overline{U_s} + |a_s|^2\mathbf{P_c}\phi^2 U_p + ... + \mathbf{P_c}|U_s|^2 U_p\right], && (64)
\end{aligned}$$

where $H = -\partial_x^2 + V(x)$ and $H\phi = \lambda\phi$.

Corollary 3.1 *The fundamental modes are slowly varying with the rate at most ϵ*

$$|\partial_t |a_s||, |\partial_t |a_p|| \leq C\epsilon.$$

Proof: This follows from Proposition 3.1 and equations (61-62). Indeed, all terms on the right hand-side are bounded since $U_p, U_s, \phi \in H^1$. The terms on the left hand-side can change only phase (not the amplitude). This ends the proof of the Corollary.

The following estimates are used in the proof of the theorem and can be easily verified.

Lemma 3.1

$$||\mathbf{P_c} f||_2 \leq ||f||_2$$

$$||\mathbf{P_c} f||_\infty \leq ||f||_\infty (1 + ||\phi||_1 \cdot ||\phi||_\infty)$$

$$\left|\left|(H-\lambda)^{-1} e^{i(H-\lambda)t} \mathbf{P_c} f\right|\right|_2 \leq C||f||_2.$$

$$||(H-\lambda)^{-1}\mathbf{P_c}||_{H^1} \leq \frac{1}{\mathrm{dist}(H_c, \lambda)} \leq \frac{1}{|\lambda|}$$

Proof of the Theorem 3.1:
We now make transformation $a_s = e^{-i\lambda t} A_s$ and $U_s = \epsilon e^{-iHt} W_s$ to remove rapid oscillations and explicitly show the smallness of radiation. By hypothesis of Theorem 3.1, we have $||W_{s,p}(0)||_{H^1} \leq C$. Note that by the bounds of Proposition 3.1 we have $||W_{s,p}||_{H^1} \leq C/\epsilon$ and $|A_{s,p}| \leq C$.
The slowly varying amplitudes A_s, A_p satisfy

$$\begin{aligned} \partial_t A_s = {} & \epsilon |A_p|^2 A_s \langle \phi^3 | \phi \rangle + \epsilon^2 \overline{A_p} A_s \langle \phi^3 | e^{-i(H-\lambda)t} W_p \rangle + \epsilon^2 A_p A_s \langle \phi^3 | e^{i(H-\lambda)t} \overline{W_p} \rangle + \\ & \epsilon^2 |A_p|^2 \langle \phi^3 | e^{-i(H-\lambda)t} W_s | \phi \rangle + \ldots + \epsilon^4 \langle |e^{-iHt} W_p|^2 e^{-i(H-\lambda)t} W_s \rangle \end{aligned} \tag{65}$$

$$\begin{aligned} \partial_t A_p = {} & -\epsilon |A_s|^2 A_p \langle \phi^3 | \phi \rangle - \epsilon^2 \overline{A_s} A_p \langle \phi^3 | e^{-i(H-\lambda)t} W_s \rangle - \epsilon^2 A_p A_s \langle \phi^3 | e^{i(H-\lambda)t} \overline{W_s} \rangle - \\ & \epsilon^2 |A_s|^2 \langle \phi^3 | e^{-i(H-\lambda)t} W_p \rangle - \ldots - \epsilon^4 \langle |e^{-iHt} W_s|^2 e^{-i(H-\lambda)t} W_p | \phi \rangle. \end{aligned} \tag{66}$$

Further $W_{s,p}$ satisfy

$$\partial_t W_s = e^{i(H-\lambda)t} A_s |A_p|^2 \mathbf{P_c} \phi^3 +$$

$$\begin{aligned} e^{iHt} \Big[& \epsilon \overline{A_p} A_s \mathbf{P_c} \phi^2 e^{-iHt} W_p + \epsilon A_p A_s \mathbf{P_c} \phi^2 e^{iHt} \overline{W_p} + \epsilon |A_p|^2 \mathbf{P_c} \phi^2 e^{-iHt} W_s + \ldots + \\ & \epsilon^3 \mathbf{P_c} |e^{-iHt} W_p|^2 e^{-iHt} W_s \Big] \end{aligned} \tag{67}$$

$$\partial_t W_p = e^{i(H-\lambda)t} A_p |A_s|^2 \mathbf{P_c} \phi^3 +$$

$$\begin{aligned} e^{iHt} \Big[& \epsilon \overline{A_s} A_p \mathbf{P_c} \phi^2 e^{-iHt} W_s + \epsilon A_p A_s \mathbf{P_c} \phi^2 e^{iHt} \overline{W_s} + \epsilon |A_s|^2 \mathbf{P_c} \phi^2 e^{-iHt} W_p + \ldots + \\ & \epsilon^3 \mathbf{P_c} |e^{-iHt} W_s|^2 e^{-iHt} W_p \Big] \end{aligned} \tag{68}$$

The goal is now to show that given initial data where W is $\mathcal{O}(1)$ (which corresponds to radiation of order ϵ) during the evolution W will remain $\mathcal{O}(1)$ on time interval of length $1/\epsilon$.

In order to do this we integrate the above equations and using that $W_{p,s}(0) = \mathcal{O}(\epsilon)$, we obtain

$$W_s(t) = W_s(0) + \int_0^{T/\epsilon} e^{i(H-\lambda)s} A_s|A_p|^2 \mathbf{P_c}\phi^3 ds + \epsilon \int_0^{T/\epsilon} R_s ds, \tag{69}$$

where ϵR_s is the $\epsilon-$order part in equation (67), *i.e.* the second and the third lines. Integrating by parts, we obtain

$$W_s(t) = \frac{e^{i(H-\lambda)T/\epsilon} - 1}{i(H-\lambda)} A_s|A_p|^2 \mathbf{P_c}\phi^3 - \int_0^{T/\epsilon} \frac{e^{i(H-\lambda)s}}{i(H-\lambda)} \partial_s(A_s|A_p|^2)\mathbf{P_c}\phi^3 ds + \epsilon \int_0^{T/\epsilon} R_s ds. \tag{70}$$

Now, we estimate

$$\mathrm{sup}_{t\in[0,T/\epsilon]} ||W_s(t)||_{H^1} \leq \frac{C}{|\lambda|}||\phi^3||_{H^1} + \frac{CT}{|\lambda|}||\phi^3||_{H^1} + T\mathrm{sup}_{t\in[0,T/\epsilon]}||R_s||_{H^1}. \tag{71}$$

It is easy to see that since $||W_{s,p}||_{H^1} \leq C/\epsilon$ and using the property $||f \cdot g||_{H^s} \leq ||f||_{H^s} \cdot ||g||_{H^s}$ for $s \geq n/2$ the last term can be estimated $||R_s||_{H^1} \leq C$, which proves the bound $||W_s||_{H^1} \leq C$. The corresponding bound on W_p is obtained in a similar way.

4 Two-dimensional problem

We now consider the Raman system in the case of two transverse spatial dimensions

$$i\partial_t u_s \quad - \quad \beta H u_s = i|u_p|^2 u_s \tag{72}$$
$$i\partial_t u_s \quad - \quad \beta H u_s = -i|u_s|^2 u_p, \tag{73}$$

where $H = -\Delta + V(x,y)$ and prove analogous existence results and energy transfer estimates. Our strategy in the two-dimensional case is similar to the one-dimensional case, therefore, we omit some calculations which can be found in the previous sections.

In two-dimensional case we require stronger conditions on the potential.

Assumption 4.1 *The potential $V(x,y)$ is twice differentiable and*

$$|D^\alpha V| \leq C_\alpha(1 + x^2 + y^2)^{-a},$$

where $a > 6$ and $|\alpha| \geq 2$.

Assumption 4.2 *We assume that potential $V(x,y)$ has no zero energy eigenvalues or resonances.*

These assumptions are required to obtain space-time estimates in the next section.

4.1 Space-time estimates for the propagator

The definition of admissible pair is modified:
(q,r) is admissible (in dimension $n=2$) if

$$\frac{1}{q}=\frac{1}{2}-\frac{1}{r}, \qquad r\in[2,\infty].$$

Theorem 4.1 *Assume that the potential V satisfies both assumptions and let (q,r) be an admissible pair. Then for any $f\in L^2$ we have $U_0(t)f$ and $U(t)\mathbf{P_c}f$ are in $L^q(\mathcal{R},L^r)$ and*

$$\begin{aligned} \|U_0(\cdot)f\|_{L^q(R,L^r)} &\le C\ \|f\|_{L^2}, \\ \|U(\cdot)P_cf\|_{L^q(R,L^r)} &\le C\ \|f\|_{L^2}, \end{aligned} \tag{74}$$

where C depends only on q.

Theorem 4.2 *Assume that the potential V satisfies both assumptions and let (γ,ρ) be an admissible pair and let $f\in L^{\gamma'}([0,T],L^{\rho'})$, where (γ',ρ') is conjugate to (γ,ρ), then for any admissible pair (q,r)*

$$\begin{aligned} \left\|\int_0^t U_0(t-\tau)f(\tau)d\tau\right\|_{L^q([0,T],L^r)} &\le C\|f\|_{L^{\gamma'}([0,T],L^{\rho'})}, \\ \left\|\int_0^t U(t-\tau)\mathbf{P_c}f(\tau)d\tau\right\|_{L^q([0,T],L^r)} &\le C\|f\|_{L^{\gamma'}([0,T],L^{\rho'})}, \end{aligned} \tag{75}$$

where C depends only on q, γ.

Both Theorems 4.1 and 4.2 are proven by using Yajima's results [12] on $W^{k,p}$ continuity of wave operators. The argument follows the proof of Proposition 2.5 and is omitted here. Finally, Corollary 2.1 is valid in the current setting without any changes as the scaling is independent of the space dimension.

Remark 4.1 *The application of Yajima's results on $W^{k,p}$ continuity of wave operators is the origin of the more restrictive smoothness assumptions on the potential $V(x)$. In one space dimension smoothness of $V(x)$ is not required [8].*

4.2 *Á priori* space-time estimates

The same argument as in one-dimensional case applies here and we obtain the *á priori* bound of Proposition 2.1, as well as the bound (31) on nonlinear terms.

Theorem 4.3 (Á priori bounds in $L^q([0,T],L^r)$) *Let (q,r) be an admissible pair. Then any solution (u_s,u_p) satisfies the bounds*

$$||u_s||_{L^q([0,T],L^r)} \leq C{\mathcal{P}_0}^{\frac{1}{2}} \tag{76}$$
$$||u_p||_{L^q([0,T],L^r)} \leq C{\mathcal{P}_0}^{\frac{1}{2}} \tag{77}$$

Proof: As in the one-dimensional case, this estimate is proved by a straightforward application of space-time estimate for Schrödinger equation with a potential and using *á priori* estimate on nonlinear terms. Note, however that in two-dimensional case the bound is uniform in time, T. This seems to be related to the criticality of the fourth power in two-dimensional case.[2]

4.3 Local existence in $H^2(\mathcal{R}^2)$.

We now prove local existence of solutions in H^2 and will extend it to a global solution using our space time estimates.

Theorem 4.4 *(***Local Existence***)*
(1) Given initial data $(u_s(0),u_p(0)) \in H^2(\mathcal{R}^2)$, there exists $T>0$ and a unique solution $(u_s,u_p) \in L^\infty([0,T],H^2(\mathcal{R}^2))$.
(2) Let $T_{\max}>0$ denote the maximal time of existence. Then, either $T_{\max}=\infty$ (global existence in time) or

$$T_{\max} < \infty \text{ and } \limsup_{t\to T_{\max}} ||(u_s,u_p)||_{L^\infty([0,T],H^2(\mathcal{R}^2))} = \infty$$

To prove the local existence theorem we have to show that a ball in H^2 is mapped into itself and that the mapping is a contraction.

Consider the same mapping as in the one-dimensional case (41)-(42). We will first show that it maps a ball into a ball.

$$||A_\beta^{(s)}(u_s,u_p)||_{H^2} \leq ||U_\beta(t)u_s(0)||_{H^2} + \left|\left|\int_0^t U_\beta(t-\tau)|u_p|^2u_s\right|\right|_{H^2} \tag{78}$$

The first term is estimated as follows

$$||U_\beta(t)u_s(0)||_{H^2} = ||(I-\Delta)U_\beta(t)u_s(0)||_{L^2} = ||(I-\Delta)(H+i)(H+i)^{-1}U_\beta(t)u_s(0)||_{L^2} \leq$$
$$||(I-\Delta)(H+i)^{-1}||_{\mathcal{B}(L^2,L^2)}||(H+i)U_\beta(t)u_s(0)||_{L^2}.$$

Note that the operator $(I-\Delta)(H+i)^{-1}$ is bounded in L^2-operator norm. this follows from the identity

$$(I-\Delta)(H+i)^{-1} = -I+(I+i-V)(H+i)^{-1},$$

[2]Indeed, the corresponding Strichartz estimate takes the form $\int_{\mathcal{R}^3}|U_0f|^4dxdydt \leq C\int_{\mathcal{R}^2}|f|^2dxdy$.

the fact that $V \in L^\infty$, the boundedness of $(H+i)^{-1}$ in L^2.

Next, we have to establish the bound for

$$||(H+i)U_\beta(t)u_s(0)||_{L^2} = ||(H+i)u_s(0)||_{L^2} = ||(H+i)(I-\Delta)^{-1}(I-\Delta)u_s(0)||_{L^2} \leq$$

$$||(H+i)(I-\Delta)^{-1}||_{\mathcal{B}(L^2,L^2)}||(I-\Delta)u_s(0)||_{L^2} \leq C||u_s(0)||_{H^2},$$

where the operator $(H+i)(I-\Delta)^{-1}$ is bounded by similar calculations as for $(I-\Delta)(H+i)^{-1}$.

Now, we estimate the second term in (78):

$$\left|\left|\int_0^t U_\beta(t-\tau)|u_p|^2 u_s d\tau\right|\right|_{H^2} \leq \left|\left|\int_0^t (H+i)U_\beta(t-\tau)|u_p|^2 u_s d\tau\right|\right|_{L^2}.$$

To bound this term we write its square

$$\int dx \left\{\int_0^t\int_0^t (H+i)U_\beta(t-\tau_1)|u_p(\tau_1)|^2 u_s(\tau_1)(H-i)\overline{U_\beta}(t-\tau_2)|u_p(\tau_2)|^2\overline{u_s}(\tau_2)d\tau_1 d\tau_2 \leq\right\}$$

(using Holder inequality and isometry of U_β in L^2)

$$\int_0^t\int_0^t d\tau_1 d\tau_2 \left|\int \left|(H+i)|u_p(\tau_1)|^2 u_s(\tau_1)\right|^2 dx\right|^{\frac{1}{2}} \left|\int \left|(H+i)|u_p(\tau_2)|^2 u_s(\tau_2)\right|^2 dx\right|^{\frac{1}{2}} \leq$$

$$t^2 \sup_{\tau\in[0,t]} ||(H+i)u_p^2(\tau)u_s(\tau)||_2 \leq Ct^2 \sup_{\tau\in[0,t]} ||u_p^2(\tau)u_s(\tau)||_{H^2}^2 \leq Ct^2 \sup_{\tau\in[0,t]} ||u_p(\tau)||_{H^2}^4||u_s(\tau)||_{H^2}^2$$

Therefore, we finally obtain the bound on the s-part of the map

$$||A_\beta^{(s)}(u_s,u_p)||_{H_2} \leq C||u_s(0)||_{H^2} + Ct \sup_{\tau\in[0,t]} ||u_p(\tau)||_{H^2}^2||u_s(\tau)||_{H^2}$$

and the full map

$$||A_\beta(u_s,u_p)||_{L^\infty([0,t],H^2)} \leq C\left(||u_s(0)||_{H^2} + ||u_p(0)||_{H^2}\right) +$$

$$Ct\left(||u_p||_{L^\infty([0,t],H^2)}^2||u_s||_{L^\infty([0,t],H^2)} + ||u_p||_{L^\infty([0,t],H^2)}||u_s||_{L^\infty([0,t],H^2)}^2.\right)$$

With the obtained inequality it is easy to establish

Proposition 4.1 *Let $(u_s(0),u_p(0))$ be in H^2. Define the ball in $L^\infty([0,T],H^2)$ as*

$$\mathcal{B}(T) = \left\{(u_s,u_p) \in L^\infty([0,T],H^2) \ : \ ||(u_s,u_p)||_{L^\infty([0,T],H^2)} \leq 4C(||u_s(0)||_{H^2} + ||u_p(0)||_{H^2})\right\}.$$

then there exists $T_0 > 0$ such that for any $T < T_0$, the ball is mapped into itself, i.e. $A_\beta(\mathcal{B}(T)) \subset \mathcal{B}(T)$ with T_0 independent of β.

Next, we have to show that the mapping is a contraction in $L^\infty([0,T],H^2)$. Consider the first component of the map applied to two different pairs (u_s,u_p) and (v_s,v_p) with the same initial data $(u_s(0),u_p(0)) = (v_s(0),v_p(0))$:

$$||A_\beta^{(s)}(u_s,u_p) - A_\beta^{(s)}(v_s,v_p)||_{L^\infty([0,t],H^2)} \leq CT \sup_{t\in[0,T]} |||u_p(t)|^2 u_s(t) - |v_p(t)|^2 v_s(t)||_{H^2} \leq$$

$$CT \sup_{t\in[0,T]} \Big(\| \; u_p^2(u_s - v_s) \; \|_{H^2} \; + \; \|u_p v_s(u_p - v_p)\|_{H^2} \; + \; \|v_p v_s(u_p - v_p)\|_{H^2} \Big) \leq$$

$$C(||u_p,u_s,v_p,v_s||_{L^\infty([0,T],H^2)})T||(u_p - v_p, u_s - v_s)||_{L^\infty([0,T],H^2)}$$

Adding both s and p components of the map we obtain

$$||A_\beta(u_p,u_s) - A_\beta(v_p,v_s)||_{L^\infty([0,T],H^2)} \leq TC(||u_p,u_s,v_p,v_s||_{L^\infty([0,T],H^2)})||(u_p - v_p, u_s - v_s)||_{L^\infty([0,T],H^2)}.$$

This inequality implies

Proposition 4.2 *There exists $T_1 : 0 < T_1 < T_0$ sufficiently small such that the map A_β is a contraction in the ball $\mathcal{B}(T)$ for any $T : 0 < T < T_1$*

$$||A_\beta(u_p,u_s) - A_\beta(v_p,v_s)||_{L^\infty([0,T],H^2)} \leq q||(u_p - v_p, u_s - v_s)||_{L^\infty([0,T],H^2)},$$

where $q < 1$.

Remark 4.2 *Whenever there exists a local solution on $t \in [0,T]$, it is bounded in $L^q([0,T],L^r(\mathcal{R}^2))$. Indeed, a solution in $L^\infty([0,T],H^2)$ is also in $L^q([0,T],L^r(\mathcal{R}^2))$ and the earlier obtained* á priori *bounds apply.*

4.4 Global existence in $H^2(\mathcal{R}^2)$

In this section we will show that the local solution obtained via the contraction mapping principle in the previous section can be extended to a global solution using space-time estimates. We start with establishing uniform bound in H^1 space.

***Á priori* estimates in H^1** Proceeding as in the one-dimensional problem, we obtain H^1 uniform bound. Since $(\infty, 2)$ is an admissible pair, we obtain that the solutions are even uniformly bounded in time on the interval of existence of a local solution

$$||(u_s,u_p)||_{L^\infty([0,T],H^1)} \leq C\,(||u_s(0),u_p(0)||_2)\;\; ||(u_s(0),u_p(0))||_{H^1}, \tag{79}$$

where C is independent of time. Continuation to a global solution using Theorem 4.4 requires an H^2 estimate, which we now derive.

Global existence in H^2 Having established uniform bound in H^1, we are ready to obtain a bound in H^2, which will rule out the first alternative in the Theorem 4.4 (on local existence) and, thus, leave the global existence as the only possibility.

Applying $H+I$ to the s-component of (73) and using energy estimates, we obtain:

$$\partial_t \int |(H+I)u_s|^2 \leq \int |(H+I)u_s| \cdot |(H+I)(|u_p|^2 u_s)| \leq ||(H+I)u_s||_2 ||(H+I)(|u_p|^2 u_s)||_2. \quad (80)$$

To estimate the last term, we write

$$\int |\Delta(|u_p|^2 u_s)|^2 = \int |u_p|^4 |\Delta u_s|^2 + 2\int |\nabla u_p|^2 |\nabla u_s|^2 |u_p|^2 + \int |\nabla u_p|^4 |u_s|^2 \leq$$

$$C(||u_p||_\infty^4 \cdot ||u_s||_{H^2}^2 + ||u_s||_\infty^2 \cdot ||u_p||_{H^2}^2 \cdot ||u_p||_{H^1}^2 + ||u_p||_\infty^2 \cdot ||u_s||_{H^2}^2 \cdot ||u_s||_{H^1}^2 + ||u_p||_\infty^2 \cdot ||u_p||_{H^2}^2 \cdot ||u_p||_{H^1}^2)$$

and therefore we have

$$\int |\Delta(|u_p|^2 u_s)|^2 \leq C\left(1 + ||u_p||_\infty^4 + ||u_s||_\infty^4\right) ||(u_s,u_p)||_{H^1}^2 \cdot ||(u_s,u_p)||_{H^2}^2.$$

Now, adding the s and p components, we obtain

$$\partial_t \int \left(|(H+I)u_s|^2 + |(H+I)u_s|^2\right) \leq C(1 + ||u_p||_\infty^2 + ||u_s||_\infty^2)\left(||(H+I)u_s||_{H^2}^2 + ||(H+I)u_p||_{H^2}^2\right),$$

where we have used that $||(u_s,u_p)||_{H^1}$ is uniformly bounded (see previous paragraph on H^1 estimates).

Since, $(2,\infty)$ is an admissible pair, both $\int ||u_p||_\infty^2 dt$ and $\int ||u_s||_\infty^2 dt$ are bounded, and we obtain the required bound:

$$||(H+I)u_s||_2^2 + ||(H+I)u_p||_2^2 \leq Ce^{kT}\left(||(H+I)u_s(0)||_2^2 + ||(H+I)u_p(0)||_2^2\right)$$

on the interval $t \in [0,T]$. Because of the norm equivalence, we also obtain:

$$||u_s(t)||_{H^2}^2 + ||u_p(t)||_{H^2}^2 \leq Ce^{kt}\left(||u_s(0)||_{H^2}^2 + ||u_p(0)||_{H^2}^2\right).$$

This bound implies the global existence of solutions in H^2.

Radiation losses in two-dimensional amplification model The proof of boundedness of radiation carries over from the one-dimensional case.

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A Application to optical communications

In this section we provide the details on the Raman model in optical communication systems as well as the derivation of the reaction-dispersion system. In modern long-haul optical communication systems the signal propagates in the fundamental mode of a single mode fiber. In an "ideal" lossless optical fiber waveguide the transverse shape of the wave envelope does not change and there is no transfer of energy from the fundamental mode to radiation modes. However, in the real systems Rayleigh scattering causes attenuation of signal power, thus requiring periodic amplification [3]. A current strategy for amplification of a signal is based on the stimulated Raman effect. Here, light of a second *pump* wavelength is co- or counter-propagated in the medium. The stimulated Raman effect is a parametric process in which light of the pump frequency is transferred to that of the signal frequency. This amplification process is inherently nonlinear and therefore is expected to cause deformation of the transverse mode shape.

Approximate equations for the Raman interaction of signal and pump in the waveguide have been derived in [5]. These authors derive a pair of coupled ODEs for the signal and pump intensities, based on the assumption that all energy is contained in the fundamental modes. This model compares well with experiment [5] (see also [3] and references therein). The authors [5] also discussed why the approximation was so accurate. They suggested that radiative losses (energy transfer from bound to radiation modes) is negligible due to the fact that "the wave-guiding action of the fiber reforms the pump and Stokes waves so that they always have intensities distributions which are close approximations to those which would exist in the absence of Raman interaction". However, this explanation has some limitations as the energy transfer could occur adiabatically (*e.g.* like the ionization of an atom). In other words, a weak process may lead to non-negligible changes after sufficiently long time. In particular, one might expect that the modes would undergo continuous deformation while also shedding radiation, so that after the full energy exchange a non-negligible amount of energy would accumulate in radiative modes and would constitute a significant loss.

To understand these effects, equations which take into account the effects of diffraction, wave-guiding and amplification should be studied. Naturally, the model will contain a small parameter: the ratio of diffraction and amplification lengths.

Raman stimulated emission describes the amplification of signal photons (with frequency ω_s) with Stokes down shifted pump photons (with frequency ω_p) and is governed by [5]

$$\frac{\partial n_s}{\partial z} = g(\omega_p - \omega_s) n_s n_p \tag{81}$$

$$\frac{\partial n_p}{\partial z} = -g(\omega_p - \omega_s) n_s n_p, \tag{82}$$

where n_s, n_p are the number densities of signal and pump photons respectively, and the total number of photons per unit volume is conserved $n_s + n_p = N$. Since we wish to focus on the effects due to the resonant coupling of the two wave fields (pump and signal) we ignore other effects, such as Amplified Spontaneous emission [3] which is always present, though a small effect.

Introducing the intensities

$$I_s = h\omega_s n_s, \quad I_p = h\omega_p n_p, \tag{83}$$

where h denotes Planck's constant, we obtain the corresponding equations

$$\frac{\partial I_s}{\partial z} = \frac{g(\omega_p - \omega_s)}{h\omega_p} I_s I_p \tag{84}$$

$$\frac{\partial I_p}{\partial z} = -\frac{g(\omega_p - \omega_s)}{h\omega_s} I_s I_p. \tag{85}$$

These equations satisfy the photon number conservation relation

$$\frac{I_s}{\omega_s} + \frac{I_p}{\omega_p} = \text{constant}. \tag{86}$$

In the case of radiative loss, this conservation law would be violated, since some photons would be lost from the bound waveguide mode to radiation modes. Equations (84)-(85) describe the plane wave Raman interaction.

Consider now the propagation of light in a dielectric cylinder waveguide with longitudinal coordinate, z. Maxwell's equations [3], imply

$$\Delta \mathbf{E} - \frac{1}{c^2}\mathbf{E}_{tt} - \nabla(\nabla \cdot \mathbf{E}) = \frac{1}{c^2}\left[\chi^{(1)}(\mathbf{r}, t) * \mathbf{E}_{tt}\right]_{tt} = 0,$$

where $\mathbf{E} \in \mathcal{R}^3$ is the electric field and $\chi^{(1)}(\mathbf{r}, t)$ is the linear susceptibility. Neglecting vector effects (see, *e.g.* [3]), we find that the time Fourier transform of $\mathbf{E}$, $\hat{\mathbf{E}}$, satisfies

$$\Delta \hat{\mathbf{E}} + \frac{\omega^2 n^2(\mathbf{x}_\perp, \omega)}{c^2}\hat{\mathbf{E}} = 0.$$

Each component of $\mathbf{E}$ satisfies

$$E_{zz} + \Delta_\perp E + \frac{\omega^2 n^2(\mathbf{x}_\perp, \omega)}{c^2} E = 0. \tag{87}$$

Next, we introduce the paraxial approximation. Let δ be a small parameter, and assume the following structure for the refraction index dependence on $\mathbf{x}_\perp$

$$n^2(\mathbf{x}_\perp, \omega) = n_0^2(\omega) + \delta^2 n_1^2(\delta \mathbf{x}_\perp/\lambda_0, \omega). \tag{88}$$

We also seek E, in the form:

$$E = A((\delta/\lambda_0)\mathbf{x}_\perp, (\delta^2/\lambda_0)z)e^{ikz},$$

where λ_0 is the light wavelength and

$$\frac{2\pi}{\lambda_0} = k = \omega n_0(\omega)/c. \tag{89}$$

Thus, E varies more rapidly in the transverse than longitudinal directions.

Substituting (88),(A) into equation (87) and multiplying by $\lambda_0^2\delta^{-4}$ we obtain

$$A_{ZZ} + 2ik\lambda_0\delta^{-2}A_Z + \delta^{-2}\Delta_\perp A + \delta^{-2}\lambda_0^2(\omega^2/c^2)n_1^2(\mathbf{X}_\perp) = 0,$$

where $Z = (\delta^2/\lambda_0)z$, $\mathbf{X}_\perp = (\delta/\lambda_0)\mathbf{x}_\perp$ and $(*)_\perp$ denotes differentiation with respect to $\mathbf{X}_\perp$. For δ small, we keep the dominant terms, those of order δ^{-2} and obtain, after using the relation between k and λ_0:

$$i4\pi A_Z + \Delta_\perp A + \frac{4\pi^2 n_1(\mathbf{X}_\perp, \omega)}{n_0^2}A = 0. \tag{90}$$

Equation (90) governs the linear propagation of any light field (signal or pump) in the paraxial approximation. To obtain a model governing the Raman interaction of signal and pump fields, u_s and u_p, we argue as follows. The signal field envelope propagates through a medium with refractive index (88) corrected by an imaginary term proportional to $i|u_p|^2$ corresponding to the Raman amplification by pump. The pump field envelope, u_p, propagates through a medium with refractive index (88) corrected by an imaginary term proportional to $-i|u_s|^2$ corresponding to pump depletion by the signal. The coupled signal and pump envelopes are then taken to satisfy the system:

$$i\partial_z u_s + \Delta_\perp u_s - V(\omega_s, \mathbf{x}_\perp)u_s = \ i\epsilon_s|u_p|^2 u_s \tag{91}$$
$$i\partial_z u_p + \Delta_\perp u_p - V(\omega_p, \mathbf{x}_\perp)u_p = -i\epsilon_p|u_s|^2 u_p, \tag{92}$$

where $\epsilon_{p,s}$ is the parameter which measures the ratio of the diffraction and nonlinear lengths. Usually, $\epsilon_{p,s}$ is very small [3]. We further assume[3] $\epsilon = \epsilon_s = \epsilon_p$ and neglect the dependence of the refractive index on frequency, *i.e.* $V(x_\perp, \omega) = V(x_\perp)$.

The system (92) models the Raman energy exchange between the two continuous waves. We have not included the effects of losses due to the Rayleigh scattering in order not to burden the exposition. In reality, the Raman amplification length might be comparable to the effective (loss) length (20 km).

Thus, we have

$$\begin{aligned} i\partial_t u_s - H\ u_s &= i\epsilon|u_p|^2 u_s \\ i\partial_t u_p - H\ u_p &= -i\epsilon|u_s|^2 u_p, \end{aligned} \tag{93}$$

[3]Nonlinear coefficient ϵ is then the same for both fields. Therefore, in this system the energy is conserved rather than photon number. This is done to simplify the presentation. All results hold for the "true" model, as well.

where we use "t'' to denote the "time-like" direction, z, $x_\perp = x$ and

$$H \;=\; -\Delta + V(x). \tag{94}$$

We study the system (93) in the case where H has spectrum consisting of one point eigenvalue, $\lambda < 0$, with corresponding eigenfunction ϕ, $||\phi||_{L^2} = 1$. The components of u_s and u_p, which are orthogonal to ϕ, are called radiative components. Our goal is to prove if for $t = 0$ the order 1 energy is concentrated in ϕ alone then on time scales of order ϵ^{-1} the energy in radiative components is at most of order ϵ.

B Normal form theorem

In this section we state a normal form result on the absence of the terms driving the radiation to any order of the perturbation theory. While, in the main part of the article we have used only the fact that these terms can be removed at the first order, we find this result important and potentially useful in the search for sharper estimates of radiation growth. The results in this section are formal in the sense that we do not verify the validity of transformations and of obtained systems.

For the Raman energy exchange system

$$i\partial_t u_s - Hu_s \;=\; i\epsilon|u_p|^2 u_s \tag{95}$$

$$i\partial_t u_p - Hu_p \;=\; -i\epsilon|u_s|^2 u_p, \tag{96}$$

we now use representation

$$u_s = u_0^s\phi_0^s(x) + \int_0^\infty u_\lambda^s \phi_\lambda^s(x) d\lambda$$

$$u_p = u_0^p\phi_0^p(x) + \int_0^\infty u_\lambda^p \phi_\lambda^p(x) d\lambda,$$

where H becomes diagonal

$$H\phi_0 = \lambda_0\phi_0,$$

$$H\phi_\lambda = \lambda\phi_\lambda, \;\; \langle u_\lambda, \phi_\lambda\rangle = u_\lambda$$

where $\lambda_0 < 0$ corresponds to the fundamental mode and the remaining part of the spectrum ($\lambda > 0$) corresponds to the continuous spectrum. In this representation the equations take the form

$$i\partial_t u_0^s - \lambda_0 u_0^s = i\epsilon C_{00}^{00} u_0^p \overline{u}_0^p u_0^s + i\epsilon \int_0^\infty C_{\lambda_1 0}^{00} u_0^p \overline{u}_0^p u_{\lambda_1}^s d\lambda_1 + ... + i\epsilon \int_0^\infty \int_0^\infty \int_0^\infty C_{\lambda_1 0}^{\mu_1\mu_2} u_{\mu_1}^p \overline{u}_{\mu_2}^p u_{\lambda_1}^s d\mu_1 d\mu_2 d\lambda_1$$

$$i\partial_t u_0^p - \mu_0 u_0^p = i\epsilon C_{00}^{00} u_0^s \overline{u}_0^s u_0^p + i\epsilon \int_0^\infty C_{00}^{\mu_1 0} u_0^s \overline{u}_0^s u_{\mu_1}^p d\mu_1 + ... + i\epsilon \int_0^\infty \int_0^\infty \int_0^\infty C_{\lambda_1\lambda_2}^{\mu_1 0} u_{\lambda_1}^s \overline{u}_{\lambda_2}^s u_{\mu_1}^p d\lambda_1 d\lambda_2 d\mu_1$$

$$i\partial_t u_\lambda^s - \lambda u_\lambda^s = i\epsilon C_{0\lambda}^{00} u_0^p \overline{u}_0^p u_0^s + i\epsilon \int_0^\infty C_{\lambda_1\lambda}^{00} u_0^p \overline{u}_0^p u_{\lambda_1}^s d\lambda_1 + ... + i\epsilon \int_0^\infty \int_0^\infty \int_0^\infty C_{\lambda_1\lambda}^{\mu_1\mu_2} u_{\mu_1}^p \overline{u}_{\mu_2}^p u_{\lambda_1}^s d\mu_1 d\mu_2 d\lambda_1$$

$$i\partial_t u_\mu^p - \mu u_\mu^p = i\epsilon C_{00}^{0\mu} u_0^s \overline{u}_0^s u_0^p + i\epsilon \int_0^\infty C_{00}^{\mu_1\mu} u_0^s \overline{u}_0^s u_{\mu_1}^p d\mu_1 + ... + i\epsilon \int_0^\infty \int_0^\infty \int_0^\infty C_{\lambda_1\lambda_2}^{\mu_1\mu} u_{\lambda_1}^s \overline{u}_{\lambda_2}^s u_{\mu_1}^p d\lambda_1 d\lambda_2 d\mu_1,$$

where

$$C^{\mu_1\mu_2}_{\lambda_1\lambda_2} = \int_{-\infty}^{+\infty} \phi^p_{\mu_1}\overline{\phi}^p_{\mu_2}\phi^s_{\lambda_1}\overline{\phi}^s_{\lambda_2}dx.$$

The natural consequence of the stimulated emission process is the invariance with respect to the phase shifts of both the signal and the pump modes:

$$u^s_{\lambda_i}, u^p_{\mu_j} \to u^s_{\lambda_i}e^{i\psi_s}, u^p_{\mu_j}e^{i\psi_p}.$$

This torus action will be called G_{sp}-action below.

Consider the class of polynomial vector fields which stay invariant under this group action. We are going to consider near-identity transformations, which commute with the torus action. Therefore these transformations map a G_{sp}-invariant vector field to another G_{sp}-invariant vector field. We will now invoke these transformations to remove non-resonant terms from the equations. We first observe that a polynomial vector field that is invariant with respect to the $G_{sp}-$action is generated by the monomials of this form

$$\mathbf{e}^{\mathbf{s}}_{\lambda}u^p_{\mu_1}\overline{u}^p_{\mu_2}...u^p_{\mu_{m-1}}\overline{u}^p_{\mu_m}u^s_{\lambda_1}\overline{u}^s_{\lambda_2}...u^s_{\lambda_{k-1}}\overline{u}^s_{\lambda_k}u^s_{\lambda_{k+1}} \tag{97}$$

$$\mathbf{e}^{\mathbf{p}}_{\mu}u^s_{\lambda_1}\overline{u}^s_{\lambda_2}...u^s_{\lambda_{m-1}}\overline{u}^s_{\lambda_m}u^p_{\mu_1}\overline{u}^p_{\mu_2}...u^p_{\mu_{k-1}}\overline{u}^p_{\mu_k}u^p_{\mu_{k+1}}, \tag{98}$$

where m and k are even numbers.

Definition: *The monomial of type (97) is called resonant if*

$$\mu_1 - \mu_2 + ... + \mu_{m-1} - \mu_m + \lambda_1 - \lambda_2 + ... + \lambda_{k-1} - \lambda_k + \lambda_{k+1} - \lambda = 0$$

and the monomial of type (98) is called resonant if

$$\lambda_1 - \lambda_2 + ... + \lambda_{m-1} - \lambda_m + \mu_1 - \mu_2 + ... + \mu_{k-1} - \mu_k + \mu_{k+1} - \mu = 0.$$

If in a G_{sp}-invariant vector field initially all the energy is concentrated in fundamental modes, then the radiative modes can be excited only through the terms

$$\mathbf{e}^{\mathbf{s}}_{\lambda}|u^p_0|^k|u^s_0|^l u^s_0 \text{ and } \mathbf{e}^{\mathbf{p}}_{\mu}|u^s_0|^k|u^p_0|^l u^p_0,$$

while no other radiation driving terms can appear after application of a $G_{sp}-$invariant transformations. These terms are non-resonant, since the corresponding arithmetic combinations are $\lambda - \lambda_0$ and $\mu - \lambda_0$, where $\lambda, \mu > 0$ and $\lambda_0 < 0$. Therefore, $|\lambda - \lambda_0| > |\lambda_0| > 0$ and $|\mu - \lambda_0| > |\lambda_0| > 0$.

According to the standard normal form procedure, these terms can be removed by employing the transformations of the form [4]

$$u^s_{\lambda} = U^s_{\lambda} + \epsilon^{\frac{k+l}{2}}C^s_{\lambda}|U^p_0|^k|U^s_0|^l U^s_0 \tag{99}$$

$$u^p_{\mu} = U^p_{\mu} + \epsilon^{\frac{k+l}{2}}C^p_{\mu}|U^s_0|^k|U^p_0|^l U^p_0, \tag{100}$$

[4]Capitals denote new variables while small ones denote old variables.

Now we formulate the normal form theorem

Theorem B.1 *For any $N \geq 1$ there exists a sequence of transformations of the form (99)-(100), which bring the system to the form*

$$\begin{aligned}
i\partial_t U_0^s - \lambda_0 U_0^s &= i\left[\epsilon C_{00}^{00} U_0^p \overline{U}_0^p U_0^s + \sum_{n_1,n_2>0}^{n_1+n_2<N+1} \epsilon^{\frac{n_1+n_2}{2}} C_{n1,n_2}^s |U_0^p|^{n_1} |U_0^s|^{n_2} U_0^s\right] + \epsilon R_0^s(U,\epsilon) + O(\epsilon^{N+1}) \\
i\partial_t U_0^p - \lambda_0 U_0^p &= -i\left[\epsilon C_{00}^{00} U_0^s \overline{U}_0^s U_0^p + \sum_{n_1,n_2>0}^{n_1+n_2<N+1} \epsilon^{\frac{n_1+n_2}{2}} C_{n1,n_2}^p |U_0^p|^{n_1} |U_0^s|^{n_2} U_0^p\right] + \epsilon R_0^p(U,\epsilon) + O(\epsilon^{N+1}) \\
i\partial_t U_\lambda^s - \lambda U_\lambda^s &= \epsilon R_\lambda^s(U,\epsilon) + O(\epsilon^{N+1}) \\
i\partial_t U_\mu^p - \mu U_\mu^p &= \epsilon R_\mu^p(U,\epsilon) + O(\epsilon^{N+1}),
\end{aligned}$$

where $R_*^{s,p}(U,\epsilon)$ are the terms which vanish if there is no energy in radiative modes ($U_\lambda^s = U_\mu^p = 0$ for all $\lambda, \mu > 0 \Rightarrow R = 0$).

Proof:
We will prove the theorem by applying a series of nearly-identical transformations. We start with the transformation

$$u_\lambda^s = U_\lambda^s + \epsilon C_\lambda^s U_0^p \overline{U_0^p} U_0^s.$$

Straightforward calculations show that in order to remove the corresponding radiation driving terms, the coefficient C_λ^s must be of the form

$$C_\lambda^s = \frac{C_{0\lambda}^{00}}{i(\lambda_0 - \lambda)}$$

and similarly to remove pump radiation driving terms, we apply

$$u_\mu^p = U_\mu^p + \epsilon C_\mu^p U_0^s \overline{U_0^s} U_0^p.$$

with

$$C_\mu^p = \frac{C_{0\mu}^{00}}{i(\mu_0 - \mu)}.$$

Our results from the previous sections indicate that the transformations are valid and the new system is well defined. Indeed, with the first order radiation driving terms removed, the obtained system is equivalent to (67-68).

Next, we formally remove quadratic radiation driving terms. We observe that all transformations are near-identity ones differing only by fundamental mode amplitudes. No small denominators arise due to the gap in the spectrum (between the eigenvalue and the continuum spectrum). Continuing these transformations, we remove higher order radiation driving terms to order N.

C Numerical simulations

We verify some of the results obtained in this paper by carrying out numerical simulations. We simulate the system (2) in one dimension, where the potential is chosen to be $V = \mathrm{sech}^2(x)$, so that the fundamental mode can be explicitely calculated. We use the initial data with all energy contained in fundamental modes.

The numerical simulation uses a Fourier split-step scheme, where evolution due to dispersive, potential, and nonlinear interactions are calculated separately. Nonlinear interaction is solved exactly using the standard solution [3]. Time step is chosen to be $\Delta t = 0.01$ and there are 2^{12} Fourier modes. In Figure 1, the L^2 norm of total radiation is calculated after sufficiently long evolution, so that power exchange between the fields is almost complete. This is done for $\epsilon \in [0.005, 0.05]$. One can see that the time interval is sufficiently long from Figure 2, where for the smallest $\epsilon = 0.005$, the power of both fields contained in the fundamental modes is computed. Even in this case with the smallest ϵ (so that the energy exchange takes longer) there is enough time for the pump field to transfer almost all the power to the signal field.

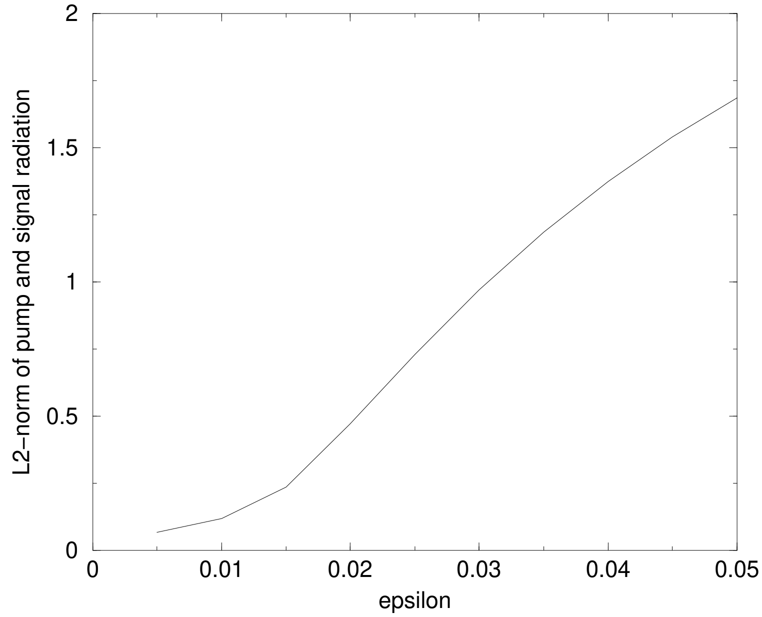

Figure 1: Dependence of radiation power on time. The radiation is “measured” after the evolution for 75 units of dimensionless time.

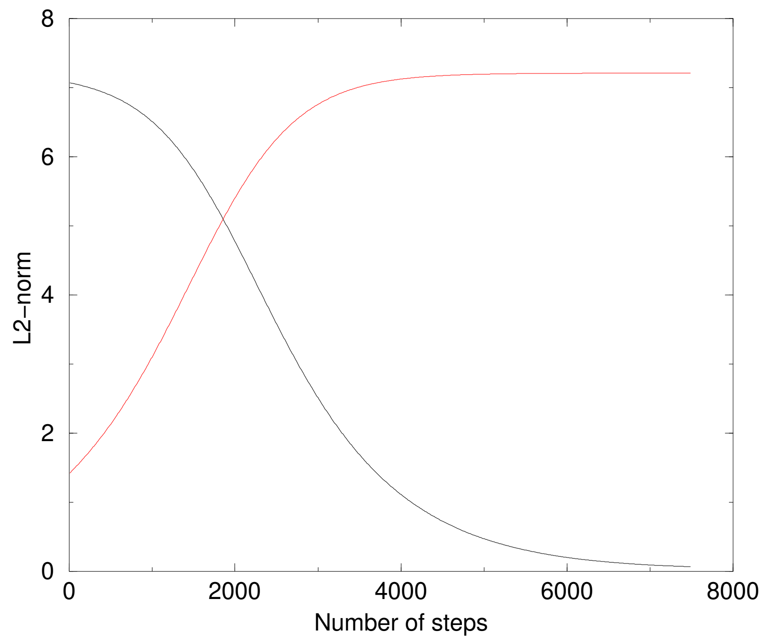

Figure 2: Power exchange between fundamental modes for the smallest $\epsilon = 0.005$ after the evolution for the same time $T = 75$ (it corresponds to 7500 steps). Note that the pump field is almost completely "depleted". This indicates that time interval $T = 75$ is of sufficient length for the power exchange to take place.