Scattering Resonances of Microstructures and Homogenization Theory

S.E. Golowich and M.I. Weinstein
Mathematical Sciences Research
Bell Laboratories, Lucent Technologies
Murray Hill, NJ 07974

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Abstract

Scattering resonances are the eigenvalues and corresponding eigenmodes which solve the Schrödinger equation $H\psi = E\psi$ for a Hamiltonian, H, subject to the condition of outward going radiation at infinity. We consider the scattering resonance problem for potentials which are rapidly varying in space and are not necessarily small in a pointwise sense. Such are of interest in many applications in quantum, electromagnetic and acoustic scattering, where the environment consists of microstructure, *e.g.* rapidly varying material properties. Of particular, interest in applications are *high contrast* microstructures, *e.g.* large pointwise variations of material properties.

We develop a perturbation theory for the scattering resonances and eigenvalues of such high contrast and oscillatory potentials. The expansion is proved to be convergent in a norm which encodes the degree of oscillation in the microstructure. Next, we consider the concrete example of two-dimensional microstructure potentials. These correspond, for example, to a class of photonic wave guides with transverse microstructures. The leading order behavior is given by the scattering resonances of a suitable averaged potential, as predicted by classical homogenization theory. We show that the next term in the expansion agrees with that given by higher order a homogenization multiple-scale expansion, with an error term determined by the regularity of the potential. The higher order corrections, which take into account the detailed microstructure, have been shown by the authors to be important for efficient and accurate numerical computation of radiation rates.

Key words: scattering frequency, scattering resonance, quasi-mode, radiation, photonic crystal, leaky mode, homogenization, tunneling, eigenvalue perturbation theory

Contents

1. Introduction and overview

1.1. Scattering resonances and microstructures

Consider the time-independent Schrödinger equation

$$H\psi = E\psi$$

where H is the *Hamiltonian*

$$H = -\Delta + V(x). \tag{1.1}$$

Scattering resonances of (1.1) are solutions (E, ψ), with $\psi \neq 0$, which satisfy an *outgoing radiation condition* at spatial infinity. The condition of outgoing radiation is a non-selfadjoint boundary condition and therefore the scattering resonance energies, E, are typically complex numbers. A precise formulation of this radiation condition and the scattering resonance problem is given in section 4.

In this paper, we are interested in the scattering resonances for equation (1.1) in the case where the *potential*, $V(x)$ has a background (slowly varying or "mean") part, $V_0(x)$, and a rapidly oscillatory perturbation, $\delta V(x)$, which is not necessarily pointwise small:

$$V(x) = V_0(x) + \delta V(x). \tag{1.2}$$

We call such potentials *potentials with microstructure*, and refer to $\delta V(x)$ as the *microstructure part* of the potential. If the variations in the values of the potential are large, we refer to a *high contrast* potential. The Hamiltonian with potential $V_0(x)$ will be denoted H_0 and that with perturbed potential is denoted H. A consequence of the results of this paper is a rigorous analytical foundation for the work in [7], where we derived a multiple scale expansion for certain microstructured potentials, which arise as photonic microstructures, and used it as part of an accurate and efficient scheme for numerically computing scattering resonances; see subsection 1.2. A detailed discussion of the application to photonic microstructures as well as an announcement of some of the current analytical work is contained in [8].

The scattering resonance problem is of great mathematical interest and physical importance, arising in quantum, acoustic, elastic and electromagnetic scattering. A basic problem in each field is to consider the situation of a compact region of space which is occupied by a collection of inhomogeneities or "scatterers," while outside this region the medium is homogeneous. Thus, outside this compact region the elementary propagating states are plane waves, $e^{i(k\cdot x - \omega t)}$, where $\omega = \omega(k)$ is the dispersion relation of the homogeneous medium ($k = |k|$, $k = 2\pi/\lambda$, $\lambda =$ wavelength). For fixed wavevector, k, one considers a plane wave incident on a collection of inhomogeneities or "scatterers" and calculates the scattered field. The mapping from incident to scattered field is determined by the scattering matrix, $S(k)$. In many situations, $S(k)$ can be analytically continued from the positive real axis into the complex plane. Its poles in the lower half plane are called *resonances*. Corresponding to these resonances are (non-L^2) solutions of (1.1) with $E = k^2$. The scattering resonance energy plays an important role in the dynamics of waves. Namely, a general spatially localized initial condition incident on our collection of scatterers will typically interact for some time, and give rise to a spray of radiation. The local energy, in a neighborhood of the scatterers, will die away as time evolves. The long time transient exponential rate of local energy decay is determined by the imaginary parts of the resonances. In particular, one expects this decay to be limited by the resonance with the smallest imaginary part. For a given resonance, one refers to its imaginary part as the line-width or reciprocal lifetime of the associated state.

Being based on the Schrödinger equation (1.1), the analysis of this paper applies to a non-relativistic quantum particle in a rapidly varying landscape [13], the acoustic propagation of small amplitude pressure fluctuations about an equilibrium in an inhomogeneous medium [17], and to electromagnetic waves, in the scalar approximation, propagating in an inhomogeneous dielectric medium [24, 14].

Our initial motivation comes from the electromagnetic context, where there is a great deal of interest in the analysis of optical properties of media comprised of micro- or nanostructures. Recent advances in fabrication technology have made possible, through variations in material contrast and distribution of microfeatures, the manipulation of optical properties of composite media in a manner analogous to the way the electrical properties of materials have been manipulated for many years. Examples range from engineering the dispersion properties of photonic waveguides [20, 11] to cavity QED experiments, with a view toward applications to quantum computers [26]. Scattering resonances are central importance in the behavior of such structures. The real parts are relatively insensitive to the detailed microstructure and are approximated by an averaged potential model, leading order homogenization theory. As illustrated below, and extensively in [7], the imaginary parts are highly sensitive to the detailed microstructure, and we require higher order homogenization theory to estimate them accurately. Imaginary parts of scattering resonances of these structures correspond to the "Q-factor" of the effective cavity resonator.

1.2. An application

A specific example, considered in detail in [7] is an optical fibre waveguide with transverse microstructure. Such waveguides are often called photonic crystal fibres or holey fibers. In the scalar approximation, the modes of the waveguide are governed by the Helmholtz wave equation:

$$\left(\Delta + k^2 n^2(x)\right)\psi = 0, \tag{1.3}$$

where $k = 2\pi/\lambda$, λ denotes the wavelength of light, and $n(x)$ denotes the refractive index profile. Let n_g denote a background refractive index. Then, the Helmholtz equation can be rewritten as a Schrödinger equation (1.1) with the definitions:

$$V(x) = k^2\left(n_g^2 - n^2(x)\right), \quad E = k^2 n_g^2 - \beta^2. \tag{1.4}$$

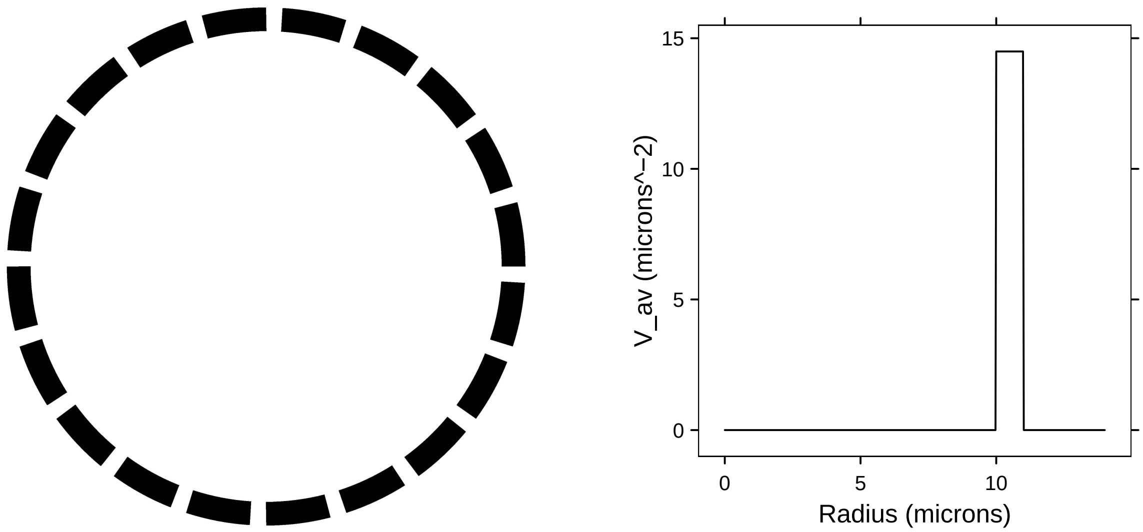
Here, x denotes the 2-dimensional spatial coordinate in the plane transverse to the waveguide.

A class of examples corresponding to a large family of microstructured waveguides of interest, treated in detail in section 6, is:

$$V(x) = V_N(x) = V_0(|x|) + \delta V_N(x) = V_0(r) + \delta V(r, N\theta). \tag{1.5}$$

Here, (r, θ) are polar coordinates for $x \in \mathbb{R}^2$. The potential defined by (1.5) corresponds to a N-fold symmetric structure. The functions $V_0(r)$ and $\delta V(r, \Theta)$ are compactly supported locally L^2 functions. The parameter N is a positive integer. When N is large, the potential V_N is a rapidly varying perturbation of V_0. This perturbation does not tend to zero pointwise, but does tend to zero in a weak sense. Examples of such high contrast microstructure potentials appear in figures 1 and 2.

We have derived and numerically implemented homogenization and corrected homogenization theories of scattering resonances and compared the results with direct numerical simulation [7]. Numerical results indicate that the real parts of scattering resonances are well approximated by those associated with the averaged potential, $V_0(r)$. However, the situation with imaginary parts of scattering resonances is very different. Figure 3 contrasts predictions for the imaginary parts of scattering resonances for structures of the type illustrated in figure 1. Shown are results for five different structures parametrized by N and "fill fraction", f. The imaginary part of a scattering resonance is plotted as a function of wavelength, λ. The crosses correspond to direct numerical simulation. The two curves correspond to predictions obtained from the average potential (dashed curve) and the 2^{nd} order corrected homogenization theory (solid curve), proved in this work; see discussion below. Corrected homogenization gives excellent agreement with direct numerical simulations over wavelength ranges of interest in the particular application. For structures of the type shown in figure 2, the corrected homogenization theory gives an a even more dramatic improvement over the averaged (homogenization)theory. Clearly and in general, numerical methods informed by a analytical estimates on resonances can play an important role in finding optimal structures, *e.g.* maximizing "Q-factors".



(a) (b)

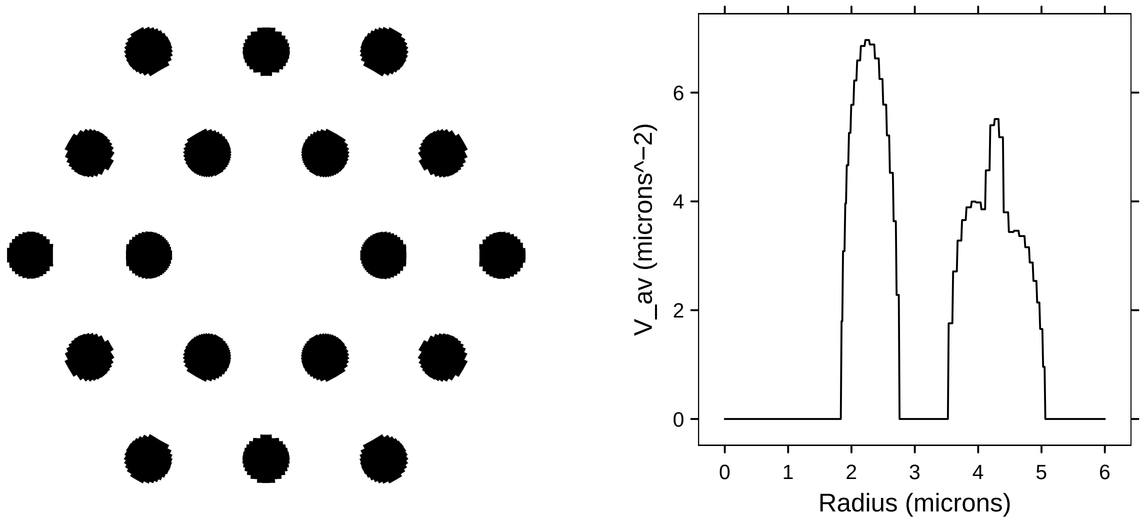
Figure 1: (a) Black regions form the support of an $N-$ fold symmetric potential ($N = 20$). $V(x)$ takes on one constant value on these regions and is zero outside. (b) The averaged potential $V_0(|x|) = (2\pi)^{-1} \int_0^{2\pi} V(r, \Theta) d\Theta$, where $V(x) = V_0(r) + \delta V(r, N\theta)$
.

The current work implies analytical justification for the homogenization expansion and numerical approach summarized above and presented in detail in [7]. In particular, Theorems 4.1 and 7.1 imply validity of second order homogenization provided a particular norm of the perturbation, $|||\delta V_N|||$, is sufficiently small. By Theorem 6.1, this smallness condition can be expressed as

$$\frac{1}{\lambda^2} \cdot \frac{1}{N} \cdot C_* \text{ sufficiently small,} \tag{1.6}$$

where the constant $C_* = \sup_M M \; |||\delta V_M||$ (see Theorem 6.1), which depends on the details of the microstructure. Our analysis applies to the case of fixed λ and N sufficiently large. However, the condition (1.6) suggests validity for fixed N (fixed microstructure) and λ large. Results which encompass this limit as well will be presented in a separate publication.

We conclude this subsection with a discussion of two observations on departures of approximate theories from direct numerical simulation. (1) Note that as the wavelength is decreased, there is an increasing departure of approximate theoretical results from simulation results. The effect is most pronounced for the structure whose parameters are $N = 3$ and $f = .8$. This trend can be understood by noting that for a fixed structure, as the wavelength is decreased, one approaches the regime of geometrical optics, where the ray picture rather than an average wave picture. Nevertheless, we find good agreement in our simulations for a ratio of wavelength to microfeature size of about 3/2. (2) The plots in figure



(a) (b)

Figure 2: (a) An 18-hole subset of a hexagonal lattice ($N = 6$) with (b) The averaged potential $V_0(|x|)$

3 show a systematic underestimation of the "exact" imaginary parts. Note that there are two essential mechanisms of scattering loss: *free space diffraction or ballistic propagation* - propagation concentrated on rays where the potential is zero in between the barriers, where V is positive and *tunnelling* - propagation along rays which impinge on regions where V is positive. Homogenization theory replaces a problem in which both mechanisms are present by an effective problem, defined by the effective potential barrier, $V_0(|x|)$, for which tunneling is the only mechanism. Although, still rooted in homogenization ideas, our higher order theory gives a very significant improvement in the approximation even for relatively small values of N and wavelength to microfeature size ratios. Though we have found a natural small parameter, $|||\delta V|||$, for measuring the "size" of the microstructure perturbation, we believe a related *mode-dependent* intrinsic parameter exists, which should give yet deeper insight. This is currently under investigation.

1.3. An illustrative and elementary example

In this subsection we present a simple example which motivates our approach to spectral problems with high contrast microstructure. We present only a sketch of the ideas. All analytical details are contained within our full implemention for the resonance problem, which is far more involved.

Consider the Schrödinger eigenvalue problem with periodic boundary conditions for an

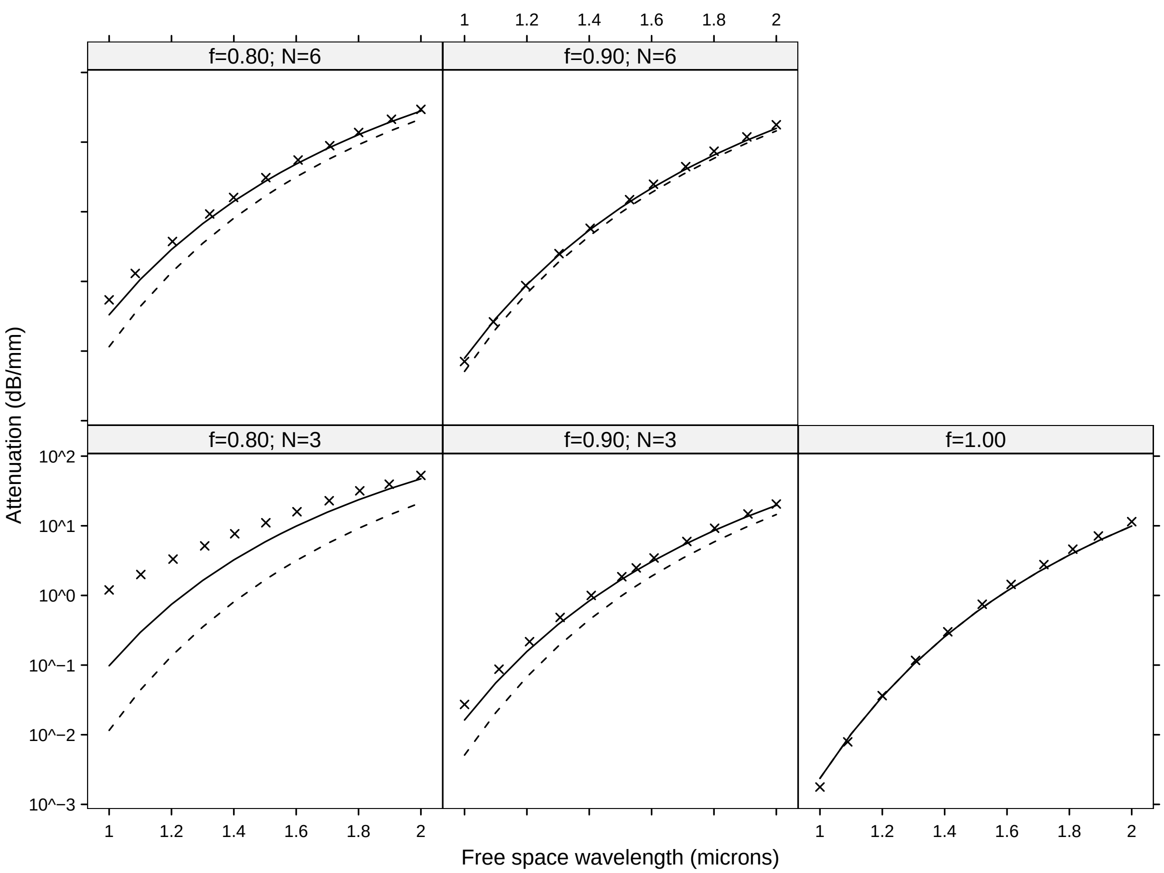


Figure 3: Imaginary parts of scattering resonances, corresponding to scattering attentuation rate in the optical waveguide context, for the lowest order (fundamental, or LP_{01}) resonance. Calculation is for $N-$ fold symmetric structures of the type shown in figure 1, for different fill-fractions, f, and $N = 3$ and $N = 6$. The calculations were performed for a range of wavelengths, λ. Solid curves are attenuations computed using $2^{\rm nd}$ order, $\mathcal{O}(N^{-2})$, homogenization, proved herein to valid for large N. Dashed curves correspond to leading order homogenization (averaging) theory. Crosses correspond to the results of direct numerical simulation in presented in figure 3a of [19].

eigenfunction $u \in L^2(S^1)$ and eigenvalue, E:

$$\begin{aligned} \partial_\theta^2 u(\theta) \ + \ V(\theta)u(\theta) \ &= \ E\ u(\theta) \\ u(0) \ = \ u(1), \ \partial_\theta u(0) \ &= \ \partial_\theta u(1) \end{aligned} \tag{1.7}$$

Here, $V(\theta) \in L^\infty_{\text{periodic}}$. We rewrite (1.7) the equivalent equation of Lippman-Schwinger type:

$$\begin{aligned} &\big(\ I \ - \ (1+E)(I-\partial_\theta^2)^{-1}\ \big)\,U \ + \ \mathcal{Q}_V\ U \ = \ 0, \\ &U \ \equiv \ (I-\partial_\theta^2)^{\frac{1}{2}} u \\ &\mathcal{Q}_V U \ \equiv \ (I-\partial_\theta^2)^{-\frac{1}{2}} V (I-\partial_\theta^2)^{-\frac{1}{2}}\ U \end{aligned} \tag{1.8}$$

Using the implicit function theorem, one can prove:

Theorem 1.1. *Let (U_0, E_0) be a solution of the eigenvalue problem (1.8) with potential V_0 and consider the perturbed eigenvalue problem with potential V. Let $\delta V \equiv V - V_0$ denote the perturbation. If the mapping $U \mapsto \mathcal{Q}_{\delta V} U$ has small norm as an operator from L^2 to L^2, i.e.*

$$\|\ (I-\partial_\theta^2)^{-\frac{1}{2}} \delta V (I-\partial_\theta^2)^{-\frac{1}{2}}\ \|_{\mathcal{B}(L^2)} \text{ sufficiently small,} \tag{1.9}$$

then eigenvalues of the eigenvalue problem with potential $V = V_0(\theta)$ problem perturb analytically, to nearby solution, $(U(V), E(V))$, of the eigenvalue problem.

Remark 1.1. *While the operator norm in (1.9) can be small if δV is pointwise small, e.g. $\delta V(\theta) = \varepsilon Q(\theta),\ 0 < \varepsilon << 1$, it can also be small if δV is pointwise large (high contrast) but sufficiently oscillatory (microstructure). For example, if $\delta V_N(\theta) = Q(N\theta)$, then the operator norm (1.9) is $\mathcal{O}(N^{-1})$ by the following Lemma (see also Theorem 6.1):*

Lemma 1.1. *Let $f(\theta)$ be a 2π period function. Then,*

$$\begin{aligned} &\text{The mapping } f(\theta) \ \mapsto \ e^{iN\theta}\ f(\theta) \text{ has } \ L^2 \text{ norm one} \\ &\text{The mapping } f(\theta) \ \mapsto \ \langle\partial_\theta\rangle^{-1}\ e^{iN\theta}\ \langle\partial_\theta\rangle^{-1}\ f(\theta) \text{ has } \ L^2 \text{ norm } \mathcal{O}(N^{-1}) \end{aligned} \tag{1.10}$$

1.4. Goals, results and overview

Perturbation problem: Suppose the (E_0, ψ_0) is a solution to the scattering resonance problem for the Hamiltonian H_0 with potential $V_0(x)$. Our goals are to address the following issues.

(1) For $\delta V(x)$ is sufficiently oscillatory (has sufficiently fine microstructure) and high contrast, show that the scattering resonance problem for H with potential $V(x) = V_0(x) + \delta V(x)$ have a "nearby" solution $(E(V), \psi(V))$.
(2) Develop a resonance perturbation theory in terms of a natural parameter, measuring the *fineness* of the microstructure?
(3) Relate (1) and (2) to homogenization theory.

In sections 3 and 4 we reformulate the scattering resonance problem as a "preconditioned" Lippman-Schwinger type equation (Theorem 3.1):

$$(I + T_R(E)T_V)\,\Psi = 0,$$

where the operators $T_R(E)$ and T_V are defined in section 3. We then obtain a stability and perturbation theory of scattering resonances in a norm appropriate for the study of microstructure perturbations which are not necessarily pointwise small (Theorem 4.1). The norm of the perturbation, $|||\delta V|||$ gives a natural measure of the degree of fineness of the microstructure and is defined for a compactly supported L^2 perturbation, δV by

$$\| T_{\delta V} \|_{\mathcal{B}(L^2)} \equiv ||| \delta V ||| = \| \langle D \rangle^{-1} \delta\tilde{V} \langle D \rangle^{-1} \|_{\mathcal{B}(L^2)}; \tag{1.11}$$

see also (4.1). Here, $\delta\tilde{V} = \chi^{-1}\delta V \chi^{-1}$, $\chi(x) = \mathcal{O}(e^{-\alpha|x|})$, for some $\alpha > 0$ and $\langle D \rangle = (m^2 - \Delta)^{\frac{1}{2}}$.

If δV is pointwise large but very oscillatory $||| \delta V |||$ may be small due to the operator $\langle D \rangle^{-1}$ which gives small weight to the part of δV supported at high wavenumbers; see the discussion of section 1.3. For the class of examples represented by (1.5) we have (Theorem 6.1):

$$||| \delta V_N ||| = \mathcal{O}\left(\frac{1}{N}\right)$$

(Note that the norm defined in (1.11) can be small as well even when δV is not microstructure-like.)

The main results of this paper are the following:

1. Theorem 3.1 introduces a characterization of solutions to the scattering resonance problem, (non-normalizable) scattering resonance modes and complex scattering resonance energies, as solutions of a "preconditioned" Lippman-Schwinger equation defined on L^2.

2. Theorem 4.1 is a stability theorem showing that if the potential, V_0, is such $(\psi(V_0), E(V_0))$ is a (simple) scattering resonance pair and V_0 is perturbed to a "nearby" potential

V, *i.e.* $|||V - V_0|||$ is small, then the scattering resonance problem with potential V has a nearby scattering resonance pair $(\psi(V), E(V))$. Furthermore, the mapping $V \mapsto (\psi(V), E(V))$ is analytic in V. The norm, $|||\cdot|||$, has been constructed so that if $V - V_0$ is not pointwise small, but is very oscillatory, then $|||V - V_0|||$ is small. In Theorem 4.2 we treat the case of degeneracies arising when V_0 is a radial potential, $V_0 = V_0(r)$.

3. Theorem 7.1 demonstrates the connection with homogenization theory. For the class of microstructures, corresponding to high-contrast microstructures of the type arising in physical problems, we show that the N^{-2} corrections to homogenization theory agree with the resonance perturbation theory of Theorem 4.1 with an error of order $N^{-2-\tau}$, $\tau > 0$. The number τ depends on the regularity of the potential. In particular, if V is C^2 then $\tau = 2$, but if as is often the case in applications, V has jump discontinuities (*e.g.* an interface between two different materials at which there is a jump in refractive index), then $\tau = 1$.

4. Theorems 2.1 and 2.2 encompass the (simpler) perturbation theory of eigenvalues for microstructure potentials.

Although, the problem of stability of eigenvalues for self-adjoint operators (see sections 1.3 and 2) has many of the structural features of the resonance problem, the resonance problem requires a much more technical treatment for the following reasons:

- While eigenvalues of a self-adjoint operator, H, are poles of the resolvent (Green's function) $(H - E)^{-1}$, scattering resonance energies are poles, E ($\Im E \neq 0$), of the analytic continuation of the resolvent across the continuous spectrum (branch cut) to a "non-physical" sheet; see section 3.

- Solutions of the eigenvalue equation $H\psi = E\psi$, where E is a scattering resonance energy do not lie in L^2 and in fact grow exponentially at infinity.

- For high contrast potentials (V pointwise large), resonances are complex numbers E with very small imaginary part. These resonance energies very close to the continuous spectrum of H, the branch cut for $(H-E)^{-1}$. A theory which is useful for high contrast potentials requires very detailed information on the analytic continuation of $(H-E)^{-1}$ in a neighborhood of the branch cut.

Overview

- In section 2 we discuss microstructure perturbations of the eigenvalue problem.

• In section 3 we formulate the scattering resonance problem.
• In section 4 we state and prove our theorem on microstructure perturbations of scattering resonances.
• In section 5 we explicitly calculate the expansion of the scattering resonance pair $(E(V), \psi(V))$ about the case $V = V_0$.
• In section 6 we show that the theory applies to potentials of the form $V = V_0(r)+\delta V(r, N\theta)$, where $V_0(r)$ supports scattering resonances and N is sufficiently large.
• In section 7 the homogenization expansion of [7] is reviewed and it is proved that the leading term and lowest nonzero correction in N^{-1} agree with those of the expansion displayed in section 5. As demonstrated in [7] the imaginary parts of scattering resonances (leakage rates) are very sensitive to the detailed microstructure. Therefore, it is important to understand the corrections to leading order (effective medium) homogenization theory. Our second order homogenization expansion leads, in examples of physical interest, to an efficient and accurate numerical method, giving very good agreement with full simulation by Fourier methods and multipole methods. The real parts of scattering resonances are far less sensitive to the specific microstructure, and often the first term in the expansion (the averaged or homogenized problem) gives a good approximation.

Finally, we wish to mention other work on spectral problems in the setting of microstructure. In [23] and [18] the validity of first order corrections to homogenized eigenvalues for divergence form (self-adjoint) elliptic operators with rapidly varying coefficients on bounded domains is analyzed. The problem of scattering resonances for the Helmholtz resonator problem in [5], [6] and in other works cited therein.

1.5. Definitions, Notation and Conventions

All integrals are assumed to be over all space ($\mathbb{R}^n$) unless otherwise noted.
(r, θ) denote polar coordinates in $\mathbb{R}^2$
$\Re z$, the real part of z; $\Im z$, the imaginary part of z
Both $\overline{z}$ and z^* are used to denote the complex conjugate of z.
A^* denotes the adjoint of the operator A.
$[A, B] \ = \ AB - BA$, the commutator of A and B.
The inner product for functions $f, g \in L^2(\mathbb{R}^n)$ is denoted by:

$$\langle f, g\rangle \ = \ \int \overline{f(x)} g(x) \ dx$$

For *radial* functions $f(r), g(r) \in L^2(\mathbb{R}^n)$ on $\mathbb{R}^n$ we denote the radial inner product by:

$$\langle f, g\rangle_{\rm rad} = \int_0^\infty \overline{f(r)} g(r)\ r^{n-1}\ dr.$$

<u>Fourier transform:</u>

$$\begin{aligned} \mathcal{F}[g](\xi) &\equiv \hat{g}(\xi) \equiv \int e^{-ix\cdot\xi} g(x)\ dx, \\ g(x) &= \mathcal{F}^{-1}\left[\mathcal{F}[g]\right](x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\hat{g}(\xi)\ d\xi \end{aligned} \tag{1.12}$$

$\langle\xi\rangle = (m^2+\xi^2)^{1/2}$, $\xi^2 = \xi\cdot\xi$.
$D = -i\nabla$, $\Delta = \nabla\cdot\nabla$, the Laplace operator
<u>Functional calculus:</u>

$$f(D)g \equiv \mathcal{F}^{-1}\left[f\mathcal{F}[g](\cdot)\right] = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi} f(\xi)\hat{g}(\xi)\ d\xi \tag{1.13}$$

$$\langle D\rangle g = \left(m^2 I - \Delta\right)^{1/2} g = \frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\langle\xi\rangle\ \hat{g}(\xi)\ d\xi \tag{1.14}$$

<u>Exponentially localized weights:</u> Let $0 < \tilde{M} < M$.

$$\begin{aligned} &\chi(x) = \mathcal{F}^{-1}[\hat{\chi}] \text{ and } \tilde{\chi}(x) = \mathcal{F}^{-1}[\hat{\tilde{\chi}}], \text{ where} \\ &\hat{\chi}(\xi) = (M^2+\xi^2)^{-4} \text{ and } \hat{\tilde{\chi}}(\xi) = (\tilde{M}^2+\xi^2)^{-4} \end{aligned} \tag{1.15}$$

Note that $0 < \chi(x) = \mathcal{O}(e^{-M|x|}$ and $0 < \tilde{\chi}(x) = \mathcal{O}(e^{-\tilde{M}|x|})$. The exponential upper bound follows from deformation of the ξ integral contour. Note that $\chi(x)$ is essentially the Bessel potential $G_8(x)$, which is strictly positive; see equation (26) in [25].

For a radial function, $g(r)$ defined on $\mathbb{R}^2$,

$$f(D_l)g \equiv e^{-il\theta} f(D) e^{il\theta} g$$

$\langle D_l\rangle = \left(I - \Delta_r + \frac{l^2}{r^2} \right)^{\frac{1}{2}}$
<u>Young's inequality:</u> Let $\alpha\star\beta$ denote the convolution of functions α and β.

$$\|\alpha\star\beta\|_{L^p} \le \|\alpha\|_{L^1}\|\beta\|_{L^p},\ p\ge 1. \tag{1.16}$$

<u>Bessel functions:</u> For any integer ℓ,

$$J_\ell(z) = \frac{i^{-\ell}}{\pi}\int_0^\pi e^{iz\cos\theta}\cos(\ell\theta)\ d\theta. \tag{1.17}$$

Fourier transform of uniform measure on S^{n-1} See, for example, [3]. For, $n \geq 2$

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{S^{n-1}} e^{i\omega\cdot z}\, d\omega \;=\; \frac{1}{|z|^{\frac{n-2}{2}}}\; J_{\frac{n-2}{2}}(|z|) \tag{1.18}$$

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2. Microstructure perturbations of eigenvalues

We will study the time-independent Schrödinger equation

$$H\psi = E\psi \tag{2.1}$$

where

$$H = -\Delta + V(x).$$

Throughout this paper, we require fairly stringent conditions on the potential, namely that $V(x) \in L^\infty(\mathbb{R}^n)$ and $V(x)$ has compact support. Minor modifications in our arguments can be made to treat the case of potentials which decay exponentially as $|x| \to \infty$. For simplicity, we will work in spatial dimension $n = 2$ or 3, though we believe that generalization to higher dimensions can be done. Standard arguments [21, 9] imply that H is self-adjoint on the domain $D(H) = D(-\Delta) = H^2(\mathbb{R}^n)$, and the essential spectrum $\sigma_{\text{ess}}(H) = \sigma(-\Delta) = [0, \infty)$.

In this section we focus on the eigenvalue problem for equation (2.1), requiring $\psi \in L^2$ and for the remainder of the paper the scattering resonance problem, corresponding to equation (2.1), with ψ "outgoing" at ∞.

The class of potentials we consider can be written as

$$V(x) = V_0(x) + \delta V(x)$$

where V_0 is "slowly varying" and δV is a perturbation which is "rapidly varying," but possibly a pointwise large perturbation, in a sense which is made precise below.

A solution to the eigenvalue problem

$$(-\Delta + V)\,\psi \;=\; E\psi \tag{2.2}$$

is a pair (E, ψ), where ψ is nontrivial and $\psi \in H^2(\mathbb{R}^n)$.

We now reformulate the eigenvalue problem in a manner which is well-suited to treating microstructure perturbations; see section 1.3. Let $E = \mu^2$. For $\Im\mu > 0$ ($E \notin [0,\infty)$), $(-\Delta - \mu^2)^{-1}$ is bounded on L^2 and so (2.2) implies

$$\psi \ + \ (-\Delta - \mu^2)^{-1} V \psi \ = \ 0$$

or equivalently

$$\left(I \ + \ \underbrace{(-\Delta - \mu^2)^{-1}(I - \Delta)}_{S_R(\mu)} \ \underbrace{\langle D\rangle^{-1} V \langle D\rangle^{-1}}_{S_V} \right) \langle D\rangle\psi \tag{2.3}$$

Therefore, we have the following simple correspondence between solutions of (2.2) and a Lipmann-Schwinger type integral equation:

Theorem 2.1. *The pair $(E = \mu^2, \psi)$ is a solution to the eigenvalue problem if and only if $(\mu, \tilde{\psi})$, where $\tilde{\psi} = \langle D\rangle\psi \in L^2$, solves*

$$(\ I \ + \ S_R(\mu) S_V\ )\, \tilde{\psi} \ = \ 0. \tag{2.4}$$

If E_0 is an eigenvalue, then $E_0 = \mu_0^2 < 0$, for some μ_0 on the imaginary axis in the upper half plane. We have the following perturbation theorem for simple eigenvalues:

Theorem 2.2. *(a) Let V_0 denote a potential for which $|||V_0|||_1$ is finite, where*

$$|||V_0||| \ \equiv \ \|S_{V_0}\|_{\mathcal{B}(L^2)} \ = \ \| \ \langle D\rangle^{-1} \ V_0 \langle D\rangle^{-1}\|_{\mathcal{B}(L^2)}. \tag{2.5}$$

Let (E_0, ψ_0) denote a solution of the eigenvalue problem corresponding to the potential, $V_0(x)$, and assume that E_0 is a simple eigenvalue, for which $\tilde{\psi}_0 = \langle D\rangle\psi_0$ correspondingly spans the null space of $I \ + \ S_R(\mu_0) S_{V_0}$. Assume that

$$\langle S_{V_0}\tilde{\psi}_0, S_R'(\mu_0) S_{V_0}\tilde{\psi}_0\rangle \ = \ \|R_0(\mu_0) V_0 \psi_0\|_2^2 \neq 0. \tag{2.6}$$

Then, there exists ε_0 such that for any potential V satisfying $|||V - V_0||| < \varepsilon_0$, there corresponds a unique solution $(E(V), \psi(V))$ of the eigenvalue problem which lies near (E_0, ψ_0).
(b) The mapping

$$V \mapsto (E, \psi) \in \mathbb{R}^1 \times H^2,$$

which associates to a potential V a solution $(E(V), \psi(V))$ of the eigenvalue problem, is analytic in the norm $|||V|||$ in a neighborhood $|||V - V_0||| < \varepsilon_0$ of V_0.

The proof of this theorem is a very much simplified version of the proof of the analogous result (Theorem 4.1) for the case of scattering resonances. We implement the ideas in the latter context. However, we do make a few remarks presently.

Remark 2.1. *The proofs are based on the implicit function theorem. Recall that an eigenvalue E_0 of $-\Delta+V_0$ is such that $E_0=\mu_0^2$, where μ is purely imaginary and in the upper half plane. Also, as can easily be seen using the Fourier transform, the operators $\partial_\mu^j S_R(\mu),\ j\geq 0$ are bounded in L^2 in a neighborhood of μ_0. Therefore, one expects the continuity of eigenvalues in the norm for potentials varying in a neighborhood of V_0 in the norm (2.5).*

Remark 2.2. *The results worked out in sections 5- 7 apply as well for the case of the eigenvalue problem of this section, which much simpler proofs. These include:*

1. *a convergent perturbation expansion of eigenvalues in the $V-V_0$; compare section 5*
2. *results for the special case of $N-$ fold symmetric perturbations, $V-V_0=\delta\tilde{V}(r,N\theta)$; compare section 6*
3. *a homogenization / multiple scale expansion of eigenvalues as well as comparison and agreement with the eigenvalue perturbation theory through second order in N^{-1}; compare section 7*

3. Formulation of the scattering resonance problem

3.1. Elementary review of resonances

Our particular interest is in the resonances of H, which we will construct as perturbations of the the resonances of $H_0=-\Delta+V_0$. Therefore, we must first consider the latter resonances. Their existence for the restricted class of potentials we consider is well known [2]. We will briefly review the arguments as they will serve as a starting point for the proofs of our main results.

Proposition 3.1. *The resolvent $R_{V_0}(\mu)=(-\Delta+V_0-\mu^2)^{-1}$ can be analytically continued from the half plane $\Im\mu>0$ to the entire plane $\mathbb{C}$, when n is odd, or to the logarithmic Riemann surface Λ when n is even. The analytic continuation of the resolvent is meromorphic in μ, with residues at the poles corresponding to finite rank operators associated with nontrivial solutions of*

$$(-\Delta+V_0-\mu^2)u=0 \tag{3.1}$$

that satisfy either outgoing or incoming boundary conditions at infinity.

The half plane $\Im\mu > 0$ is often called the "physical sheet" and $\Im\mu < 0$, the "unphysical sheet". As seen below, poles of $R_{V_0}(\mu)$ on the physical sheet correspond to eigenvalues, $E = \mu^2 < 0$ ($\Re\mu = 0, \Im\mu > 0$). Those on the unphysical sheet correspond to outgoing ($\Re\mu > 0, \Im\mu < 0$) or incoming ($\Re\mu < 0, \Im\mu < 0$) solutions, which are not L^2. These are called *resonances*. See figure 4.

Proof of Proposition 3.1: It is shown in [10] that in the physical upper half plane $\Im\mu > 0$, the resolvent R_{V_0} is meromorphic, with poles possibly existing on the imaginary axis. Poles, μ, in the upper half plane correspond to eigenvalues, $E = \mu^2 < 0$, of H_0. Away from its poles, the resolvent $R_{V_0}(\mu)$ satisfies

$$(-\Delta + V_0 - \mu^2)R_{V_0}(\mu) = I$$

which, since the domains of $-\Delta + V_0$ and Δ are equal, can be rewritten as

$$(I + R_0(\mu)V_0)R_{V_0}(\mu) = R_0(\mu). \tag{3.2}$$

The kernel of the free resolvent $R_0(\mu)$ is known explicitly [16]

$$R_0(\mu; x, y) = \frac{1}{4i}\left(\frac{\mu}{2\pi|x-y|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n-2}{2}}(\mu|x-y|) \tag{3.3}$$

where $H^{(1)}_{\frac{n-2}{2}}$ is a Hankel function [1]. In odd dimensions $n \geq 3$, this Hankel function is entire as a function of μ, while in even dimensions it is entire on the logarithmic covering Λ of $\mathbb{C}$. The large $|z|$ asymptotics of Hankel functions are displayed in appendix A. $H^{(1)}_\nu(z)$ satisfies an *outgoing* radiation condition as $|z| \to \infty$.

In order to analytically continue $R_{V_0}(\mu)$ from the physical half-plane to $\Im\mu < 0$, it is evident from (3.3) and the asymptotic behavior of $H^{(1)}_\nu$, (A.4) that we will need to work in a larger Hilbert space, since $H^{(1)}(\mu|x-y|)$ maps the space of square integrable functions, which are compactly supported in $|x| \leq r_0$, $L^2_c(r_0)$ to $e^{\alpha|x|}L^2(\mathbb{R}^n)$, with $\alpha > |\Im\mu|$. It is possible to formulate the analytic continuation in $L^2(\mathbb{R}^n)$ by explicitly introducing localizing functions $\chi(x)$ satisfying

$$0 < \chi(x) = \mathcal{O}(e^{-\alpha|x|}); \tag{3.4}$$

see (1.15). Multiplying (3.2) on the left by $\chi(x)$, we obtain

$$(I + \chi R_0(\mu)V_0\chi^{-1})\chi R_{V_0}(\mu) \;=\; \chi R_0(\mu). \tag{3.5}$$

Both sides of (3.5) are considered as mappings from $L^2_c(\mathbb{R}^n)$, the space of compactly supported L^2 functions, to $L^2(\mathbb{R}^n)$.

For $|\Im\mu| < \alpha$, it can be shown by direct estimation of the kernel associated with the operator $\chi R_0 V_0 \chi^{-1}$ using (3.3) that [2]

$$\chi R_0(\mu) V_0 \chi^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is Hilbert-Schmidt, and therefore compact. Furthermore, it is also analytic in this region, and its norm tends to zero as $\Im\mu \to +\infty$ in the physical sheet Therefore, the analytic Fredholm theorem [22] tells us that $(I + \chi R_0 V_0 \chi^{-1})^{-1}$ is meromorphic in the domain of analyticity of μ, and that the residues at the poles are finite-rank operators. From (3.5) and (3.3) is it clear that $\chi R_{V_0}(\mu)$ inherits the analyticity of $(I + \chi R_0(\mu) V_0 \chi^{-1})^{-1} \chi R_0(\mu)$.

Furthermore, if μ is at a pole, then

$$(I + \chi R_0 V_0 \chi^{-1}) F_0 = 0 \tag{3.6}$$

has a nontrivial solution $F_0 \in L^2(\mathbb{R}^n)$. Setting $\psi_0 = \chi^{-1} F_0$, we see that

$$\psi_0 = -R_0 V_0 \psi_0 \tag{3.7}$$

which is equivalent to (3.1). Also, if the analytic continuation in μ is from the physical half-plane to the fourth quadrant across the positive real axis, then (3.7) along with the asymptotic behavior (A.4) of the Hankel function in (3.3) allows us to conclude that the solution is outgoing at infinity; if the continuation is across the negative real axis to the third quadrant, then it is incoming. □

3.2. Resonances of microstructure potentials

We can now begin our study of resonances of microstructured potentials $V = V_0 + \delta V$. The guiding intuition is that a resonance pair (E_0, ψ_0) of V_0 should perturb to a nearby pair (E, ψ) of V, if δV is sufficiently oscillatory, depending on the pair (E, ψ). To make these statements precise, we take (3.6) as the starting point. We assume that (μ^2, Ψ) are a resonance pair, with $\Psi \in L^2(\mathbb{R}^2)$, and write $\psi = \chi^{-1}\Psi$. We then introduce the smoothing operator $\langle D\rangle^{-1}$ and its inverse, $\langle D\rangle$, given in (1.14) to obtain

$$(I + \underbrace{\langle D\rangle \chi R_0(\mu) \chi \langle D\rangle}_{T_R(\mu)} \underbrace{\langle D\rangle^{-1} \chi^{-1} V \chi^{-1} \langle D\rangle^{-1}}_{T_V}) (\langle D\rangle \chi \psi) \tag{3.8}$$

Theorem 3.1. *Resonances are solutions (Ψ, μ), with $0 \neq \Psi \in L^2$, and $\Im\mu < 0$ of*

$$(\ I \ + \ T_R(\mu) T_V \ ) \ \Psi \ = \ 0, \tag{3.9}$$

where $T_R(\mu)$ and T_V are defined in (3.8).

Remark 3.1. *(1) The properties of $T_R(\mu)$ and T_V, which validate (3.9) as an alternative formulation of the resonance problem are stated and proved below.*
(2) The localizing operators χ are used to transfer decay from the potential to localization of the free resolvent, which facilitates analytic continuation in μ to the lower half plane, where resonance energies are found.
(3) The operators $\langle D\rangle^{-1}$ and $\langle D\rangle$ transfer smoothness from the free resolvent R_0 to act on the microstructured potential V. This latter property enables the perturbative treatment of high contrast microstructures, as explained in section 1.3.

To prove Theorem 3.1 and to work with the formulation (3.9) we require the following two lemmas:

Lemma 3.1. *The operator*

$$T_V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is Hilbert-Schmidt, and therefore compact, for $n \le 3$.

Lemma 3.2. *Assume $n \le 3$ and let $k \ge 0$ be arbitrary. The operator*

$$\partial_\mu^k T_R(\mu) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

is defined, analytic and bounded for $\Im\mu > -\tilde{M} > -m$.

Corollary 3.1. *The operator*

$$T_R T_V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is compact for $\Im\mu > -\tilde{M} > -m$ and $n \le 3$.

Proof of Lemma 3.1: Let

$$\tilde{V} = \chi^{-1} V \chi^{-1} \tag{3.10}$$

Note that $\tilde{V}$ has the same support as V. To prove that T_V is compact, note that

$$\langle D\rangle^{-1}\tilde{V}\langle D\rangle^{-1} f = \int K(x,y) f(y)\, dy,$$

where

$$K(x,y) = \int \frac{d\xi}{(2\pi)^n} \frac{d\eta}{(2\pi)^n}\, e^{i(x\cdot\xi - y\cdot\eta)} \frac{\hat{\tilde{V}}(\xi-\eta)}{\langle\xi\rangle\langle\eta\rangle}, \tag{3.11}$$

where $\langle D\rangle^{-1} = (m^2 - \Delta)^{-1/2}$ and $\langle\xi\rangle = (m^2 + \xi^2)^{1/2}$. We claim that

$$\int\int |K(x,y)|^2 \, dx\, dy < \infty, \tag{3.12}$$

implying that T_V is a Hilbert Schmidt operator and therefore compact on L^2.

We now verify (3.12). The Fourier transform of $K(x,y)$ is given by:

$$\hat{K}(\xi,\eta) = \frac{\hat{\tilde{V}}(\xi+\eta)}{(m^2+|\xi|^2)^{\frac{1}{2}}\ (m^2+|\eta|^2)^{\frac{1}{2}}} \tag{3.13}$$

By the Plancherel theorem

$$\begin{aligned} \int dx\, dy\ |K(x,y)|^2 &= (2\pi)^{2n} \int d\xi\, d\eta\, |\hat{K}(\xi,\eta)|^2 \\ &= (2\pi)^{2n} \int d\xi\, d\eta\, \frac{|\hat{\tilde{V}}(\xi+\eta)|^2}{(m^2+|\xi|^2)\ (m^2+|\eta|^2)} \\ &= (2\pi)^{2n} \int \frac{1}{(m^2+|\xi|^2)}\, d\xi \int |\hat{\tilde{V}}(\eta+\xi)|^2 \frac{1}{(m^2+|\eta|^2)}\, d\eta \end{aligned} \tag{3.14}$$

The inner integral $(d\eta)$ is the convolution of an L^1 function, $|\hat{\tilde{V}}|^2$ and, for dimension $n < 4$, an L^2 function, $\langle\eta\rangle^{-2}$, and is therefore in $L^2(d\eta)$ by Young's inequality, (1.16). It follows that the integrand of the outer integral $(d\xi)$ is the product of L^2 functions and by the Cauchy-Schwarz inequality

$$\int dx\, dy\, |K(x,y)|^2 \leq (2\pi)^{2n} \|\langle\xi\rangle^{-2}\|_2^2\ \|\tilde{V}\|_2^2\ < \infty. \tag{3.15}$$

Lemma 3.1 now follows. □

The following lemma, concerning analyticity properties of the free resolvent kernel, is central to analytic extension of $T_R(\mu)$ into the lower half plane and the proof of Lemma 3.2.

Lemma 3.3. *Let $n \geq 2$. For $j = 0, 1$, the operator*

$$\mathcal{L}_j(\mu) = R_0(\mu)\langle D\rangle^j \tag{3.16}$$

has a kernel $\mathcal{L}_j(\mu; x, y)$ that can be analytically continued from the (physical) upper half plane to the lower half plane, with a branch cut starting at $-im$. The kernel is given by

$$\mathcal{L}_j(\mu; x, y) = m^{j+n-2} G_{2-j}\big(m(x-y)\big) + (m^2+\mu^2) m^{j+n-4} G_{4-j}\big(m(x-y)\big) + \mathcal{R}_j(\mu; x, y). \tag{3.17}$$

Here, $G_\alpha(x)$ denotes the kernel associated with the Bessel potential $(I-\Delta)^{-\frac{\alpha}{2}}$, define in terms of the Fourier transform [25]

$$\hat{G}_\alpha(\xi) = (m^2+\xi^2)^{-\frac{\alpha}{2}}. \tag{3.18}$$

For μ in the physical upper half plane, $\mathcal{R}_j$ is defined by

$$\mathcal{R}_j(\mu;x,y) = \frac{(m^2+\mu^2)^2}{(2\pi)^n} e^{i\xi\cdot(x-y)} \frac{d\xi}{(\xi^2-\mu^2)(m^2+\xi^2)^{2-j/2}}.$$

The first two terms in (3.17) form a quadratic polynomial in μ and is therefore an entire function of μ. The analytic continuation of $\mathcal{R}_j(\mu;x,y)$ from $\Im\mu>0$ to $\Im\mu<0$ is given by

$$\begin{aligned}\mathcal{R}_j(\mu;x,y) &= \mathcal{R}_j(-\mu;x,y) + \\ &\qquad (m^2+\mu^2)^{j/2}\frac{\pi i}{(2\pi)^n}\left(\frac{\mu}{|x-y|}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\mu|x-y|),\end{aligned} \tag{3.19}$$

where J_α is the Bessel function [1]. Furthermore, $\mathcal{R}_j(\mu;x,y)$ satisfies

$$\mathcal{R}_j(\mu;x,y) = (m^2+\mu^2)^{j/2} R_0(\mu;x,y) + \mathcal{R}_j^{(2)}(\mu;x-y) \tag{3.20}$$

where, $R_0(\mu;x,y)$ denotes the free resolvent kernel given by (3.3) and

$$|\partial_\mu^k \mathcal{R}_j^{(2)}(\mu;x,y)| \le C(b,k)e^{-b|x-y|}, \quad \Im\mu\ge 0 \tag{3.21}$$

for any $b<m$ and any $k\ge 0$.

Proof of Lemma 3.3: For $\Im\mu>0$ we represent the kernel of $\mathcal{L}_j(\mu)$ using the Fourier transform:

$$\mathcal{L}_j(\mu;x,y) = \int \frac{d\xi}{(2\pi)^n} e^{i\xi\cdot(x-y)} \frac{(m^2+\xi^2)^{j/2}}{\xi^2-\mu^2}. \tag{3.22}$$

Using the algebraic identity

$$\frac{1}{\xi^2-\mu^2} = \frac{1}{m^2+\xi^2} + \frac{m^2+\mu^2}{(\xi^2-\mu^2)(m^2+\xi^2)}, \tag{3.23}$$

we have

$$\begin{aligned}\mathcal{L}_j(\mu;x,y) &= \int \frac{d\xi}{(2\pi)^n} e^{i\xi\cdot(x-y)} \Big[\frac{1}{(m^2+\xi^2)^{1-j/2}} + \frac{(m^2+\mu^2)}{(m^2+\xi^2)^{2-j/2}} \\ &\qquad + \frac{(m^2+\mu^2)^2}{(\xi^2-\mu^2)(m^2+\xi^2)^{2-j/2}}\Big].\end{aligned}$$

The expansion (3.17) now follows from definition (3.18).

The first two terms of (3.17) are entire in μ. To analytically continue the remaining term $\mathcal{R}_j(\mu;x,y)$ we will adapt an argument in [15]. We begin by rewriting $\mathcal{R}_j$ as

$$\mathcal{R}_j(\mu;x,y) = \frac{(m^2+\mu^2)^2}{(2\pi)^n} \int_{S^{n-1}} d\omega \int_0^\infty d\rho\, \rho^{n-1} e^{i\rho\omega\cdot(x-y)} \frac{1}{(\rho^2-\mu^2)(m^2+\rho^2)^{2-j/2}}$$

We now focus on the ρ integral. Its integrand is analytic in ρ except for poles at $\rho = \pm\mu$ and, for the case of $j = 1$, a branch cut starting at $\rho = \pm im$. Let $\mu_1 = \mu' + i\epsilon_1$ and $\mu_2 = -\mu' + i\epsilon_2$, where $\mu' \in \mathbb{R}_+$ and $\epsilon_j > 0$, denote points in the first and second quadrants. We begin by considering $\mathcal{R}(\mu_1; x, y)$ and $\mathcal{R}(\mu_2; x, y)$ separately. $\mathcal{R}(\mu_1; x, y)$ can be expressed as an integral over the contour in figure 5, part of which is the lower half of a circle about μ' traversed counterclockwise, $\gamma_-(\mu')$. $\mathcal{R}(\mu_2; x, y)$ can be expressed as an integral over the contour in figure 6, part of which is the upper half of a circle about μ' traversed clockwise, $\gamma_+(\mu')$. In this representation of $\mathcal{R}(\mu_2; x, y)$, we let ϵ_2 approach $-\epsilon_1$ ($\mu_2 \to -\mu_1$) to obtain a representation of $\mathcal{R}(-\mu_1; x, y)$, in terms of integration over a contour containing γ_+; see figure 7. The difference, $\mathcal{R}(\mu_1; x, y) - \mathcal{R}(-\mu_1; x, y)$, has an integral representation over the contour homotopic to $\gamma_-(\mu') - \gamma_+(\mu')$, a circle about μ' traversed counterclockwise; see figure 8. Thus

$$\begin{aligned}
&\mathcal{R}_j(\mu_1; x, y) - \mathcal{R}_j(-\mu_1; x, y) \\
&\quad = (m^2 + \mu_1^2)^2 (2\pi)^{-n} \int_{S^{n-1}} d\omega \oint_{\gamma_+ - \gamma_-} d\rho\, \rho^{n-1} e^{i\rho\omega\cdot(x-y)} \frac{1}{(\rho^2 - \mu_1^2)(m^2 + \rho^2)^{2-j/2}} \\
&\quad = (m^2 + \mu_1^2)^{j/2} \mu_1^{n-2} (\pi i)(2\pi)^{-n} \int_{S^{n-1}} d\omega\, e^{i\mu_1\omega\cdot(x-y)}. \\
&\quad = (m^2 + \mu_1^2)^{j/2} (\pi i)(2\pi)^{-n/2} \left(\frac{\mu_1}{|x-y|}\right)^{\frac{n-2}{2}} J_{\frac{n-2}{2}}(\mu_1|x-y|).
\end{aligned} \tag{3.24}$$

In the second equality of (3.24), we have performed the contour integral and in the third we have used the well-known expression (1.18) for the Fourier transform of the uniform measure on the sphere, S^{n-1}, in terms of Bessel functions. This completes the proof of (3.19).

To prove (3.20), we rewrite the kernel $\mathcal{R}_j$ as

$$\begin{aligned}
\mathcal{R}_j(\mu; x, y) &= (m^2 + \mu^2)^{j/2} \int \frac{d\xi}{(2\pi)^n} e^{i\xi\cdot(x-y)} \frac{1}{\xi^2 - \mu^2} + \\
&\quad (m^2 + \mu^2)^2 \int \frac{d\xi}{(2\pi)^n} e^{i\xi\cdot(x-y)} \left(\frac{(m^2 + \xi^2)^{j/2-2} - (m^2 + \mu^2)^{j/2-2}}{\xi^2 - \mu^2} \right).
\end{aligned} \tag{3.25}$$

We recognize the first term in (3.25) as a multiple of a Hankel function, which for $\Im\mu > 0$, by a contour deformation decays exponentially as $|x - y| \to \infty$. It is therefore proportional to the outgoing Hankel function. The integrand in the second term, along with any number of derivatives with respect to μ, is analytic in ξ in a strip around the real axes, and has sufficiently rapid decay as $|\xi| \to \infty$ so that a Paley-Wiener type theorem (e.g. Theorem IX.13 of [22]) applies to prove (3.21) for $n = 2, 3$. □

Proof of Lemma 3.2

For $\Im\mu > 0$ the boundedness of T_R in L^2 follows from bounding its Fourier transform, which is simple because the resolvent kernel decays exponentially for $\Im\mu > 0$. However, the analytic continuation of the resolvent kernel to $\Im\mu < 0$ grows exponentially, so we must proceed more carefully. We use a combination of techniques; see, for example, [2], [15].

By commuting $\langle D\rangle$, we rewrite T_R as a sum of operators, whose kernels can be analytically continued. The proof uses two technical results, Lemmata 3.4 and 3.5, which are stated and proved at the end of this section.

First consider the case $k = 0$.

$$\begin{aligned}
T_R &\equiv \langle D\rangle\chi R_0(\mu)\chi\langle D\rangle \\
&= \big(\chi\langle D\rangle + [\langle D\rangle,\chi]\big)R_0(\mu)\big(\langle D\rangle\chi + [\chi,\langle D\rangle]\big) \\
&= \chi\langle D\rangle R_0\langle D\rangle\chi + \big([\langle D\rangle,\chi]\tilde{\chi}^{-1}\big)\big(\tilde{\chi}R_0\langle D\rangle\chi\big) + \big(\chi\langle D\rangle R_0\tilde{\chi}\big)\big(\tilde{\chi}^{-1}[\chi,\langle D\rangle]\big) \\
&\qquad + \big([\langle D\rangle,\chi]\tilde{\chi}^{-1}\big)\big(\tilde{\chi}R_0\tilde{\chi}\big)\big(\tilde{\chi}^{-1}[\chi,\langle D\rangle]\big) \\
&= A_{\mathrm{I}} + A_{\mathrm{II}}^{(a)} + A_{\mathrm{II}}^{(b)} + A_{\mathrm{III}}
\end{aligned} \tag{3.26}$$

and estimate each of the terms in succession.

We begin with A_{I}. For μ in the upper half plane, the Fourier representation of $R_0(\mu)$ is valid, and we may therefore commute one of the $\langle D\rangle$ operators through the resolvent and apply the identity

$$\chi R_0(\mu)\langle D\rangle^2\chi = \chi I\chi + (m^2+\mu^2)\chi R_0(\mu)\chi. \tag{3.27}$$

The expression (3.27) enables us to carry out the analytic continuation of A_{I} from the upper half plane to the lower half plane. Boundedness on L^2 in the required region, follows from Lemma 3.4 with $k=0$ below and (3.3).

Moving to the next two terms, we observe that it is sufficient to discuss only one of $A_{\mathrm{II}}^{(a)}$ and $A_{\mathrm{II}}^{(b)}$, since $\langle D\rangle$ commutes with R_0 in the upper half plane, and the analytic continuation is given by Lemma 3.3. We decompose

$$A_{\mathrm{II}}^{(a)} = \big([\langle D\rangle,\chi]\tilde{\chi}^{-1}\big)\ \big(\tilde{\chi}R_0\langle D\rangle\chi\big)$$

and note that the first term is bounded by Lemma 3.5. We show boundedness of the second term in two steps. We first consider $\Im\mu \geq 0$. Then (3.17) and (3.20) allow us to express $R_0(\mu)\langle D\rangle$ as a sum of four operators. The two containing Bessel kernels are clearly bounded, while the term containing R_0 is bounded by Lemma 3.4. That $\mathcal{R}_j^{(2)}$ is bounded can be seen by computing the $L^1 \to L^\infty$ norm of its kernel, using (3.21).

Next, choose μ such that $\Im\mu > -m$. Then (3.19) allows us to express $R_0(\mu)\langle D\rangle$ as a sum of two terms, the first of which, $R_0(-\mu)\langle D\rangle$, we have already bounded, and the second of which is bounded by Lemma 3.4.

Finally, the last term A_{III} is bounded by application of Lemmas 3.5 and 3.4.

The result for $k > 0$ is obtained by differentiating the kernels R_0 and $\mathcal{L}_1$ in (3.26) and proceeding as above, noting that Lemmas 3.3 and 3.4 can be applied for any $k > 0$. □

Lemma 3.4. *The operator mapping $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by the kernel*

$$\partial_\mu^k \left[\chi(x) \left(\frac{\mu}{|x-y|} \right)^{\frac{n-2}{2}} \mathcal{J}_{\frac{n-2}{2}}(\mu|x-y|)\chi(y) \right] \tag{3.28}$$

is Hilbert-Schmidt, and therefore compact and, in particular, bounded, for $\Im\mu > -m$ and any $k \geq 0$. In (3.28), $\mathcal{J}$ may be any of the Bessel functions J, Y, $H^{(1)}$, or $H^{(2)}$.

This result, for dimension $n = 3$, follows from the proof in [2]; see, in particular, the analysis of the operator denoted T_κ. The case of general n follows the same line of argument.

Lemma 3.5. *Let χ and $\tilde{\chi}$ be exponentially localized functions with exponential rates M and $\tilde{M}$ as introduced in (1.15). Then, the operator*

$$\mathcal{C} = [\langle D\rangle, \chi]\tilde{\chi}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \tag{3.29}$$

is bounded when $\tilde{M} < \min(m, M/\sqrt{2})$. For any $f \in L^2(\mathbb{R}^n)$

$$\overline{\mathcal{C}f} = \mathcal{C}\overline{f}. \tag{3.30}$$

The adjoint of $\mathcal{C}$, $\mathcal{C}^*$, is given by

$$\mathcal{C}^* = \tilde{\chi}^{-1}[\chi, \langle D\rangle]. \tag{3.31}$$

Proof of Lemma 3.5: We first show that $\mathcal{C}$ is bounded. Let f denote any member of $L^2(\mathbb{R}^n)$. Then

$$\mathcal{C}f \equiv F(x) = \int dy \int \frac{d\zeta}{(2\pi)^n} \frac{d\eta}{(2\pi)^n}\, e^{i\zeta\cdot x - i\eta\cdot y} \hat{\chi}(\zeta - \eta)\big(\langle\zeta\rangle - \langle\eta\rangle\big)\tilde{\chi}^{-1}(y) f(y). \tag{3.32}$$

Since $\|\mathcal{C}f\|_2 = \|\hat{F}\|_2$ it suffices to show that $\|\hat{F}\|_2$ is bounded above by $C\|f\|_2$ for some postive constant, C. The integrand in (3.32) is analytic in ζ varying over the product of strips around the real axes, $-\infty < \zeta_j < \infty$, and has sufficient decay as $|\zeta_j| \to \infty$ so that we may shift the ζ contours $\zeta_j \to \zeta_j + i\gamma_j$, with $|\gamma| < \min(m, M)$. We next write

$$\tilde{\chi}^{-1}(y) = \sum_{s \in \{-1,1\}^n} \mathbf{1}_{\mathrm{sgn}(y)=s}(y)\tilde{\chi}^{-1}(y)$$

and define $\gamma_j = \gamma_j(s) = -vs_j$ where v is chosen to satisfy

$$n^{-1/2}\tilde{M} < v < n^{-1/2}\min(m, M). \tag{3.33}$$

We note that the first inequality in (3.33) will ensure that[1]

$$\|\mathbf{1}_{\mathrm{sgn}(y)=s}(y)\,\tilde{\chi}^{-1}(y)\,e^{-v|y|_1}\,f(y)\|_2 \le C\|f\|_2, \tag{3.34}$$

while the second ensures that $|\gamma| < \min(m, M)$, so that the contour shift is permitted.

After the contour shift, we have

$$\begin{aligned}\hat{F}(\xi) \;=\; \sum_{s\in\{-1,1\}^n} \int dy \int \frac{d\eta}{(2\pi)^n}\, e^{-i\eta\cdot y} \hat{\chi}(\xi-\eta-i\gamma(s))\big(\langle\xi\rangle - \langle\eta+i\gamma(s)\rangle\big) \times \\ \mathbf{1}_{\mathrm{sgn}(y)=s}(y)\tilde{\chi}^{-1}(y)\,e^{-v|y|_1}\,f(y).\end{aligned}$$

To continue, we define

$$g_s(y) = \mathbf{1}_{\mathrm{sgn}(y)=s}(y)\tilde{\chi}^{-1}(y)e^{-v|y|_1}$$

and carry out the dy integral:

$$\hat{F}(\xi) = \sum_{s\in\{-1,1\}^n} \int \frac{d\eta}{(2\pi)^n}\, \underbrace{\hat{\chi}(\xi-\eta-i\gamma)}_{F_1} \underbrace{\big(\langle\xi\rangle - \langle\eta+i\gamma\rangle\big)}_{F_2} \hat{g}_s(\eta). \tag{3.35}$$

We may bound F_1 of (3.35) as follows. Choose $\gamma^2 = (\delta/2)M^2$, with $\delta \in (0,1)$, and set $x = \xi - \eta$. Then

$$\begin{aligned}|M^2 + (x-i\gamma)^2|^2 \;&=\; (M^2+x^2)^2 + \gamma^4 - 2\gamma^2(M^2+x^2) + 4(x\cdot\gamma)^2 \\ &\ge\; (M^2+x^2)(M^2+x^2-2\gamma^2) \\ &\ge\; \big((1-\delta)M^2 + x^2\big)^2. \end{aligned} \tag{3.36}$$

and it follows that

$$|F_1| \le \big((1-\delta)M^2 + (\xi-\eta)^2\big)^{-4}. \tag{3.37}$$

We bound F_2 of (3.35) as follows. Write $F_2 = F_2^{(1)} + F_2^{(2)}$, with

$$\begin{aligned}F_2^{(1)} \;&=\; \big(m^2+\xi^2\big)^{1/2} - \big(m^2+\eta^2\big)^{1/2} \\ F_2^{(2)} \;&=\; \big(m^2+\eta^2\big)^{1/2} - \big(m^2+(\eta+i\gamma)^2\big)^{1/2}.\end{aligned}$$

We observe that $|F_2^{(1)}| \le C|\xi-\eta|$. We may bound $|F_2^{(2)}|$ by noting that it can be written as

$$F_2^{(2)} = |\eta|\,(1+f_1(\eta)) - |\eta|\,(1+f_2(\eta))$$

[1] For $y \in \mathbb{R}^n$ $|y|_1 = |y_1| + \cdots + |y_n|$.

where

$$|f_1(\eta)| \le O(|\eta|^{-2}), \quad |f_2(\eta)| \le O(|\eta|^{-1}).$$

Then clearly $|F_2^{(2)}|$ is bounded by a constant, and

$$|F_2| \le C + |\xi - \eta|. \tag{3.38}$$

Finally, we assemble (3.37) and (3.38) by writing

$$|\hat{F}(\xi)| \le (2\pi)^{-n} \sum_{s\in\{-1,1\}^n} \big(h \star |\hat{g}_s|\big)(\xi)$$

with

$$h(\xi-\eta) = \big((1-\delta)M^2 + (\xi-\eta)^2\big)^{-4}\left(C + |\xi-\eta|\right).$$

Then by Young's inequality

$$\|\mathcal{C}f\|_2 = \|\hat{F}\|_2 \le \pi^{-n}\,\|h\|_1 \sup_{s\in\{-1,1\}^n} \|\hat{g}_s\|_2 \le C\|f\|_2.$$

The expression (3.30) follows from (3.32), as does (3.31) for the adjoint $\mathcal{C}^*$, along with the fact that $\mathcal{C}$ is bounded. □

4. Perturbation theory of scattering resonances

In this section we state and prove a perturbation theorem for solutions of the scattering resonance problem. We first consider the case where the resonance subspace being perturbed has dimension one. This result can be generalized in a straightforward manner to higher multiplicity. We then turn to the case the two-dimensional radial case, $V_0 = V_0(r)$ and show how the perturbation theory of degenerate pairs of resonances (occuring for angular momentum $|l| \ge 1$) can be reduced to the above case. The latter results are straightforward to generalize.

4.1. Perturbations of simple scattering resonances

Theorem 4.1. *(a) Let V_0 denote a potential for which $|||V_0|||$ is finite, where*

$$|||V_0||| \;\equiv\; \|T_{V_0}\|_{\mathcal{B}(L^2)} \;=\; \|\,\langle D\rangle^{-1}\,V_0\langle D\rangle^{-1}\|_{\mathcal{B}(L^2)}. \tag{4.1}$$

Let (E_0,ψ_0) denote a solution of the corresponding scattering resonance problem. Assume

$$\dim\left\{\ker\left(I + T_R(E_0)\,T_{V_0}\right)\right\} = 1 \tag{4.2}$$

and this subspace is spanned by the function

$$\Psi_0 = \langle D\rangle \chi\psi_0;$$

(see (3.8)). Furthermore, assume the condition

$$\langle T_{V_0}\overline{\Psi}_0, T_R'(E_0)T_{V_0}\Psi_0\rangle \neq 0. \tag{4.3}$$

Then, there exists ε_0 such that for any potential V satisfying $|||V - V_0||| < \varepsilon_0$, there corresponds a unique solution $(E(V), \psi(V))$ of the scattering resonance problem which lies near (E_0, ψ_0).
(b) The mapping

$$V \mapsto (E, \psi) \in \mathbb{C} \times H^1(\chi(x)dx),$$

which associates to a potential V a solution $(E(V), \psi(V))$ of the scattering resonance problem, is analytic in the norm $|||V|||$ in a neighborhood $|||V - V_0||| < \varepsilon_0$ of V_0.

Remark 4.1. *In the application of section 6, we take $V = V_0(r)$, $r = |x|$, $x \in \mathbb{R}^2$. Radial symmetry of the potential implies a degeneracy of modes corresponding to non-trivial angular momentum, i.e. $\psi_0 = \psi_{0,l}e^{\pm il\theta}$. In Theorem 4.2 we show how this kind of degeneracy can be treated by a reduction to Theorem 4.1.*

Proof of Theorem 4.1: We formulate the problem so the result follows from the implicit function theorem For $|||V - V_0|||$ small we seek a solution of the scattering resonance problem

$$\left(I + T_R(E)T_V\right)\Psi = 0 \tag{4.4}$$

in the form:

$$E = E_0 + \delta E \tag{4.5}$$

$$\Psi = \Psi_0 + \delta\Psi, \tag{4.6}$$

where

$$\left(I + T_R(E_0)T_{V_0}\right)\Psi_0 = 0. \tag{4.7}$$

Substitution of (4.5,4.6) into (4.4) and use of (4.7) yields:

$$\begin{aligned}\left(I + T_R(E_0)T_{V_0}\right)\delta\Psi &= -T_R(E_0)T_{\delta V}\Psi_0 - \delta T_R(E_0, \delta E)T_{V_0}\Psi_0 \\ &\quad -\mathcal{Q}(\delta V, \delta\Psi, \delta E; V_0, \Psi_0, E_0).\end{aligned} \tag{4.8}$$

with

$$\delta T_R(E_0, \delta E) = T_R(E_0 + \delta E) - T_R(E_0)$$

and $\mathcal{Q}$ consisting only of the quadratic and higher order terms:

$$\begin{aligned} \mathcal{Q}(\delta V, \delta\Psi, \delta E; V_0, \Psi_0, E_0) &= \big(T_R(E_0)T_{\delta V} + \delta T_R(E_0, \delta E)T_{V_0}\big)\delta\Psi \\ &\quad + \delta T_R(E_0, \delta E)T_{\delta V}\left(\Psi_0 + \delta\Psi\right). \end{aligned} \tag{4.9}$$

We will apply the analytic Fredholm theorem to (4.8). The solvability condition requires an understanding of the adjoint operator: $I + T_{V_0}^* T_R(E_0)^*$. Note that since V_0 is real valued T_{V_0} is self-adjoint.

Lemma 4.1. *Under condition (4.2) of Theorem 4.1, the adjoint operator $I + T_{V_0} T_R(E_0)^*$ has a one dimensional null space spanned by the function*

$$\Psi_0^{\#} = T_{V_0} \overline{\Psi}_0. \tag{4.10}$$

Therefore, the inhomogeneous problem

$$\left(I + T_R(E_0) T_{V_0}\right) U = S$$

has a solution for $S \in L^2$ if and only if

$$\langle T_{V_0} \overline{\Psi}_0, S \rangle = 0 \tag{4.11}$$

Proof of Lemma 4.1: Since $T_R(E_0)T_{V_0}$ is compact the dimensions of the null spaces of $I + T_R(E_0)T_{V_0}$ and $I + T_{V_0}T_R(E_0)^*$ are equal [12].

To construct the adjoint null space, observe that an element of the null space satisfies:

$$\left(I + T_R(E_0) T_{V_0}\right) \Psi_0 = 0 \tag{4.12}$$

Applying the operator T_{V_0} we get

$$\left(I + T_{V_0} T_R(E_0)\right) T_{V_0} \Psi_0 = 0. \tag{4.13}$$

Taking the complex conjugate of (4.13) yields:

$$\overline{T_{V_0} \Psi_0} + \overline{T_{V_0} T_R(E_0) T_{V_0} \Psi_0} = 0. \tag{4.14}$$

The lemma follows from the observation that $\overline{T_{V_0} f} = T_{V_0} \overline{f}$, since V_0 is real-valued, and the conclusion of Lemma 4.2. □

Lemma 4.2. *For any $\mu \in \mathbb{C} \setminus \{0\}$ satisfying $\Im\mu > -\tilde{M} > -m$ and $f \in L^2(\mathbb{R}^n)$,*

$$\overline{T_R(\mu)f} = T_R(\mu)^* \overline{f}.$$

Proof of Lemma 4.2: We employ the decomposition (3.26) of T_R. We first note that the result for $A_{\rm I}$ follows from the use of the identity (3.27), Lemma 3.4, and the fact that the free resolvent is symmetric, as can be seen from (3.3). We next consider

$$A_{\rm II} = A_{\rm II}^{(a)} + A_{\rm II}^{(b)} = \mathcal{C}\big(\tilde{\chi}\mathcal{L}_1(\mu)\chi\big) + \big(\chi\mathcal{L}_1(\mu)\tilde{\chi}\big)\mathcal{C}^*$$

where we have used the notation of (3.29) and (3.16). The result follows from Lemma 3.5, which tells us that $\overline{\mathcal{C}f} = \mathcal{C}\overline{f}$ for $f \in L^2$; Lemma 3.3, which tells us that $\mathcal{L}_1$ has a symmetric kernel; and Lemma 3.4, which tells us that $\tilde{\chi}\mathcal{L}_1(\mu)\chi$ is a bounded operator from $L^2 \to L^2$ for $\Im\mu > -\tilde{M}$. The result for $A_{\rm III}$ follows similarly. □

Continuing with the proof of Theorem 4.1, we note that (4.11) gives us an implicit condition for the solvability of (4.8) for $\delta\Psi$, obtained by setting the inner product of the right hand side of equation (4.8) with $T_{V_0}\overline{\Psi}_0$ equal to zero:

$$\begin{aligned} \big\langle T_{V_0}\overline{\Psi}_0, \delta T_R(E_0, \delta E) T_{V_0}\Psi_0\big\rangle + \big\langle T_{V_0}\overline{\Psi}_0, T_{\delta V}\Psi_0\big\rangle & \\ + \big\langle T_{V_0}\overline{\Psi}_0, \mathcal{Q}(\delta V, \delta\Psi, \delta E; V_0, \Psi_0, E_0)\big\rangle &= 0 \end{aligned} \tag{4.15}$$

We now view the task of scattering resonance problem as that of seeking a solution $(\Psi(V), E(V))$ of the system of equations (4.8),(4.15). In compact form we write:

$$\mathbf{F}(\delta V, \delta\Psi, \delta E) = \mathbf{0}, \tag{4.16}$$

where $\mathbf{F} = (F_1, F_2)$ and

$$\begin{aligned} F_1(\delta V, \delta\Psi, \delta E) &= (I + T_R(E_0)T_{V_0})\,\delta\Psi + \big(T_R(E_0)T_{\delta V} + \delta T_R(E_0, \delta E)T_{V_0}\big)\Psi_0 \\ &\quad + \mathcal{Q}(\delta V, \delta\Psi, \delta E; V_0, \Psi_0, E_0) \qquad (4.17) \\ F_2(\delta V, \delta\Psi, \delta E,) &= \big\langle T_{V_0}\overline{\Psi}_0, \big(T_R(E_0)T_{\delta V} + \delta T_R(E_0, \delta E)T_{V_0}\big)\Psi_0 \\ &\quad + \mathcal{Q}(\delta V, \delta\Psi, \delta E; V_0, \Psi_0, E_0)\big\rangle. \qquad (4.18) \end{aligned}$$

We verify that the hypotheses of the analytic implicit function theorem hold for $\mathbf{F}$: $\mathbf{X} \times \mathbf{Y} \to \mathbf{Z}$, with

$$\begin{aligned} \mathbf{X} &= \overline{\{V \in L^2(\mathbb{R}^n) \text{ of compact support}\}} \\ \mathbf{Y} &= L^2(\mathbb{R}^n) \times \mathbb{C} \\ \mathbf{Z} &= \big\{(f, z) \in L^2(\mathbb{R}^n) \times \mathbb{C} : z = \big\langle T_{V_0}\overline{\Psi}_0, f\big\rangle\big\} \end{aligned}$$

and with the norms $|||V|||$ for $V \in \mathbf{X}$, $\|\Psi\|_{L^2} + |E|$ for $(\Psi, E) \in \mathbf{Y}$, and $\|f\|_{L^2} + |z|$ for $(f, z) \in \mathbf{Z}$.

We first compute the differential $D\mathbf{F}$ evaluated at a point $x_0 = (\tilde{V}, \tilde{\Psi}, \tilde{E})$ in a neighborhood of the origin:

$$\begin{aligned}
D_{\delta V,\delta\Psi,\delta E} F_1(x_0) &= D_{\delta V} F_1(x_0) + D_{\delta\Psi} F_1(x_0) + D_{\delta E} F_1(x_0) \\
&= T_R(E_0) T_{\delta V} \Psi_0 + T_R(E_0) T_{\delta V} \tilde{\Psi} + \delta T_R(E_0, \tilde{E}) T_{\delta V} (\Psi_0 + \tilde{\Psi}) \\
&\quad + \big[I + T_R(E_0) T_{V_0} + T_R(E_0) T_{\tilde{V}} + \delta T_R(E_0, \tilde{E}) + \delta T_R(E_0, \tilde{E}) T_{\tilde{V}}\big] \delta\Psi \\
&\quad + T_R'(E_0 + \tilde{E}) \big[T_{V_0} \Psi_0 + T_{V_0} \tilde{\Psi} + T_{\tilde{V}} (\Psi_0 + \tilde{\Psi})\big] \delta E \\
D_{\delta V,\delta\Psi,\delta E} F_2(x_0) &= D_{\delta V} F_2(x_0) + D_{\delta\Psi} F_2(x_0) + D_{\delta E} F_2(x_0) \\
&= \big\langle T_{V_0} \overline{\Psi}_0, \\
&\qquad T_R(E_0) T_{\delta V} \Psi_0 + T_R(E_0) T_{\delta V} \tilde{\Psi} + \delta T_R(E_0, \tilde{E}) T_{\delta V} (\Psi_0 + \tilde{\Psi}) \\
&\qquad + \big[T_R(E_0) T_{\tilde{V}} + \delta T_R(E_0, \tilde{E}) + \delta T_R(E_0, \tilde{E}) T_{\tilde{V}}\big] \delta\Psi \\
&\qquad + T_R'(E_0 + \tilde{E}) \big[T_{V_0} \Psi_0 + T_{V_0} \tilde{\Psi} + T_{\tilde{V}} (\Psi_0 + \tilde{\Psi})\big] \delta E \big\rangle.
\end{aligned}$$

By Lemmas 3.1 and 3.2, we see that

$$\|D_{\delta V,\delta\Psi,\delta E} \mathbf{F}(x_0)\|_{\mathbf{Z}} \le C|||\delta V||| + C\|\delta\Psi\| + C|\delta E|$$

for x_0 in a neighborhood of the origin, and therefore $\mathbf{F}$ is analytic there. We observe that $\mathbf{F}(0,0,0) = 0$, and consider the differential evaluated at the origin.

$$D_{\delta\Psi,\delta E} \mathbf{F}(0,0,0) = \begin{pmatrix} I + T_R(E_0) T_{V_0} & T_R'(E_0) T_{V_0} \Psi_0 \\ 0 & \big\langle T_{V_0} \overline{\Psi}_0, T_R'(E_0) T_{V_0} \Psi_0 \big\rangle \end{pmatrix}. \tag{4.19}$$

We now verify that the inverse of $D_{\delta\Psi,\delta E} \mathbf{F}(0,0,0)$ is defined and bounded on $\mathbf{Z}$. Consider the system of equations:

$$\begin{aligned}
(I + T_R(E_0) T_{V_0})\, \delta\Psi + T_R'(E_0) T_{V_0} \overline{\Psi}_0\, \delta E &= f_1 \\
\langle T_{V_0} \overline{\Psi}_0, T_R'(E_0) T_{V_0} \Psi_0 \rangle\, \delta E &= \langle T_{V_0} \overline{\Psi}_0, f_1 \rangle
\end{aligned} \tag{4.20}$$

By Lemma 4.1, this system can be solved uniquely for $(\delta\Psi, \delta E) \in L^2 \times \mathbb{C}$ of a function of δV. The conclusions of Theorem 4.1 now follow from the implicit function theorem.

4.2. Perturbation theory of degenerate scattering resonances for V_0 symmetric

Suppose $V = V_0 = V_0(r)$, with (r, θ) polar coordinates in $\mathbb{R}^2$. For any $l = 0, \pm 1, \pm 2, \ldots$, set

$$\psi_l(r) e^{il\theta}. \tag{4.21}$$

Substitution into equation (1.1) yields

$$\left(-\Delta_r \; - \frac{l^2}{r^2} \; + V_0 \; - \; E_0 \right) \psi_l = \; 0 \tag{4.22}$$

In general, there is a discrete set of solutions $\{\psi_{0,lm}(r; E_{0,lm}) \,:\, m \,=\, 0, 1, \ldots\}$, and corresponding resonance energies $E_{0,lm}$ satisfying equation (4.22) and the outward going radiation condition at infinity. Fix some value of l. To simplify the formulae we shall, when there is no ambiguity, denote the resonance mode by $\psi_{0,l}(r; E_{0,l})$ and resonance energy by $E_{0,l}$.

Note that the there is a degeneracy, $E_{0,l} = E_{0,-l} \equiv E_{0,|l|}$, because equation (4.22) is invariant under the change $l \mapsto -l$. Thus, the scattering resonance problem has a two dimensional subspace of solutions for energy $E_{0,|l|}$, which is spanned by the functions $\psi_{0,l}(r; E_{0,|l|})e^{il\theta}$ and $\psi_{0,l}(r; E_{0,|l|})e^{-il\theta}$. The perturbation theory of each of these degenerate modes can be carried out independently.

To see this, first consider equation (4.4), the equivalent nonlocal formulation of the scattering resonance problem. Setting

$$\Psi \;\; = \;\; e^{i\theta l} \; \left(\; \Psi_{0,l}(r; E_{0,|l|}) \; + \; \delta\Psi_l(r) \; \right) \tag{4.23}$$

$$E \;\; = \;\; E_{0,|l|} \; + \; \delta E \tag{4.24}$$

we have, after commuting factors of $e^{il\theta}$,

$$\begin{aligned} & \left(\; I \; + \; T_{R,l}(E_{0,|l|}) T_{V_0,l} \; \right) \; \delta\Psi_{0,l} \\ = \;\; & T_{R,l}(E_{0,|l|}) \; T_{l,\delta V} \Psi_0 \; + \; \left(\; T_{R,l}(E_{0,|l|} + \delta E) - T_{R,l}(E_{0,|l|}) \; \right) \; T_{V_0,l} \Psi_{0,l} \\ + \;\; & \mathcal{Q}(\delta\Psi_l, \delta E, \delta V; \Psi_{0,l}, E_{0,|l|}, V_0). \end{aligned} \tag{4.25}$$

The $l-$ indexed operators in (4.25) are given by:

$$T_{R,l}(E) \;\; = \;\; e^{-il\theta} \; T_R(E) \; e^{+il\theta} \;\; = \;\; \langle D_l \rangle \chi \left(-\Delta_l - E_+ \right)^{-1} \chi \langle D_l \rangle \tag{4.26}$$

$$T_{V,l} \;\; = \;\; e^{-il\theta} \; T_V \; e^{il\theta} \;\; = \;\; \langle D_l \rangle^{-1} \chi^{-1} V \chi^{-1} \langle D_l \rangle^{-1}, \tag{4.27}$$

where

$$\langle D_l \rangle \;\; = \;\; \left(\; I \; - \; \Delta_r \; + \; \frac{l^2}{r^2} \; \right)^{\frac{1}{2}} , \;\; \Delta_l = \langle D_l \rangle^2. \tag{4.28}$$

Finally, we introduce the $l-$ dependent norm:

$$||| \; V \; |||_l \;\; \equiv \;\; || \langle D_l \rangle^{-1} \chi^{-1} V \chi^{-1} \langle D_l \rangle^{-1} \; ||_{\mathcal{B}(L^2)}. \tag{4.29}$$

Proposition 4.1. *The adjoint operator $I \ + \ T_{V_0,l}T_{R,l}(E_{0,l})^*$ acting on radial L^2 functions, has a one dimensional null space spanned by the function*

$$\Psi^{\#}_{0,l}(r) \ = \ T_{V_0,l} \ \overline{\Psi_{l,0}}(r) \ = \ T_{V_0,l} \ \Psi_{l,0}(r; E_0^*). \tag{4.30}$$

Therefore, the inhomogeneous problem

$$(\ I \ + \ T_{R,l}(E_{0,l})T_{V_0,l} \ ) \ U \ = \ S$$

has a solution for $S \in L^2(r \ dr)$ if and only if

$$\left\langle T_{V_0,l}\overline{\Psi_{0,l}}, S \right\rangle_{\text{rad}} \ = \ 0 \tag{4.31}$$

The proof parallels that of Proposition 4.1.

Therefore, we have obtained a reduction to the case of simple (dimension one) scattering resonance and Theorem 4.1 applies. We have

Theorem 4.2. *(a) Let $V_0 = V_0(r)$ denote a radial potential defined on $\mathbb{R}^2$ for which $|||V_0|||_l$ is finite. Let $E_{0,|l|}$ denote a doubly degenerate scattering resonance energy corresponding to the pair of scattering resonance modes $\psi_{0,l}(r)e^{il\theta}$ and $\psi_{0,l}(r)e^{-il\theta}$. Assume the condition*

$$\langle T_{V_0,l}\bar{\Psi}_{0,l}, T'_{R,l}(E_{0,l})T_{V_0,l}\Psi_{0,l}\rangle \neq 0. \tag{4.32}$$

Then, there exists ε_0 such that for any potential V (not necessarily radial) satisfying $|||V - V_0|||_l < \varepsilon_0$, there corresponds a pair of solutions

$$\begin{aligned} V \quad &\mapsto \quad \left(E_+(V), \psi_l(x;V)e^{il\theta}\right) \\ V \quad &\mapsto \quad \left(E_-(V), \psi_{-l}(x;V)e^{-il\theta}\right) \end{aligned} \tag{4.33}$$

of the scattering resonance problem which lie near $(E_{0,|l|}, \psi_{0,\pm l})$ in the norm $|z| \ + \ |||V|||$. (b) As in Proposition 4.1 the mappings $V \mapsto \left(E_{\pm l}(V), \psi_{\pm l}(x;V)e^{\pm il\theta}\right)$ are analytic.

5. Scattering resonance expansion

In the previous section we proved that if the scattering resonance problem has a solution (E_0, ψ_0) corresponding to a potential V_0, then for all potentials $V = V_0 + \delta V$ in a neighborhood of V_0 ($|||\delta V|||$ small) the scattering resonance problem has a solution $(E(V_0+\delta V), \psi(V_0+\delta V))$. Moreover, we have that $E(V)$ and $\psi(V)$ can be expanded about the V_0 case. In this section we compute the first few terms of this expansion. In the first subsection we give general results for the case of simple scattering resonances. In the second subsection, results for the case of degenerate scattering resonances for a radial potential in $\mathbb{R}^2$ are given.

5.1. Expansion for simple scattering resonances

To find the explicit expansions of $E(V)$ and $\psi(V)$ we write:

$$\begin{aligned} E &= E_0 + \delta E^{(1)} + \delta E^{(2)} + \dots \\ \Psi &= \Psi_0 + \delta\Psi^{(1)} + \delta\Psi^{(2)} + \dots, \end{aligned} \tag{5.1}$$

where terms with a superscript j are formally of order j. Substitution of (5.1) into (4.8) and equating like orders or magnitude yields a hierarchy of inhomogeneous equations, the first two terms of which are:

$$\mathcal{O}(1): \ (I + T_R(E_0)T_{V_0})\,\delta\Psi^{(1)} = -\left(T_R(E_0)T_{\delta V} + \delta E^{(1)}T_R'(E_0)T_{V_0}\right)\Psi_0 \tag{5.2}$$

$$\begin{aligned} \mathcal{O}(2): \ (I + T_R(E_0)T_{V_0})\,\delta\Psi^{(2)} &= -\left(\delta E^{(2)}T_R'(E_0) + \frac{1}{2}T_R''(E_0)\left(\delta E^{(1)}\right)^2\right)T_{V_0}\Psi_0 \\ &\quad -\left(T_R(E_0)T_{\delta V} + \delta E^{(1)}T_R'(E_0)T_{V_0}\right)\delta\Psi^{(1)} \\ &\quad -\delta E^{(1)}T_R'(E_0)T_{\delta V}\Psi_0. \end{aligned} \tag{5.3}$$

At order $\mathcal{O}(m)$ in perturbation theory, $\delta E^{(m)}$ is determined by the solvability condition:

$$\langle T_{V_0}\overline{\Psi}_0, \text{right hand side of } \delta\Psi^{(m)} \text{ equation}\rangle = 0. \tag{5.4}$$

The equation for $\delta E^{(m)}$ has the form:

$$\langle T_{V_0}\overline{\Psi}_0, T_R'(E_0)T_{V_0}\Psi_0\rangle\delta E^{(m)} = \dots \tag{5.5}$$

Therefore, the determination of $\delta E^{(m)}$ at all orders depends on the non-vanishing of the (m-independent) coefficient

$$C_{dE} \equiv \left\langle T_{V_0}\overline{\Psi}_0, T_R'(E_0)T_{V_0}\Psi_0\right\rangle = \left\langle V_0\overline{\psi}_0, R_0'(E_0)V_0\psi_0\right\rangle. \tag{5.6}$$

Consider the $\mathcal{O}(1)$ equation. By Lemma 4.1 a necessary and sufficient for solvability in L^2 is:

$$\langle T_{V_0}\overline{\Psi}_0, T_R(E_0)T_{\delta V}\Psi_0 + \delta E^{(1)}T_R'(E_0)T_{V_0}\Psi_0\rangle = 0.$$

Therefore,

$$\delta E^{(1)} = -\frac{\langle T_{V_0}\overline{\Psi}_0, T_R(E_0)T_{\delta V}\Psi_0\rangle}{\langle T_{V_0}\overline{\Psi}_0, T_R'(E_0)T_{V_0}\Psi_0\rangle}. \tag{5.7}$$

or equivalently, using (5.6) and the definitions of T_V, T_R, and Ψ_0,

$$\delta E^{(1)} = -\frac{\langle V_0\overline{\psi}_0, \delta V\psi_0\rangle}{\left\langle V_0\overline{\psi}_0, R_0'(E_0)V_0\psi_0\right\rangle}. \tag{5.8}$$

If $\delta E^{(1)}$ is chosen to satisfy (5.8), then (5.2) has a unique solution $\Psi^{(1)}$.

Turning to the $\mathcal{O}(2)$ equation, we then substitute this into the right hand side of (5.3) and find, via Lemma 4.1, that $\delta E^{(2)}$ is determined by the solvability condition

$$\begin{aligned} C_{dE}\delta E^{(2)} &= -\frac{1}{2}\left(\delta E^{(1)}\right)^2 \left\langle T_{V_0}\overline{\Psi}_0, T''_R(E_0)T_{V_0}\Psi_0\right\rangle \\ &\quad - \left\langle T_{V_0}\overline{\Psi}_0, T_R(E_0)T_{\delta V}\delta\Psi^{(1)}\right\rangle - \delta E^{(1)}\left\langle T_{V_0}\overline{\Psi}_0, T'_R(E_0)T_{V_0}\delta\Psi^{(1)}\right\rangle \\ &\quad -\delta E^{(1)}\left\langle T_{V_0}\overline{\Psi}_0, T'_R(E_0)T_{\delta V}\Psi_0\right\rangle . \end{aligned} \tag{5.9}$$

The procedure can be continued to obtain a solution to any order.

Special case: vanishing first order correction; $\delta E^{(1)} = 0$

We consider a expressions for $\delta E^{(2)}$, that we use in the application of section 6, where we find $\delta E^{(1)} = 0$, a case of interest in section 6. In this case expressions for $\delta E^{(2)}$ simplifies considerably.

Lemma 5.1. *Suppose* $\delta E^{(1)} = 0$*. Then,*

$$\delta E^{(2)} = C_{dE}^{-1}\left\langle \overline{\Psi}_0, T_{\delta V}\delta\Psi^{(1)}\right\rangle \tag{5.10}$$

$$= -\frac{\left\langle T_{\delta V}\overline{\Psi}_0, (I + T_R(E_0)T_{V_0})^{-1}T_R(E_0)T_{\delta V}\Psi_0\right\rangle}{\left\langle T_{V_0}\overline{\Psi}_0, T'_R(E_0)T_{V_0}\Psi_0\right\rangle}. \tag{5.11}$$

Proof of Lemma 5.1: If $\delta E^{(1)} = 0$, then (5.9) gives:

$$\left\langle T_{V_0}\overline{\Psi}_0, \delta E^{(2)}T'_+(E_0)T_{V_0}\Psi_0 + T_R(E_0)T_{\delta V}\delta\Psi^{(1)}\right\rangle = 0. \tag{5.12}$$

Therefore,

$$\begin{aligned} \delta E^{(2)}\left\langle T_{V_0}\overline{\Psi}_0, T'_R(E_0)T_{V_0}\Psi_0\right\rangle &= -\left\langle T_{V_0}\overline{\Psi}_0, T_R(E_0)T_{\delta V}\delta\Psi^{(1)}\right\rangle \\ &= -\left\langle \overline{T_R(E_0)T_{V_0}\Psi_0}, T_{\delta V}\delta\Psi^{(1)}\right\rangle \\ &= \left\langle \overline{\Psi}_0, T_{\delta V}\delta\Psi^{(1)}\right\rangle . \end{aligned} \tag{5.13}$$

The second equality in (5.13) follows from Lemma 4.2, and the third from (4.7). Finally, (5.11) follows from (5.2). □

5.2. Expansion near degenerate scattering resonances for V_0 radial

Let $V_0 = V_0(r)$ be defined on $\mathbb{R}^2$ and recall the nature of the degeneracy as explained in section 4. We follow a procedure which parallels our reduction of the perturbation theoretic result for degenerate scattering resonance pairs (Theorem 4.2) to the result for the case of simple scattering resonances (Theorem 4.1). The results are as in the previous subsection with the following changes:

$$\begin{aligned} E_0 &\to E_{0,|l|}, \quad E(V) \to E_\pm(V) \\ T_R(E_0) &\to T_{R,l}(E_{0,|l|}) \\ T_{V_0} &\to T_{V_0,l} \\ \langle f,g\rangle &\to \langle f,g\rangle_{\rm rad} \end{aligned}$$

We summarize the formulae thus obtained for the perturbed resonance energy:

$$E_{\pm l} \equiv E_{0,|l|} + \delta E^{(1)}_{\pm l} + \delta E^{(2)}_{\pm l} + \dots, \tag{5.14}$$

Note that in the case where V_0 is radial

$$\begin{aligned} \delta E^{(1)}_{\pm l} &= -\frac{\langle T_{V_0,l}\bar\Psi_{0,l}, T_{R,l}(E_{0,l})T_{\delta V,l}\Psi_{0,l}\rangle}{\langle T_{V_0,l}\bar\Psi_{0,l}, T'_{R,l}(E_{0,l})T_{V_0,l}\Psi_{0,l}\rangle}. \\ &= -\frac{\langle V_0\overline{\psi_{0,\pm l}}, \delta V_N\psi_{0,\pm l}\rangle}{\langle T_{V_0,l}\bar\Psi_{0,l}, T'_{R,l}(E_{0,l})T_{V_0,l}\Psi_{0,l}\rangle} = 0 \end{aligned} \tag{5.15}$$

because the θ- dependence of δV_N can be factored out and integrates to zero. Therefore, Proposition 5.10 applies.

Proposition 5.1. *The degenerate pair of resonances $e^{\pm il\theta}\psi_{0,\pm l}(r)$ with energy $E_{0,|l|}$ perturbs to two branches of resonances with energies:*

$$E_{\pm l} = E_{0,|l|} + \delta E^{(2)}_{\pm l} + \dots, \tag{5.16}$$

where

$$\delta E^{(2)}_{\pm l} = -\frac{\left\langle T_{\delta V,l}\bar\Psi_{0,l}, (I+T_{R,l}(E_{l,0})T_{V_0,l})^{-1}T_{R,l}(E_{l,0})T_{\delta V,l}\Psi_{l,0}\right\rangle}{\left\langle T_{V_0,l}\bar\Psi_{0,l}, T'_{R,l}(E_{0,l})T_{V_0,l}\Psi_{0,l}\right\rangle} \tag{5.17}$$

6. The Schrödinger equation with potentials with $N-$ fold symmetric (microstructure) potentials

Let $r=|x|$ and $\theta\in[0,2\pi]$ denote polar coordinates in the plane, $\mathbb{R}^2$. We consider $V_0=V$, the averaged structure and δV_N, a *microstructure perturbation*

$$\delta V_N(x) = \delta\tilde V(r, N\theta), \tag{6.1}$$

where $\delta\tilde{V}(r,\Theta)$ is 2π-periodic in Θ. Thus we have a structure with N-fold rotational symmetry.

We want to show that to any scattering resonance of the averaged structure, there is a nearby scattering resonance of the perturbed structure, provided N is sufficiently large. In order to apply the results of Theorem 4.1 we must show that as $N \to \infty$:

$$||| \, \delta V_N \, ||| \equiv || \, T_{\delta V_N} \, ||_{\mathcal{B}(L^2)} = || \langle D \rangle^{-1} \delta \check{V}_N \langle D \rangle^{-1} ||_{\mathcal{B}(L^2)} \to 0. \tag{6.2}$$

where $\delta\check{V}_N = \chi^{-1}\delta V_N \chi^{-1}$. We next prove this and, in particular, obtain the precise estimate of $||| \, \delta V_N \, |||$ in terms of N.

Theorem 6.1. *For some positive constant, C, depending on V and χ:*

$$||| \, \delta V_N \, ||| \le C \frac{1}{N}. \tag{6.3}$$

Corollary 6.1. *By Theorems 6.1 and 4.1, there exists a positive integer $N_* > 0$, such that for $N \ge N_*$, the scattering resonance problem for $H = -\Delta + V_0 + \delta V_N$ has a solution (E_N, ψ_N) with scattering frequency E_N near E_0.*

To prove (6.3), we shall make use of the Fourier series of δV_N:

$$\delta V_N(\cdot, \theta) = \sum_{k \neq 0} \delta V_N{}^{(k)}(\cdot) e^{ik\theta}. \tag{6.4}$$

A similar expansion holds for $\delta\check{V}_N$ with $\delta V_N{}^{(k)}$ replaced by $\delta\check{V}_N{}^{(k)} \equiv \chi^{-1} \delta V_N{}^{(k)} \chi^{-1}$.

Proposition 6.1. *Let $K(\Delta)$ denote a function of the Laplacian. Then,*

$$K(\Delta)\, e^{i\ell\theta}\, f(r) \;=\; e^{i\ell\theta}\, K(\Delta_\ell) f(r) \tag{6.5}$$

where

$$\Delta_\ell \;=\; \Delta_r \;-\; \frac{\ell^2}{r^2} \tag{6.6}$$

In particular,

$$\langle D \rangle^{-1}\, e^{i\ell\theta}\, f(r) \;=\; e^{i\ell\theta}\, \langle D_\ell \rangle^{-1}\, f(r), \tag{6.7}$$

$$\langle D_\ell \rangle^{-1} \;=\; \left(m^2 I - \Delta_\ell \right)^{-\frac{1}{2}} \tag{6.8}$$

Proof of Theorem 6.1: We now estimate the norm of $T_{\delta V_N}$. Let $f \in L^2$ be arbitrary. Then,

$$\begin{aligned} \langle D\rangle^{-1}\,\delta\check{V}_N\,\langle D\rangle^{-1} f &= \sum_\ell \langle D\rangle^{-1}\,\delta\check{V}_N\,\langle D\rangle^{-1} f_\ell e^{i\ell\theta} \\ &= \sum_\ell \langle D\rangle^{-1}\,\delta\check{V}_N\, e^{i\ell\theta}\langle D_\ell\rangle^{-1} f_\ell \\ &= \sum_{\ell,k\neq 0} \langle D\rangle^{-1}\,\delta\check{V}_N^{(k)}\, e^{i(\ell+kN)\theta}\langle D_\ell\rangle^{-1} f_\ell \\ &= \sum_{\ell,k\neq 0} e^{i(\ell+kN)\theta}\langle D_{\ell+kN}\rangle^{-1}\,\delta\check{V}_N^{(k)}\,\langle D_\ell\rangle^{-1} f_\ell \end{aligned} \tag{6.9}$$

Taking the L^2 norm and using orthogonality of $\{e^{i\ell\theta} : \ell \in Z\}$ we have:

$$\begin{aligned} \|\,\langle D\rangle^{-1}\,\delta\check{V}_N\,\langle D\rangle^{-1} f\,\|_{L^2}^2 &= \sum_{\ell,k\neq 0} \|\,\langle D_{\ell+kN}\rangle^{-1}\,\delta\check{V}_N^{(k)}\langle D_\ell\rangle^{-1} f_\ell\,\|_{L^2}^2 \\ = \sum_{\ell,k\neq 0} &\|\,\langle D_{\ell+kN}\rangle^{-1}\,|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\;|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\,\mathrm{sgn}(\delta\check{V}_N^{(k)})\,\langle D_\ell\rangle^{-1} f_\ell\,\|_{L^2}^2 \\ \leq \sum_{\ell,k\neq 0} &\|\,\langle D_{\ell+kN}\rangle^{-1}\,|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\,\|_{\mathcal{B}(L^2)}^2 \||\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\,\langle D_\ell\rangle^{-1}\,\|_{\mathcal{B}(L^2)}^2\,\|f_\ell\|_{L^2}^2 \end{aligned} \tag{6.10}$$

Since the operators $|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\,\langle D_\ell\rangle^{-1}$ and $\langle D_\ell\rangle^{-1}|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}$ are adjoints, their $\mathcal{B}(L^2)$ norms are equal and we have:

$$\|\,\langle D\rangle^{-1}\,\delta\check{V}_N\,\langle D\rangle^{-1} f\,\|_{L^2}^2 \;\leq\; \sum_{\ell,k\neq 0} \Gamma_{\ell+kN,k}\,\Gamma_{\ell,k}\,\|f_\ell\|_{L^2}^2, \tag{6.11}$$

where

$$\Gamma_{q,k} \;=\; \|\,|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\;\langle D_q\rangle^{-1}\,\|_{\mathcal{B}(L^2)}^2. \tag{6.12}$$

Let

$$B_\ell \;\equiv\; \langle D_\ell\rangle \;=\; \left(m^2 I - \Delta_r + \frac{\ell^2}{r^2}\right)^{\frac{1}{2}} \tag{6.13}$$

Note that

$$\Gamma_{q,k} \;=\; \|\langle x\rangle\,|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\;\cdot\langle x\rangle^{-1}B_q^{-1}\,\|_{\mathcal{B}(L^2)}^2 \;\leq\; \gamma_k\,\|\langle x\rangle^{-1}B_q^{-1}\,\|_{\mathcal{B}(L^2)}^2, \tag{6.14}$$

where

$$\gamma_k = \sup_x\;\|\langle x\rangle\,|\delta\check{V}_N^{(k)}|^{\frac{1}{2}}\|_\infty \leq \gamma$$

where γ is independent of k since $V(r,\theta)$ is bounded and has compact support in x. Therefore,

$$\| \langle D\rangle^{-1} \delta\check{V}_N \langle D\rangle^{-1} f \|_{L^2}^2 \leq \gamma^2 \sum_{\ell,k\neq 0} \|\langle x\rangle^{-1} B_{\ell+kN}^{-1} \|_{\mathcal{B}(L^2)}^2 \|\langle x\rangle^{-1} B_\ell^{-1} \|_{\mathcal{B}(L^2)}^2 \|f_\ell\|_{L^2}^2 \tag{6.15}$$

We shall bound (6.14) using the following

Proposition 6.2.

$$\| \langle x\rangle^{-1} B_\ell^{-1} f \|_{L^2} \leq (1+\ell^2)^{-\frac{1}{2}} \|f\|_{L^2} \tag{6.16}$$

The proof is given in section 8.

We now conclude our estimation of $|||\, \delta V_N \,|||$. By Proposition 6.2 applied to the operators B_ℓ and $B_{\ell+kN}$ we have from (6.15)

$$\| \langle D\rangle^{-1} \delta V_N \langle D\rangle^{-1} f \|_{L^2}^2 \quad \leq \quad 2\gamma^2 \sum_{|k|\geq 1} \sum_{\ell} \frac{1}{1+(\ell+kN)^2} \frac{1}{1+\ell^2} \|f_\ell\|_{L^2}^2 \tag{6.17}$$

Consider the above sum over the the range $k \geq 1$. The range $k \leq -1$ is treated similarly.

$$\sum_{k\geq 1} \sum_{\ell} \frac{1}{1+(\ell+kN)^2} \frac{1}{1+\ell^2} \|f_\ell\|_{L^2}^2 \;=\; \sum_{\mathbf{A}} + \sum_{\mathbf{B}} + \sum_{\mathbf{C}}.$$

Here, we use the notation

$$\begin{aligned} &\sum_{\mathbf{A}} \equiv \sum_{k\geq 1} \sum_{-2kN\leq \ell\leq -kN/2} \{\cdots\}, \quad \sum_{\mathbf{B}} \equiv \sum_{k\geq 1} \sum_{\ell\geq -kN/2} \{\cdots\}, \text{ and} \\ &\sum_{\mathbf{C}} \equiv \sum_{k\geq 1} \sum_{\ell\leq -2kN} \{\cdots\} \end{aligned} \tag{6.18}$$

These three sums are estimated as follows:

$$\begin{aligned} \sum_{\mathbf{A}} &\leq \sum_{k\geq 1} \sum_{-2kN\leq m\leq -kN/2} \frac{1}{1+\ell^2} \|f_\ell\|_{L^2}^2 \leq \sum_{k\geq 1} \frac{1}{1+\frac{k^2N^2}{4}} \|f_\ell\|_{L^2}^2 \leq C\frac{1}{N^2} \|f\|_{L^2}^2 \\ \sum_{\mathbf{B}} &\leq \sum_{k\geq 1} \frac{1}{1+\frac{k^2N^2}{4}} \sum_{\ell\geq -kN/2} \frac{1}{1+\ell^2} \|f_\ell\|_{L^2}^2 \leq C\frac{1}{N^2} \|f\|_{L^2}^2 \\ \sum_{\mathbf{C}} &\leq \sum_{k\geq 1} \frac{1}{1+k^2N^2} \sum_{\ell\leq -2kN} \frac{1}{1+\ell^2} \|f_\ell\|_{L^2}^2 \leq C\frac{1}{N^2} \|f\|_{L^2}^2 \end{aligned}$$

This completes the proof of Proposition 6.1. □

7. Comparison of perturbation and homogenization expansions

In previous section we showed the applicability of resonance perturbation theory to the class of potentials V_N; see (1.5). In this section, we first summarize the formal homogenization expansion [7], derived and used extensively to obtain an expansion and fast numerical algorithm for computation of leakage rates (imaginary parts of scattering resonances) for a class of photonic microstructure waveguides. We then prove that the this expansion of the scattering resonance frequency, $E(V_N)$, agrees through second order with the expansion:

$$E(V) = E_0 + \delta E^{(1)} + \delta E^{(2)}$$

of the previous section.

7.1. Summary of homogenization/multiscale expansion

We begin with the a brief summary of the multiscale/homogenization expansion. For more detail, see [7]. We seek solutions of the equation

$$(-\Delta + V_N - E)\,\psi = 0, \tag{7.1}$$

that satisfy an outgoing radiation condition at infinity.

For simplicity, we take

$$V_N = V_N(r,\theta,N\theta) = V_0(r) + \delta\tilde{V}(r,N\theta), \tag{7.2}$$

where $\delta\tilde{V}(r,\Theta+2\pi) = \delta\tilde{V}(r,\Theta)$, and we define

$$\delta V(r,\theta) = \delta\tilde{V}(r,N\theta). \tag{7.3}$$

The more general case, where V_0 is not necessarily radial, can also be treated.

As in [7], we view a solution of (7.1) as a function of slow variables r,θ and a fast variable $\Theta = N\theta$:

$$\psi = \Phi(r,\theta,\Theta). \tag{7.4}$$

Equation (7.1) can be rewritten as an equation for $\Phi(r,\theta,\Theta)$ by replacing ∂_θ^2 in the Laplacian appearing in (7.1) by $(\partial_\theta + N\partial_\Theta)^2$. We then substitue into the resulting equation the expansions:

$$\begin{aligned} \Phi^{(N)} &= \Phi_0 + \frac{1}{N}\Phi_1 + \frac{1}{N^2}\Phi_2 + \frac{1}{N^3}\Phi_3 + \frac{1}{N^4}\Phi_4 + \dots \\ E^{(N)} &= E_0 + \frac{1}{N}E_1 + \frac{1}{N^2}E_2^{(\text{homog})} + \dots \end{aligned} \tag{7.5}$$

Equating like orders of N^{-1} yields a hierarchy of equations of the form:

$$\frac{1}{r^2}\partial_\psi^2 \Phi_j = \mathcal{F}_j, j = 0, 1, 2, \ldots \tag{7.6}$$

The solvability condition for (7.6) is

$$\langle \mathcal{F}_j \rangle_{\text{av}}(r) \equiv \frac{1}{2\pi}\int_0^{2\pi} \mathcal{F}_j(\cdot, \Theta) d\Theta = 0 \tag{7.7}$$

This hierarchy can be solved recursively. The equations from $\mathcal{O}(N^2)$ through $\mathcal{O}(1)$ imply that the leading order term is a solution of the scattering resonance problem for V_0 , (E_0, ψ_0), which we have taken in (7.5). Furthermore,

$$\Phi_2 = \Phi_2^{(p)}(r, \theta, \Theta) + \Phi_2^{(h)}(r, \theta), \tag{7.8}$$

where

$$\Phi_2^{(p)}(r, \theta, \Theta) = \partial_\Theta^{-2}\left[\delta\tilde{V} r^2 \psi_0\right]. \tag{7.9}$$

At order $\mathcal{O}(N^{-1})$, we find $\delta E_1 = 0$ and $\Phi_1 = 0$, and we can solve for Φ_3 in terms of $\Phi_2^{(p)}$. Finally, $E_2^{(\text{homog})}$ is determined via the solvability condition for the Φ_4 equation. This solvability equation reads

$$(-\Delta - E_0 + V_0)\,\Phi_2^{(h)} = E_2^{(\text{homog})}\psi_0 + \left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r) r^2 \psi_0. \tag{7.10}$$

Now $E_2^{(\text{homog})}$ is to be chosen so that (7.10) has a solution satisfying an outward going radiation condition at infinity. In [7] we implemented this construction of $E_2^{(\text{homog})}$ and Φ_2 for a class of "separable microstructures" as part of a numerical algorithm used to treat general microstructures, $\delta\tilde{V}$.

7.2. N^{-2} corrections: Homogenization vs. resonance perturbation theory

In this subsection we show that the expression for $E_2^{(\text{homog})}$, obtained as a solvability condition for (7.10) is, up to higher order corrections in N^{-1} the same as that given by (5.10) in Lemma 5.1.

We begin by reformulating equation (7.10) in L^2. First rewrite (7.10) as

$$(-\Delta - E_0 + V_0)\,\Phi_2^{(h)} = -E_2^{(\text{homog})} R_0(E) V_0 \psi_0 + \left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r) r^2 \psi_0, \tag{7.11}$$

where we have used the Lippman-Schwinger equation (3.7) to replace ψ_0 by $-R_0(E_0)V_0\psi_0$.

For E with $\Re E > 0$ and $\Im E > 0$, consider the equation

$$(-\Delta - E + V_0)\,\mathcal{R} = -E_2^{(\text{homog})} R_0(E) V_0 \psi_0 + \left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r) r^2 \psi_0. \tag{7.12}$$

$R_0(E)$ is a bounded operator on L^2 and applying it to both sides of (7.12) gives

$$\begin{aligned}\big(I + R_0(E)V_0\big)\mathcal{R} &= -E_2^{(\text{homog})} R_0(E)\ R_0(E)V_0\psi_0 + R_0(E)\left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r)r^2\psi_0 \\ &\quad -E_2^{(\text{homog})} R_0'(E)\ V_0\psi_0 + R_0(E)\left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r)r^2\psi_0 \\ &\equiv \mathcal{S} \qquad (7.13)\end{aligned}$$

We then define

$$\mathcal{R}_1 = \langle D\rangle\chi\mathcal{R}; \qquad \mathcal{S}_1 = \langle D\rangle\chi\mathcal{S}$$

and introduce localization and smoothing operators into (7.13) to find

$$\big(I + T_R(E)T_{V_0}\big)\mathcal{R}_1 = \mathcal{S}_1. \qquad (7.14)$$

For E in the upper half plane (7.14) is always solvable. We now analytically continue across the continuous spectrum, the half-line $E \geq 0$ (corresponding to $\mu \in \mathbb{R}$), to the lower half plane and, in particular, to a small punctured disc about $E = E_0$, $0 < |E - E_0| < \epsilon$. We note that $\mathcal{S}_1 \in L^2(\mathbb{R}^n)$, since V_0 and $\left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r)$ both have compact support. From Corollary 3.1 we have that $T_R(E_0)T_{V_0}$ is compact, and therefore by Lemma 4.1 the solution $\mathcal{R}_1$ of (7.14) exists for $E = E_0$ if and only if the following solvability condition holds:

$$\big\langle T_{V_0}\overline{\Psi}_0, G\big\rangle = 0 \qquad (7.15)$$

We may read off $E_2^{(\text{homog})}$ from (7.14). After recognizing that the coefficient of $E_2^{(\text{homog})}$ is exactly C_{dE} from (5.6), we find

$$\begin{aligned}E_2^{(\text{homog})} &= C_{dE}^{-1}\big\langle T_{V_0}\overline{\Psi}_0, T_R(E_0)T_{\mathcal{V}}\Psi_0\big\rangle \\ &= -C_{dE}^{-1}\big\langle \overline{\Psi}_0, T_{\mathcal{V}}\Psi_0\big\rangle \qquad (7.16)\end{aligned}$$

where we have used Lemma 4.2 and (3.8), and defined

$$\mathcal{V}(r) = r^2\left\langle \delta\tilde{V}\partial_\Theta^{-2}\delta\tilde{V}\right\rangle_{\text{av}}(r).$$

Theorem 7.1. *When δV is of the form (7.3), and has compact support,*

$$\left|\delta E^{(2)} - N^{-2}E_2^{(\text{homog})}\right| = \mathcal{O}(N^{-(2+\tau)}),$$

where $\tau > 0$ depends on the details of δV. In particular, if $\delta V(r,\theta)$ is twice differentiable in r, then $\tau = 2$; if $\delta V(r,\theta)$ has a finite number of jump discontinuities in r, then $\tau = 1$.

Proof of Theorem 7.1: For ease of presentation, we make the simplifying, but inessential, assumption

$$\psi_0 = \psi_0(r).$$

We begin by expressing $\delta E^{(2)}$ from (5.11) as

$$\delta E^{(2)} = -C_{dE}^{-1}\big(N_2^{(\mathrm{PT,main})} - N_2^{(\mathrm{PT,rem})}\big) \tag{7.17}$$

where

$$\begin{aligned} N_2^{(\mathrm{PT,main})} &= \big\langle T_{\delta V}\overline{\Psi}_0, T_R(E_0)T_{\delta V}\Psi_0\big\rangle \\ N_2^{(\mathrm{PT,rem})} &= \big\langle T_{\delta V}\overline{\Psi}_0, \big(I + T_R(E_0)T_{V_0}\big)^{-1}T_R(E_0)T_{V_0}T_R(E_0)T_{\delta V}\Psi_0\big\rangle. \\ C_{dE} &= \big\langle T_{V_0}\overline{\Psi}_0, T_R'(E_0)T_{V_0}\Psi_0\big\rangle \end{aligned}$$

<u>*Estimation of the remainder* $N_2^{(\mathrm{PT,rem})}$</u>*:* We begin by removing the smoothing operators:

$$N_2^{(\mathrm{PT,rem})} = \big\langle \underbrace{\chi^{-1}V_0\overline{R_0(E_0)}\delta V\overline{\psi}_0}_{f}, \underbrace{(I + \chi R_0(E_0)V_0\chi^{-1})^{-1}}_{A^{-1}}\underbrace{\chi R_0(E_0)\delta V\psi_0}_{g}\big\rangle. \tag{7.18}$$

By the proof of Proposition 3.1, the operator $(I + \chi R_0(E_0)V_0\chi^{-1})^{-1}$ is meromorphic with finite-rank residues at the poles. Because we are evaluating it at a pole, we must verify that $g \perp \ker A^*$ and $f \perp \ker A$. Under our assumption that $\ker A$ is the one-dimensional space spanned by $\chi\psi_0$, it is easy to see that $\ker A^*$ is also one-dimensional, and spanned by $\chi^{-1}V_0\overline{\psi}_0$. We may then compute

$$\begin{aligned} \langle \chi^{-1}V_0\overline{\psi}_0, g\rangle &= \langle R_0(E_0)^*V_0\overline{\psi}_0, \delta V\psi_0\rangle \\ &= 0, \end{aligned}$$

which follows because $R_0(E_0)^*V_0\overline{\psi}_0$ is a radial function while $\delta V\psi_0$ is mean-zero in θ. Similarly, we may verify that $\langle f, \chi\psi_0\rangle = 0$, and conclude that

$$\begin{aligned} \langle f, A^{-1}g\rangle &\leq C\|f\|_2\|g\|_2 \\ &= C\|\chi^{-1}V_0\chi^{-1}\,\chi R_0(E_0)^*\delta V\overline{\psi}_0\|_2\|\chi R_0(E_0)\delta V\psi_0\|_2 \\ &\leq C\|\chi R_0(E_0)\delta V\psi_0\|_2^2. \end{aligned}$$

In order to bound $\|\chi R_0(E_0)\delta V\psi_0\|_2^2$, we first express $R_0(E_0)$ as a sum of a bounded operator and an unbounded correction, using the analytic continuation formula (3.19) with $j = 0$.. This gives

$$\|\chi R_0(\mu_0)\delta V\psi_0\|_2 \leq \|\chi R_0(-\mu_0)\delta V\psi_0\|_2 + C\|\chi J_0(\mu_0|\cdot|) \star \delta V\psi_0\|_2. \tag{7.19}$$

We next expand δV in a Fourier series

$$\delta V(r,\theta) = \sum_{n\neq 0} \delta V_n(r) e^{inN\theta}. \tag{7.20}$$

We consider the first term in (7.19), and use Proposition 6.1 to see that

$$\chi R_0(-\mu_0)\delta V_n \psi_0 e^{inN\theta} = e^{inN\theta}\chi G_{nN}(\mu_0)\delta V_n \psi_0, \tag{7.21}$$

where

$$G_\ell(\mu) \;=\; \left(-\Delta_r + \frac{(nN)^2}{r^2} - \mu_0^2\right)^{-1}$$

is the *partial wave Green's function.* In section 8, we prove the following estimate on $G_\ell(\mu)$:

Proposition 7.1. *Let $\mu \in \mathbb{C}\setminus\{0\}$ with $|\Im\mu| < M$. There exists a constant, C, such that for all $l \geq 1$*

$$f \in L^2(\mathbb{R}^2) \implies \|\chi G_\ell(\mu)\chi f\|_2 \leq C\frac{\|f\|_2}{\ell^2}.$$

This result implies that

$$\|\chi R_0(-\mu_0)\delta V_n\psi_0 e^{inN\theta}\|_2 \leq C\frac{\|\chi^{-1}\delta V_n\psi_0\|_2}{(nN)^2},$$

and therefore

$$\begin{aligned}\|\chi R_0(-\mu_0)\delta V\psi_0 e^{inN\theta}\|_2 &\leq C N^{-2}\sup_n \|\chi^{-1}\delta V_n\psi_0\|_2 \\ &\leq C N^{-2}.\end{aligned}$$

We still must bound the second term in (7.19). We observe that for $f \in L^2(\mathbb{R}^2)$ and of compact support,

$$\begin{aligned}J_0(\mu_0|\cdot|)\star f(x) &= \int dy J_0(\mu_0|x-y|)f(y) \\ &= 2\pi\int dy\int_{S^1} d\omega\, e^{i\mu_0(x-y)\omega}f(y) \\ &= 2\pi\int_{S^1} d\omega\, e^{i\mu_0 x\omega}\hat{f}(\mu_0\omega)\end{aligned}$$

and therefore

$$\begin{aligned}\|\chi J_0(\mu_0|\cdot|)\star f\|_2 &\leq C\sup_{\omega\in S^1}|\hat{f}(\mu_0\omega)|\,\|\chi(x)\sup_{\omega\in S^1}|e^{i\mu_0 x\omega}|\,\|_2 \\ &\leq C\sup_{\omega\in S^1}|\hat{f}(\mu_0\omega)|.\end{aligned} \tag{7.22}$$

Next, we will need

Lemma 7.1. *If $F(r,\theta) = f(r)e^{il\theta}$, then its Fourier transform $\hat{F}$ can be written in polar coordinates (ρ,ν) as*

$$\hat{F}(\rho,\nu) = 2\pi\, e^{il(\nu+3\pi/2)} \int_0^\infty r\,dr\, f(r)\, J_l(\rho r). \tag{7.23}$$

The proof is given in appendix B.

Because we are considering the limit of large N while μ_0 is fixed, we may use the asymptotic form (A.6) of J_n along with (7.22) and Lemma 7.1 to find that

$$\begin{aligned} \|\chi J_0(\mu_0|\cdot|) \star \delta V \psi_0\|_2 &\leq \sum_n \|\chi J_0(\mu_0|\cdot|) \star \delta V_n e^{inN\theta} \psi_0\|_2 \\ &\leq C \sum_n \int r\,dr \left|\delta V_n(r) \psi_0(r) J_{nN}(|\mu_0| r)\right| \\ &\leq \sum_n \frac{(C|\mu_0|)^{nN}}{(nN)!} \\ &\leq \frac{(C|\mu_0|)^N}{N!}. \end{aligned}$$

<u>$N_2^{(\mathrm{PT,main})}$=2$^{\mathrm{nd}}$ *order homogenization + small correction:*</u> In the expression for $N_2^{(\mathrm{PT,main})}$, (7.17), we first remove the smoothing operators and move to the physical sheet using the analytic continuation formula (3.19), shedding a small correction in the process:

$$\begin{aligned} N_2^{(\mathrm{PT,main})} &= \left\langle \delta V \overline{\psi}_0, R_0(\mu_0) \delta V \psi_0 \right\rangle \\ &= \left\langle \delta V \overline{\psi}_0, R_0(-\mu_0) \delta V \psi_0 \right\rangle + \frac{i}{2} \left\langle \delta V \overline{\psi}_0, J_0(\mu_0|\cdot|) \delta V \psi_0 \right\rangle \end{aligned} \tag{7.24}$$

In order to show that the second term of (7.24) is small, we note that for $f, g \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \langle f, J_0(\mu|\cdot|) g \rangle &= \int dx\, dy\, \overline{f}(x) J_0(\mu|x-y|) g(y) \\ &= \int_{S^1} d\omega \int dx\, e^{i\mu x\cdot\omega} \overline{f}(x) \int dy\, e^{-i\mu y\cdot\omega} g(y) \\ &= \int_{S^1} d\omega\, \overline{\hat{f}(\mu\omega)} \hat{g}(\mu\omega). \end{aligned}$$

and therefore, using (7.23),

$$\left|\left\langle \delta V \overline{\psi}_0, J_0(\mu_0|\cdot|) \delta V \psi_0 \right\rangle\right| \leq \frac{(C|\mu|)^{2N}}{(N!)^2}.$$

In order to estimate the difference between the first term in (7.24) and $\left\langle \overline{\Psi}_0, T_{\mathcal{V}} \Psi_0 \right\rangle$, the numerator from (7.16), we first prove

Lemma 7.2.

$$\big\langle \delta V\overline{\psi}_0, R_0(-\mu_0)\delta V\psi_0\big\rangle = \sum_{n\neq 0}\big\langle f_n^{(L)}, R_0(-\mu_0) f_n^{(R)}\big\rangle \tag{7.25}$$

$$N^{-2}\big\langle \overline{\Psi}_0, T_{\mathcal{V}}\Psi_0\big\rangle = \sum_{n\neq 0}\big\langle f_n^{(L)}, r^2\partial_\theta^{-2} f_n^{(R)}\big\rangle. \tag{7.26}$$

where

$$\begin{aligned} f_n^{(L)}(r,\theta) &= \delta V_n(r)\overline{\psi}_0(r)e^{inN\theta} \\ f_n^{(R)}(r,\theta) &= \delta V_n(r)\psi_0(r)e^{inN\theta}. \end{aligned}$$

Proof of Lemma 7.2: We first note that (7.25) follows immediately on expanding δV in a Fourier series (7.20) and moving to a Fourier integral representation, using (7.23).

In order to see (7.26), we begin by writing

$$N^{-2}\big\langle \overline{\Psi}_0, T_{\mathcal{V}}\Psi_0\big\rangle = \big\langle \delta V\overline{\psi}_0, r^2\partial_\theta^{-2}\delta V\psi_0\big\rangle,$$

which follows directly from the definitions. We may then expand $\delta V(r,\theta)$ in a Fourier series (7.20) and observe that, again, the off-diagonal terms vanish. □

We may now complete the proof of Theorem 7.1. We begin by using Lemma 7.2 to see that

$$\begin{aligned} \left|\delta E^{(2)} - N^{-2}E_2^{(\text{homog})}\right| &= C_{dE}^{-1}\left|\big\langle \delta V\overline{\psi}_0, R_0(-\mu_0)\delta V\psi_0\big\rangle - N^{-2}\big\langle \overline{\Psi}_0, T_{\mathcal{V}}\Psi_0\big\rangle\right| + \mathcal{O}(N^{-4}) \\ &\leq C_{dE}^{-1}\sum_{n\neq 0}\left|\big\langle f_n^{(L)}, \big(R_0(-\mu_0) - r^2\partial_\theta^{-2}\big) f_n^{(R)}\big\rangle\right| + \mathcal{O}(N^{-4}). \end{aligned} \tag{7.27}$$

We may rewrite the summand in (7.27) as

$$\begin{aligned} &\left|\big\langle f_n^{(L)}, \big(R_0(-\mu_0) - r^2\partial_\theta^{-2}\big) f_n^{(R)}\big\rangle\right| \\ &\qquad = \left|\left\langle f_n^{(L)}, \left(-\Delta_r + \frac{(nN)^2}{r^2} - \mu_0^2\right)^{-1}\left(\Delta_r + \mu_0^2\right)\left(\frac{r^2}{(nN)^2}\right) f_n^{(R)}\right\rangle\right|. \end{aligned} \tag{7.28}$$

In order to estimate (7.28), we prove in section 8 the following

Proposition 7.2. *When $f, g \in L_\infty([0,R])$, $g \in C^2$, and ℓ taken large depending on μ and R,*

$$|\langle f, G_\ell(\mu)\Delta_r g\rangle| \leq \frac{C}{\ell^2}\|f\|_\infty\|\Delta_r g\|_\infty \tag{7.29}$$

while when g is piecewise C^2, with a finite number of jump discontinuities in g or g', then

$$|\langle f, G_\ell(\mu)\Delta_r g\rangle| \leq \frac{C}{\ell}\|f\|_\infty\|g\|_\infty \tag{7.30}$$

By Proposition 7.2

$$|(7.28)| \ \leq C\frac{1}{(nN)^2}\frac{1}{(nN)^\tau}$$

Use of this bound in (7.27) and summing over $n \neq 0$ implies

$$\left|\delta E^{(2)} - N^{-2}E_2^{(\mathrm{homog})}\right| \ = \ \mathcal{O}(N^{-2-\tau}) + \mathcal{O}(N^{-4})$$

This completes the proof of Theorem 7.1. □

8. Partial wave Green's function estimates

In this section we prove the weighted estimates, used in sections 6 and 7, involving the partial wave Green's function

$$G_\ell(\mu) = \left(-\Delta_r + \frac{\ell^2}{r^2} - \mu^2\right)^{-1},$$

defined for μ in the physical upper half plane. $G_\ell(\mu)$ may be analytically continued to the lower half plane by explicitly computing its kernel [16]

$$\begin{aligned} G_\ell(\mu)f(r) &= \int_0^\infty G_\ell(r,r';\mu)f(r')r'dr' \\ G_\ell(r,r';\mu) &= -\frac{\pi}{2}i\begin{cases} J_\ell(\mu r)\, H_\ell^{(1)}(\mu r') \text{ if } r' \geq r \\ H_\ell^{(1)}(\mu r)\, J_\ell(\mu r') \text{ if } r \geq r'. \end{cases} \end{aligned} \tag{8.1}$$

We first prove Propositions 7.1 and 7.2, which are used in section 7. We then prove Proposition 6.2.

Proof of Proposition 7.1: Because the kernel $A(r,r') = \chi(r)G_\ell(r,r')\chi(r')$ is symmetric, we may estimate the $L^2 \to L^2$ norm of the associated operator [4] as

$$\|A\| \leq \sup_r \int r'dr'\, |A(r,r')|$$

and use the explicit form (8.1) of G_ℓ. We define

$$\begin{aligned} A_J(r_1,r_2) &= \chi(r_2)|H_\ell^{(1)}(r_2)| \int_{r_1}^{r_2} rdr\ |J_\ell(\mu r)|\, \chi(r) \\ A_H(r_1,r_2) &= \chi(r_1)|J_\ell(r_1)| \int_{r_1}^{r_2} rdr\ \left|H_\ell^{(1)}(\mu r)\right| \chi(r). \end{aligned}$$

and so

$$\|A\| \leq \sup_{r'} \left(A_J(0,r') + A_H(r',\infty)\right).$$

It suffices to obtain the bound $\sup_{r_1,r_2} A_{J,H}(r_1,r_2) \leq C\ell^{-2}$. We treat A_J; the bounds on A_H are similar. We prove this bound by separately considering the cases: for $r_1 \leq r_2 \leq \sigma$, $r_1 \leq \sigma \leq r_2$ and $\sigma \leq r_1 \leq r_2$, where $\sigma > 0$. We shall see that the dominant contribution to the bound comes from r small which dictates that σ be sufficiently small, yet large enough as $\ell \to \infty$ so that the large ℓ asymptotics in the crossover and large r regions can be used. We set $\sigma = \sqrt{\ell}$.

$\sqrt{\ell} \leq r_1 \leq r_2$**:** An examination of the asymptotic forms (A.1) reveals that the following bounds hold for r bounded away from the origin. For $r \geq r_\#$, there exists $C_\# > 0$, such that

$$\begin{aligned} \chi(r)|J_\ell(\mu r)| &\leq C_\# e^{-(M-|\Im\mu|)r} \\ \chi(r)|H_\ell(\mu r)| &\leq C_\# e^{-(M-\Im\mu)r}. \end{aligned}$$

Therefore,

$$\begin{aligned} A_J(r_1,r_2) &\leq Ce^{-(M-\Im\mu)r_2} \int_{r_1}^{r_2} rdr\, e^{-(M-|\Im\mu|)r} \\ &\leq Ce^{-(1-\epsilon)(M-\Im\mu)r_2}, \end{aligned}$$

Therefore, if $r_2 > r_1 > \sqrt{\ell}$, then

$$A_{J,H}(r_1,r_2) \leq Ce^{-C'\sqrt{l}} \leq C\ell^{-2}. \tag{8.2}$$

$r_1 \leq r_2 \leq \sqrt{\ell}$**:** Note that the asymptotic expressions (A.6) hold, and therefore

$$\begin{aligned} A_J(r_1,r_2) &\leq C(\ell-1)!\, r_2^{-1}\, e^{-Mr_2} \int_{r_1}^{r_2} rdr\, \frac{r^\ell}{\ell!} e^{-Mr} \\ &\leq \frac{C}{\ell}\, r_2^{-\ell}\, e^{-Mr_2} \int_{r_1}^{r_2} dr\, r^{\ell+1} \\ &\leq \frac{C}{\ell^2}\, r_2^2\, e^{-Mr_2} \leq \frac{C}{\ell^2}. \end{aligned} \tag{8.3}$$

Finally, we treat the transition region
$r_1 \leq \sqrt{\ell} \leq r_2$*:*

$$A_J(r_1,r_2) \leq e^{-(M-|\Im\mu|)r_2} \int_0^{\sqrt{\ell}} r\, dr \frac{r^\ell}{\ell!} = o\left(e^{-C\sqrt{\ell}}\right) \leq \mathcal{O}(\ell^{-2})$$

This completes the proof of Proposition 7.1. □

Proof of Proposition 7.2: Because $f(r)$ and $g(r)$ are supported for $r \in [0,R]$, we may write

$$
\begin{aligned}
\langle f, G_\ell(\mu)\Delta_r g\rangle &= \int_0^R r\,dr\,\overline{f}(r)\int_0^R \tilde{r}\,d\tilde{r}\,G(r,\tilde{r})\,(\Delta_r g)\,(\tilde{r}) \\
&= C\int_0^R r\,dr\,\overline{f}(r)H_\ell^{(1)}(\mu r)\int_0^r \tilde{r}\,d\tilde{r}\,J_\ell(\mu r)\,(\Delta_r g)\,(\tilde{r}) \\
&\qquad + C\int_0^R r\,dr\,\overline{f}(r)J_\ell(\mu r)\int_r^R \tilde{r}\,d\tilde{r}\,H_\ell^{(1)}(\mu r)\,(\Delta_r g)\,(\tilde{r}) \\
&\equiv D_1 + D_2
\end{aligned}
$$

case 1: $g \in C^2$**:** We may use (8.1) and (A.6) to find

$$
\begin{aligned}
|D_1| + |D_2| &\leq \frac{C}{\ell}\Bigg[\int_0^R rdr\,|f(r)|\frac{(\ell-1)!}{(\mu r)^\ell}\int_0^r \tilde{r}d\tilde{r}\,\frac{(\mu\tilde{r})^\ell}{\ell!}\,|\Delta_r g(\tilde{r})| + \\
&\qquad \int_0^R rdr\,|f(r)|\frac{(\mu r)^\ell}{\ell!}\int_r^R \tilde{r}d\tilde{r}\,\frac{(\ell-1)!}{(\mu\tilde{r})^\ell}\,|\Delta_r g(\tilde{r})|\Bigg], \\
&\leq \frac{C}{\ell^2}\,\|f\|_\infty\,\|\Delta_r g\|_\infty,
\end{aligned} \tag{8.4}
$$

which is the desired bound (7.29).

case 2: g **or** g' **have jump discontinuities:** When g or g' has a finite number of jump discontinuities at the points r_i, we introduce a partition of unity $1 = \sum_i \chi_i(r)$, where each $\chi_i \in C^\infty$ and $\operatorname{supp}\chi_i$ contains r_i and no other breakpoint $r_{j\neq i}$. We may then represent g as

$$
g = \sum_i \chi_i(g_i + \theta(r - r_i)\tilde{g}_i)
$$

where $g_i, \tilde{g}_i \in C^2$ and $\theta(x)$ is the Heaviside function ($\theta(x) = 0,\ x \leq 0$ and $\theta(x) = 1,\ x \geq 0$). We will examine D_1 in detail; D_2 will give a similar contribution, as can be easily checked. Using (8.1),

$$
D_1 = C\sum_i \int_0^R r\,dr\,\overline{f}(r)H_\ell^{(1)}(r)\int_0^r \tilde{r}\,d\tilde{r}\,J_\ell(\tilde{r})\Delta_{\tilde{r}}\big(\chi_i(\tilde{r})\,(g_i(\tilde{r}) + \theta(\tilde{r} - r_i)\tilde{g}(\tilde{r}))\big). \tag{8.5}
$$

A number of terms result from the action of the $\Delta_{\tilde{r}}$ in the inner integral. When all of the $\tilde{g}_i(r_i) = 0$, *i.e.* the case of jumps in g' only, the result is the same as the C^2 case (7.29). When some of the $\tilde{g}_i(r_i) \neq 0$, however, the sum (8.5) can be written as a dominant term containing $\theta'' = \delta'$, where $\delta(r)$ denotes the one-dimensional Dirac delta distribution, plus contributions satisfying the same bound as in the C^2 case. That is,

$$
D_1 = C\sum_i \int_0^R r\,dr\,\overline{f}(r)H_\ell^{(1)}(\mu r)\int_0^r \tilde{r}\,d\tilde{r}\,J_\ell(\mu\tilde{r})\chi_i(\tilde{r})\tilde{g}_i(\tilde{r})\delta'(\tilde{r} - r_i) \;+\; \mathcal{O}(\ell^{-2})\|f\|_\infty\|g\|_\infty \tag{8.6}
$$

The sum in (8.6) is, in turn, dominated by the contribution from $\partial_r J_\ell$:

$$
\begin{aligned}
|D_1| &\le C\|f\|_\infty \sum_i \|\tilde{g}_i\|_\infty \left[\int_0^R r\, dr\, |H_\ell^{(1)}(\mu r)| \theta(r-r_i) r_i \tilde{g}_i(r_i) |\partial_r J_\ell(\mu r)|_{r=r_i} + \mathcal{O}(\ell^{-2}) \right] \\
&\le C\|f\|_\infty \sum_i \|\tilde{g}_i\|_\infty \left[\left(\frac{\ell r_i^{\ell-1}}{\ell!} \right) \int_{r_i}^R r\, dr\, \frac{(\ell-1)!}{r^\ell} + \mathcal{O}(\ell^{-2}) \right] \\
&\le C\|f\|_\infty \sum_i \|\tilde{g}_i\|_\infty \left[\left(\frac{r_i}{\ell} \right) + \mathcal{O}(\ell^{-2}) \right] \\
&\le \frac{C}{\ell} \|f\|_\infty \|g\|_\infty .
\end{aligned}
$$

D_2 may be estimated by similar arguments. □

We now turn to the proof of Proposition 6.2, used in section 6. We first need the following

Proposition 8.1. *Let V_0 be a non-negative, bounded and sufficiently decaying function. For any $p \in \mathbb{C}$, with $\Re p > 0$, and $f \in L^2(\mathbb{R}^2)$,*

$$
\left\| \langle x \rangle^{-1} \left(-\Delta_r + \frac{m^2}{r^2} + V_0 + p \right)^{-1} f \right\|_{L^2} \le \frac{1}{\Re p + m^2} \|f\|_{L^2} \tag{8.7}
$$

Remark 8.1. *We use this result for the case $V_0 = 0$ only. From the proof, it is clear that it holds under much less stringent hypotheses on V_0.*

Proof of Proposition 8.1: The proof is broken into two parts. We first prove estimates for the heat kernel $\exp[(\Delta - \frac{m^2}{r^2})t]$, and then reduce the desired resolvent estimate to the heat kernel estimate.

Part 1: Let u_m denote the solution of the initial value problem for the heat equation:

$$
\partial_t u_m = \left(\Delta_r - \frac{m^2}{r^2} - V_0 \right) u_m, \quad u_m(0) = f \tag{8.8}
$$

Multiplication by u_m and integration over $\mathbb{R}^2$ gives the identity:

$$
\frac{d}{dt} \int |u_m|^2\, dx = -2 \int |\nabla u_m|^2\, dx - 2m^2 \int \frac{|u_m|^2}{r^2}\, dx - 2 \int V_0 |u_m|^2\, dx \tag{8.9}
$$

Since V_0 is non-negative we can drop both the first and third terms on the right hand side of equation (8.9) to obtain

$$
\frac{d}{dt} \int |u_m|^2\, dx \le -2m^2 \int \frac{|u_m|^2}{r^2}\, dx \tag{8.10}
$$

Integration of (8.10) over the time interval from 0 to t, and using that $(1+r^2)^{-1} \le 1$ yields

$$\begin{aligned} \alpha(t) &\le \|f\|_{L^2}^2 \ - \ 2m^2 \int_0^t \alpha(s)\, ds \\ \alpha(t) &\equiv \int \frac{|u_m|^2}{1+r^2}\, dx. \end{aligned} \tag{8.11}$$

It follows that

$$\left\| \langle x\rangle^{-1} \, e^{(\Delta_r - \frac{m^2}{r^2} - V_0)t} \, f \right\|_{L^2} \le e^{-m^2 t} \, \|f\|_{L^2} \tag{8.12}$$

Part 2: First note that the resolvent and heat kernels are related by the Laplace transform: when $\Re p > 0$,

$$\left(\Delta_r - \frac{m^2}{r^2} - V_0 - p\right)^{-1} f \ = \ \int_0^\infty e^{-pt} \, e^{(\Delta_r - \frac{m^2}{r^2} - V_0)t} \, dt \ f. \tag{8.13}$$

Thus, using the weighted estimate (8.12) we have

$$\begin{aligned} &\left\| \langle x\rangle^{-1} \left(\Delta_r - \frac{m^2}{r^2} - V_0 - p\right)^{-1} f \right\|_{L^2} \\ \le &\int_0^\infty e^{-(\Re p)t} \| \, \langle x\rangle^{-1} \, e^{(\Delta_r - \frac{m^2}{r^2} - V_0)t} \, f \, \|_{L^2} \, dt \\ \le &\int_0^\infty e^{-(\Re p)t} \, e^{-m^2 t} \, \|f\|_{L^2} = \frac{1}{\Re p \ + \ m^2} \, \|f\|_{L^2} \end{aligned} \tag{8.14}$$

This completes the proof of Proposition 8.1. □

Proof of Proposition 6.2: Recall that

$$B_\ell^2 \ = \ m^2 I \ - \ \Delta_r \ + \ \frac{\ell^2}{r^2}.$$

We take $V_0 = 0$ and $p = 1$. In this case, there is no restriction on ℓ because $\Re p + \ell^2 = 1 + \ell^2$ is always positive. This gives the bound

$$\| \, \langle x\rangle^{-1} \, B_\ell^{-2} \, f\|_{L^2} \le \frac{1}{1+\ell^2} \, \|f\|_{L^2} \tag{8.15}$$

We now deduce the estimates for B_ℓ^{-1} from those obtained for B_ℓ^{-2}, through the square root formula

$$B^{-1} \ = \ \frac{1}{\pi} \int_0^\infty \frac{1}{\sqrt{w}} \frac{1}{B^2 + w} \, dw. \tag{8.16}$$

From (8.16) it follows that

$$\left\|\langle x\rangle^{-1}B_\ell^{-1}\right\|_{\mathcal{B}(L^2)} \;\leq\; \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{w}}\left\|\frac{1}{\langle x\rangle}\frac{1}{B_\ell^2+w}\right\|_{\mathcal{B}(L^2)} dw \tag{8.17}$$

By Proposition 8.1 with $V_0 = 0$ and $p = w \geq 0$ we have

$$\left\|\frac{1}{\langle x\rangle}\frac{1}{B_\ell^2+w}\right\|_{\mathcal{B}(L^2)} \leq \frac{1}{\ell^2+w}$$

Therefore,

$$\left\|\langle x\rangle^{-1}B_\ell^{-1}\right\|_{\mathcal{B}(L^2)} \;\leq\; \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{w}}\frac{1}{\ell^2+w}\,dw \leq \frac{C}{(1+\ell^2)^{\frac{1}{2}}}.$$

□

A. Asymptotics of Bessel and Hankel functions

We briefly record the well-known asymptotics of Bessel and Hankel functions.

(1) When $\ell \to \infty$, the following asymptotic expansions hold uniformly for $|\arg z| < \pi - \epsilon$, with $\epsilon > 0$.

$$\begin{aligned} J_\ell(\ell z) &\sim \left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(\frac{\mathrm{Ai}(\ell^{2/3}\zeta)}{\ell^{1/3}}\sum_{k=0}^\infty \frac{a_k(\zeta)}{\ell^{2k}} + \frac{\mathrm{Ai}'(\ell^{2/3}\zeta)}{\ell^{5/3}}\sum_{k=0}^\infty \frac{b_k(\zeta)}{\ell^{2k}}\right) \\ H_\ell^{(1)}(\ell z) &\sim 2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left(\frac{\mathrm{Ai}(e^{2\pi i/3}\ell^{2/3}\zeta)}{\ell^{1/3}}\sum_{k=0}^\infty \frac{a_k(\zeta)}{\ell^{2k}} + \right. \\ &\qquad\qquad \left. \frac{e^{2\pi i/3}\,\mathrm{Ai}'(e^{2\pi i/3}\ell^{2/3}\zeta)}{\ell^{5/3}}\sum_{k=0}^\infty \frac{b_k(\zeta)}{\ell^{2k}}\right) \end{aligned} \tag{A.1}$$

where

$$\frac{2}{3}\zeta^{3/2} = \ln\frac{1+\sqrt{1-z^2}}{z} - \sqrt{1-z^2}$$

and a_k, b_k are defined in [1].

(2) In the limit $|z| \to \infty$ with ℓ held fixed, these reduce to:

$$
\begin{aligned}
J_\nu(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) + e^{|\Im z|}\mathcal{O}\left(|z|^{-1}\right), |\arg z| < \pi \qquad (A.2)\\
Y_\nu(z) &= \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right) + e^{|\Im z|}\mathcal{O}\left(|z|^{-1}\right), |\arg z| < \pi \qquad (A.3)\\
H_\nu^{(1)}(z) &= J_\nu(z) + iY_\nu(z)\\
&\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp(i\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right), -\pi < \arg z < 2\pi \qquad (A.4)\\
H_\nu^{(2)}(z) &= J_\nu(z) - iY_\nu(z)\\
&\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \exp(-i\left(z - \frac{1}{2}\nu\pi - \frac{\pi}{4}\right), -2\pi < \arg z < \pi \qquad (A.5)
\end{aligned}
$$

(3) For $|z| \leq C\sqrt{\ell}$ and $\ell \to \infty$, (A.1) and the asymptotics or Airy functions [1] imply

$$
J_\ell(z) \sim \frac{1}{\ell!}\left(\frac{z}{2}\right)^\ell; \qquad H_\ell^{(1)}(a) \sim -\frac{i}{\pi}(\ell-1)!\left(\frac{z}{2}\right)^{-\ell}. \qquad (A.6)
$$

B. Proof of Lemma 7.1

To prove (7.23) we represent $x, k \in \mathbb{R}^2$ in polar coordinates as, repectively, (r, θ) and (ρ, ν). We wish to compute the Fourier transform

$$
\hat{F}(k) = \int dx e^{-ik\cdot x} F(x), \text{ where } F(x) = f(r)e^{i\ell\theta}.
$$

We define $\tilde{\theta} = \theta - \nu$, the angle between x and k and compute as follows:

$$
\begin{aligned}
\hat{F}(k) &= \int_0^\infty rdr \int_0^{2\pi} d\tilde{\theta}\, e^{-i\rho r\cos\tilde{\theta}} f(r) e^{i\ell(\tilde{\theta}+\nu)}\\
&= e^{i\ell(\nu+\pi)} \int_0^\infty f(r)rdr \int_{-\pi}^{\pi} d\tilde{\theta}\, e^{i\rho r\cos\psi + i\ell\psi}\\
&= 2\pi e^{i\ell(\nu+3\pi/2)} \int_0^\infty f(r) J_\ell(\rho r) rdr, \qquad (B.1)
\end{aligned}
$$

where we have used (1.17). □

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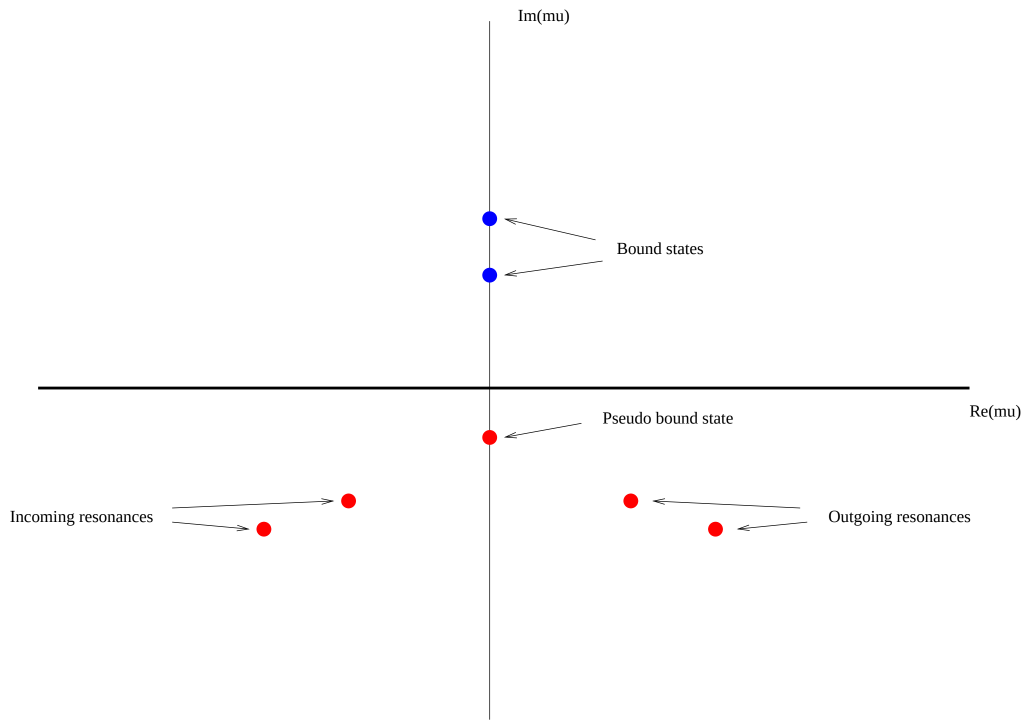

Figure 4: Complex μ plane. Outgoing / incoming resonances correspond to solutions satisfying outgoing / incoming radiation conditions at infinity.

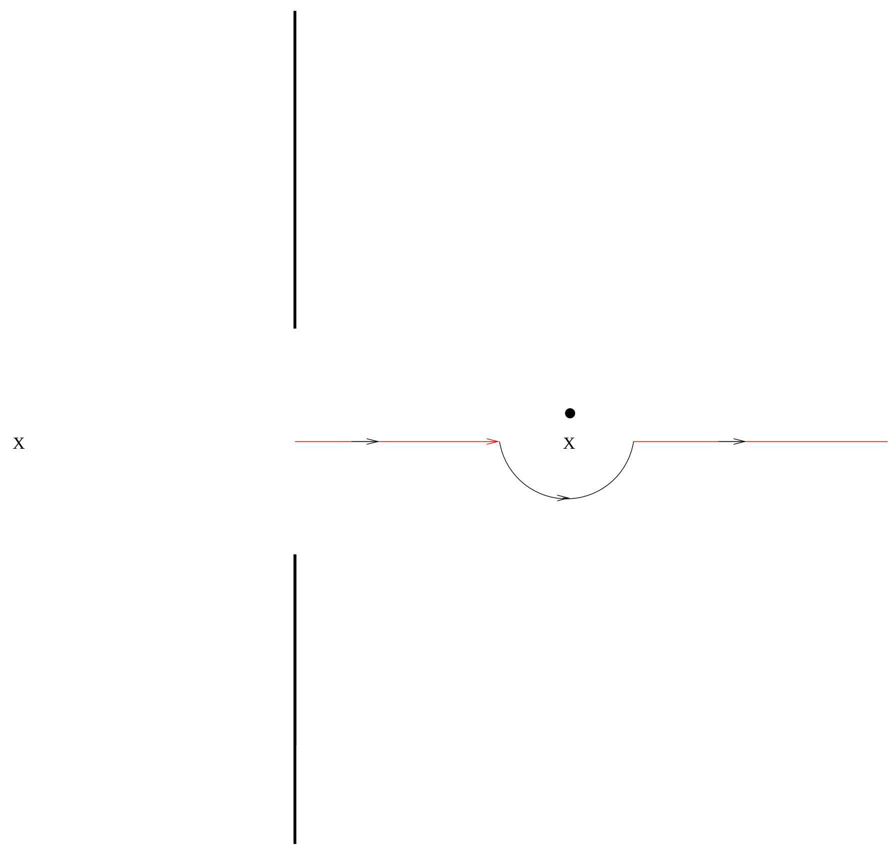

Figure 5: Complex ρ integration contour for $\mathcal{R}(\mu_1; x, y)$. The points $\pm\mu'$ are denoted by "x". The point $\mu_1 = \mu' + i\epsilon_1,\ \epsilon_1 > 0$ is denoted by a circle in the upper half plane. Vertical semi-infinite lines are branch cuts extending from $\pm im$ to $\pm i\infty$.

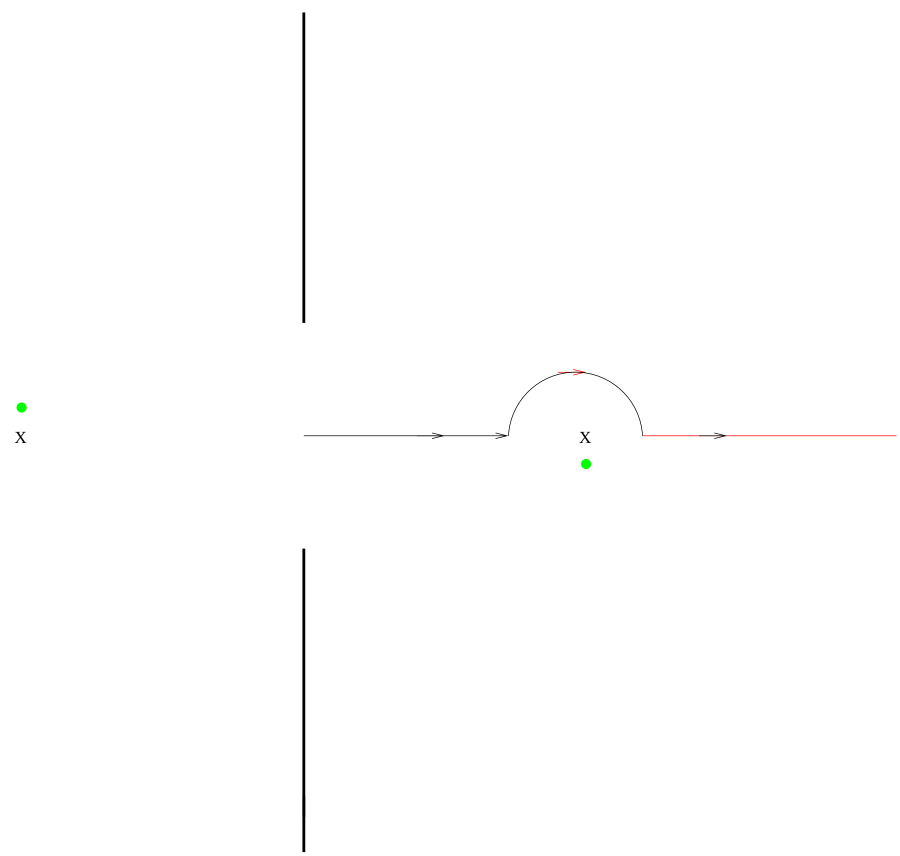

Figure 6: Integration contour for $\mathcal{R}(\mu_2; x, y)$. The points $\mu_2 = -\mu' + i\epsilon_2,\ \epsilon_2 > 0$, and $-\mu_2$, are denoted by circles.

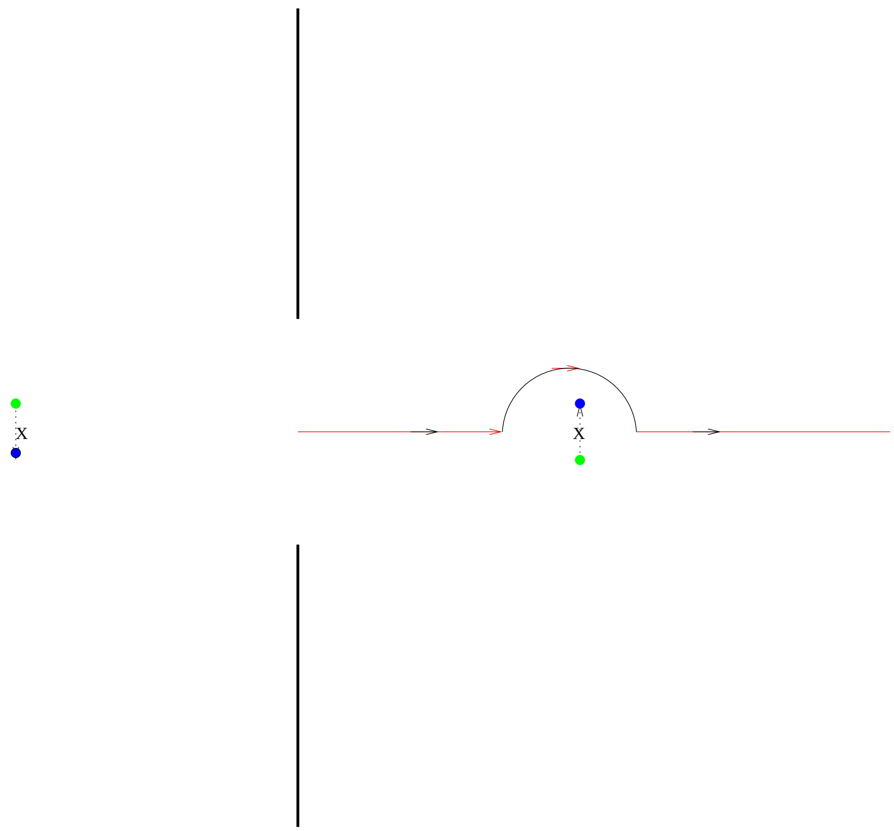

Figure 7: We let ϵ_2 approach $-\epsilon_1$, corresponding to $\mu_2 \to -\mu_1$. Integration contour shown is for $\mathcal{R}(-\mu_1; x, y)$.

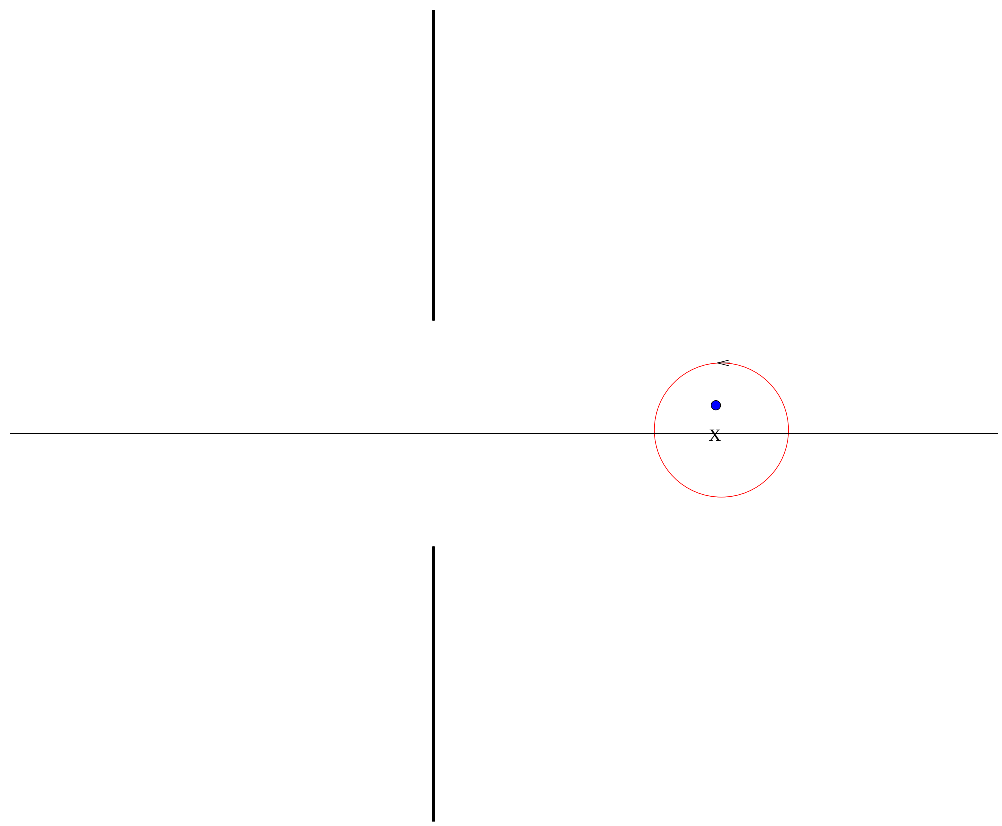

Figure 8: The jump in $\mathcal{R}(\mu; x, y)$ from its value at μ_1 on the 1^{st} Riemann sheet to its value on the 2^{nd} Riemann sheet, $\mathcal{R}(\mu_1; x, y) - \mathcal{R}(-\mu_1; x, y)$, is represented as an integral over the circular contour shown.