Optical Switch Dimensioning and the Classical Occupancy Problem

Vincenzo Eramo, Marco Listanti,
University of Rome "La Sapienza", INFOCOM Dept., Via Eudossina, 18-00184 Roma, Italy.

Carl Nuzman, Phil Whiting
Bell Laboratories, Lucent Technologies P.O. Box 636, Murray Hill, NJ 07974 USA.

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Abstract

Results for optical switch dimensioning are obtained by analysing an urn occupancy problem in which a random number of balls is used. This analysis is applied to a high speed bufferless optical switch which uses tuneable wavelength converters to resolve contention between packets at the output fibres. Under symmetric packet routing the urn problem reduces to the classical occupancy problem. Since the problem is large scale and the desired loss probabilities are very small ($\approx 10^{-10}$) we outline a large deviations approximation as an alternative to exact analysis.

1 Introduction

Future communication networks are projected to have dramatic increases in bandwidth demand. These demands can be met by the use of optical fibres carrying data at very high line speeds and by sending signals across multiple wavelengths (wavelength division multiplexing). The switching of these optical signals, however, represents a severe technical challenge which may be met by the use of all-optical devices which avoid the electronic bottle-neck. There is, therefore, considerable interest in all optical packet switches which can allocate packets to output fibres and assign wavelengths on the micro-second time-scale. The design of such switches gives rise to a range of problems including the one considered here.

We consider an application of the classical occupancy problem to the dimensioning of an $n \times n$ packet switch with an all optical fabric. The switch has synchronous inputs so that all packets for a given timeslot are presented at the inputs simultaneously. The switch is bufferless so that packets not routed to an output are lost. The switch is controlled by an electronic unit basing its decisions on header information tapped from the input packet. The switch was first described in [2] and is illustrated in figure 1.

A key problem for packet switch design is the management of contention between packets routed to the same output fibre. Among the options for managing contention are (i) buffering which under current optical technology can be accomplished only by bundles of fibre delay lines [8],[9], (ii) deflection routing, which depends on the network topology and is better suited to networks with high inter-connectivity [3], (iii) wavelength translation [5] and (iv) dimensioning of traffic on wavelengths at the network level, [4], this latter approach relies on a high number of tuneable wavelength converters.

In the wavelength translation approach, if two packets of the same wavelength are destined to the same output fibre, one will be converted to an unused wavelength on that fibre. In the simplest architecture a large number of converters are required. In the switch considered here, wavelength converters are provisioned as a shareable resource at the switch output with the outputs of the converters themselves connected to dedicated switch inputs. This allows a considerable reduction in the number of converters needed at the expense of some increase in switch complexity.

The focus of this paper is thus on the number of wavelength converters needed in order to ensure a target packet loss. This loss includes packets for which a wavelength assignment cannot be made at the given output and those for which an assignment has been made but cannot be converted because all the converters are already taken. In order to ensure low probability for packet retransmission we take a target value for overall loss of 10^{-10}.

Our analysis supposes that the input traffic to the switch is stationary. Since operation is synchronised and there is no buffering, mean packet loss is then *determined by the marginal distribution of packets over input-output pairs.* Furthermore, to provide a simplified presentation, we suppose a packet is present at a particular input fibre and wavelength with probability a independent of all other fibres and wavelengths. Finally, it will also be supposed that the packets are routed to output fibres independently with equal probability of a particular output being selected. These assumptions may be reasonable if the packet arrivals are generated from a diverse number of independent sources. The analysis we present permits the relaxation of both of these assumptions.

Under these assumptions, the dimensioning of the wavelength converters can be couched in terms of an occupancy problem, with a random number of balls (corresponding to packets in a given timeslot) thrown into a fixed number of urns (corresponding to output fibres).

We will be interested in the probability that an unusually small number of urns are occupied when a random number of balls are thrown into them. This corresponds to an exceptionally high requirement for wavelength converters.
Whilst exact formulas, based on the exclusion-inclusion principle, can be used in principle, see [6] the approach suffers from the curse of dimensionality as it involves repeated convolutions. Rounding error is a further problem as inclusion-exclusion involves alternating sums of quantities with varying magnitude. It is also difficult to generalise under some relaxations of the assumptions.
The large deviations approach is much more numerically tractable. Moreover it is asymptotically exact in the case of a large number of inputs and outputs. Since the loss probabilities are very small, the approach turns out to be quite accurate. The approach provides insight into loss, as it provides formulas for the limiting exponent of loss. It admits the following generalisations.

i) Multiple fibres serving the same edge
ii) Switches with small n but large F
iii) Frequency converters with constrained output
iv) Other traffic distributions than binomial (including non-uniform traffic)

The rest of the paper is as follows. Section 2 details the switch architecture and discusses some implications of the design. Section **??** centres on results for the classical occupancy problem, the key result being a large deviations approximation for the number of balls needed to occupy a given fraction of a large number of urns. In section 3 results are given for packet loss in a switch with an unlimited number of converters. Subsequently the results of section **??** are extended to the case that a random number of balls are thrown (packet arrivals) which allow results for overall packet loss to be obtained, including that due to limitations on the number of wavelength converters. In section **??** we examine the accuracy of the large deviations approximation and indicate the numbers of wavelength converters which are needed in a practical switch. Finally we present our conclusions.

2 The Switch Architecture

The switch is an all-optical design described in detail in [2]. There are n input fibres and n output fibres, and packets arrive on one of F wavelengths. There are thus nF fibre-wavelength pairs which we refer to as ports. Operation is time-slotted so that packets, which are of fixed size, arrive in parallel over the various input ports. Any packet may be sent to any output fibre so there is the possibility of contention at the output because more than F packets can be addressed to a given output fibre. Since the packets cannot be buffered the

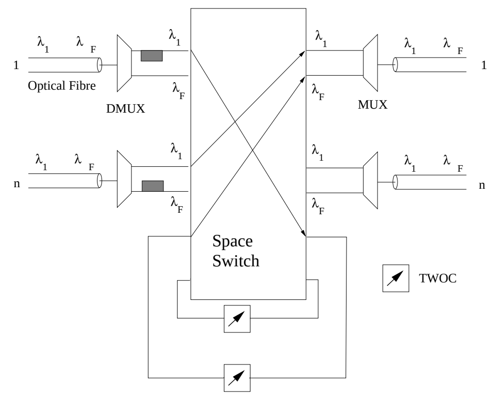

Figure 1: A Bufferless Optical Switch with Shareable Wavelength Conversion

excess packets must be dropped. Contention occurs also because packets using the same wavelength are routed to the same output. The switch resolves this contention by switching packets through a tuneable wavelength optical converter (TWOC). There are W converters provided as a shareable resource at the switch output, as shown in figure 1. Converters in the pool have the capability to convert any input wavelength to any output wavelength and have to be rapidly tuneable for this application. The output of each wavelength converter is connected to the input of the switch so that the packet on its new wavelength can be switched to the desired output fibre.

This switch design is intended to provide a happy medium between two alternate architectures. In the first, there is no provision for wavelength conversion. The switch fabric simplifies to F $(n) \times (n)$ switches operating in parallel. Denoting by a the probability that an input port has a packet in a given time slot, the number of packets N contending for an output port in that time slot is Binomial(n,a/n). The blocking probability is

$$L^0 = \frac{1}{a}\mathbb{E}[[N-1]_+] \approx \frac{a}{2},$$

where the approximation uses the Poisson approximation to the binomial for large n, and a is assumed small.

To mitigate this severe blocking, full wavelength conversion can be used. Here a wavelength converter is provisioned at the output port of each of the nF channels, and a full $(nF)\times(nF)$ swithc fabric is employed. Blocking only occurs now if more than F packets contend for the F wavelengths carried by a given fiber. The number N of such packets is Binomial(nF,a/n), and the blocking probability is

$$L^{nF} = \frac{1}{aF}\mathbb{E}[[N-F]_+]$$

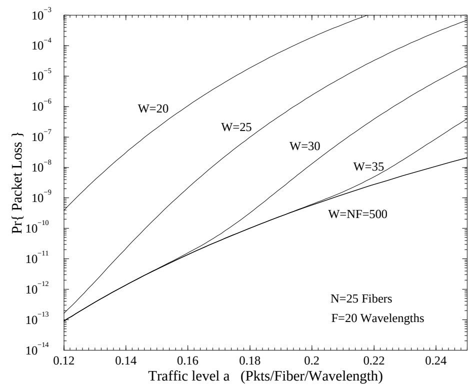

Figure 2: Packet loss with shared wavelength conversion

$$= \frac{1}{aF} \sum_{k=F+1}^{nF} (k-F) \left(\frac{a}{n}\right)^k \left(1-\frac{a}{n}\right)^{nF-k}$$

$$\approx \sum_{k=F+1}^{\infty} \frac{(aF)^k}{k!} e^{-aF}(k-F), \qquad (1)$$

where we have again used the Poisson approximation to get the last expression. In this case the blocking probability falls of exponentially with increasing F, and blocking on the order of 10^{-10} becomes feasible when several wavelengths are employed.

The design depicted in Figure 1 provides an intermediate point in the tradeoffs between no wavelength conversion and full conversion. Increasing the number of converters W increases the cost of the switch fabric and converters, but decreases the blocking probability.

Figure 2 examines the blocking behavior for a particular switch, using the analysis developed in the remainder of this paper. This switch has $n = 25$ input and output fibers, with each fiber carrying up to $F = 20$ wavelength multiplexed packets. The lower curve depicts the blocking L^{nF} achievable using full conversion. The other curves depict the blocking L^W achievable via a shared pool of converters, for $W = 20, 25, 30, 35$. If the goal is to operate at 10^{-10} blocking probability, then the load must be capped at $a < 0.2$, in which case only $W = 35$ converters perform just as well as 500. Hence a significant cost savings may be possible using the shared design. One disadvantage of the shared design over either of the other two is that converted packets pass through the switch fabric twice and hence incur additional loss.

3 Loss Analysis

Whether the switch uses full or shared conversion, packets are dropped if more than F packets require the same output fiber. If shared conversion is used, additional packets are dropped when more than W packets require wavelength conversion. In this section, we use large deviations to compute the distribution of the number of wavelength converters required per time slot, whence the blocking probability.

Remark: If packets must be dropped due to an overloaded output fiber, then in an optimal algorithm, any wavelength converters which would have been required for those packets become available. To simplify our analysis, do not take this effect into account, and hence the number of wavelength converters that we compute is an upper bound. As long as the probability of a full output fiber is small, this simplification has a negligible impact, and the bound is tight.

3.1 Converter Requirement from a Single Wavelength

Let us focus on a packets of a given wavelength λ. The number of packets R_λ of that wavelength is a Binomial(n,a) random variable. Denote by $0 \leq N_\lambda \leq R_\lambda$ the number of distinct output fibres to which these packets are assigned. Then the number of wavelength converters required is $W_\lambda = R_\lambda - N_\lambda$.

The variables R_λ and N_λ can be interpreted directly in terms of the classical occupancy problem; that is, we throw R_λ balls at random into n urns, and wish to know the distribution of the number of occupied urns N_λ.

For a fixed number of packets $R_\lambda = r$, the large deviations behavior of N_λ in n was given in [**?**]. If r and n increase together so that $\theta = r/n$ is fixed, then the probability of an unusually low number of occupied urns decreases exponentially, with exponent given by

$$\lim_{n\to\infty} -\frac{1}{n}\mathbb{P}\{N_\lambda^{(n)}/n < \xi | R_\lambda^{(n)}/n = \theta\} = J(\xi,\theta)$$

where

$$J(\xi,\theta) = (\theta - \xi)\log\rho + (1-\xi)\log(1-\xi) - \frac{(1-\rho\xi)}{\rho}\log(1-\rho\xi)$$

and ρ is the unique root of $\theta\rho = -\log(1-\rho\xi)$.

The large deviations behavior of R_λ as n grows is given informally by the expression

$$\lim_{n\to\infty} -\frac{1}{n}\mathbb{P}\{R_\lambda^{(n)}/n \approx \theta\} = l_a(\theta) = \theta\log\frac{\theta}{a} + (1-\theta)\log\frac{1-\theta}{1-a}.$$

An unusually high number of converters $W_\lambda = \alpha n$ typically occurs when there are unusually many packets R_λ occupying unusually few fibers N_λ. In the large deviations limit, the most probable combination of R_λ and N_λ meeting a given constraint will dominate. As we show in the appendix, this leads to

$$\lim_{n\to\infty} -\frac{1}{n}\mathbb{P}\{W_\lambda^{(n)}/n \geq \alpha\} = \inf_{\theta\geq\alpha} J(\theta-\alpha,\theta) + l_a(\theta) \equiv \mathcal{E}_a(\alpha).$$

The computation of $\mathcal{E}_a(\alpha)$ for a particular switch is depicted in Figure **??**. Here there are $n = 50$ fibres, there are $a = 0.4$ packets per port, and we are interested in the probability that 15 or more packets of a particular wavelength λ require conversion, so that $\alpha = 15/50 = 0.3$. The solid line in the figure depicts $J(\theta-\alpha,\theta)$ as a function of θ, the dashed line depicts $l_a(\theta)$, and the line marked with dots represents the sum. The sum of the two exponents is minimized at $\theta \approx 0.64$, with the minimum value being 0.24. The probability of this event is therefore approximately $\exp(-50\times 0.24) \approx 7.1\times 10^{-6}$. Exact calculation of this probability using inclusion/exclusion formulas yields 9.1×10^{-7}. The minimum number of converters required to achieve less than 10^{-10} is 20, according to the exact formula, and 21, by the large deviations estimate.

3.2 Total Converter Requirement

Modulo the remark at the beginning of this section, the total number of converters required to avoid blocking in any time slot is given by the sum of requirements over all wavelengths, $W = \sum_\lambda W_\lambda$. Because R_λ and N_λ are independent across wavelengths, it follows that

$$\lim_{n\to\infty} -\frac{1}{n}\log\mathbb{P}\{W^{(n)}/n > \alpha\} = \inf_{\sum_\lambda \alpha_\lambda=\alpha} \sum_\lambda \mathcal{E}_{a_\lambda}(\alpha_\lambda),$$

where the traffic load for each wavelength is a_λ. When the traffic loads are all equal, the minimizing vector of converter requirements is $\alpha_\lambda = \alpha/F$ and the minimum exponent is simply $F\mathcal{E}_a(\alpha/F)$. This fact is due to the convexity of $\mathcal{E}(\alpha)$ [**?**].

Because of this convexity, the tail of the wavelength converter distribution grows more slowly than the number of wavelengths F as F increases. In the numerical example above, we had $\mathbb{P}\{W_\lambda > 21\} \approx 10^{-10}$. If there $F = 10$ wavelengths, then $\mathbb{P}\{\sum_\lambda W_\lambda > 81\} \approx 10^{-10}$, so that four times as many converters are needed to service ten times as many packets.

3.3 Packet Loss due to Insufficient Converters

For a given number of converters w, the expected fraction of packets which must be dropped due to insufficient converters is given by

$$P = (naF)^{-1}\mathbb{E}[(W-w)_+] = \sum_{k=w+1}^{nF} \mathbb{P}\{W \geq k\}$$

$$= \sum_{k=w+1}^{nF} e^{-nF\mathcal{E}(k/nF)}.$$

For large n this last sum approximates a Riemann integral. Because $\mathcal{E}(\alpha)$ is increasing and convex, the integral can be upper bounded as follows:

$$\begin{aligned} P &\approx a^{-1} \int_{w/nF}^{1} e^{-nF\mathcal{E}(\alpha)}\, d\alpha \\ &\leq a^{-1} \int_{w/nF}^{\infty} e^{-nF(\mathcal{E}(w/nF)+(\alpha-w/nF)\mathcal{E}'(w/nF))}\, d\alpha \\ &= \frac{e^{-nF\mathcal{E}(w/nF)}}{naF\mathcal{E}'(w/nF)}. \end{aligned} \qquad (2)$$

As mentioned in Section 2, the lower curve in Figure 2 depicts the blocking probability for a switch with full conversion, computed from (1). The other curves were obtained by adding to the lower curve the additional blocking probability caused by insufficient wavelength converters, via (2).

4 Conclusions

We have analysed a bufferless optical wavelength-multiplexed packet switch with wavelength converters provided as a shareable resource at the switch output. Our analysis indicates that, when a given low blocking probability is targeted, the number of wavelength converters w required may be much smaller than the total number of output channels nF. The ratio w/nF decreases as the number of fibers n and number of wavelengths F increases.

Our results also indicate that a large deviations approximation for survival and loss probabilities (at the converter) is reasonably accurate, even for small switch sizes (of the order $n = 50$) and for low blocking. The large deviations analysis may be readily generalised to other assumptions on the arrival distribution, beyond the independent binomial result obtained here, by modifying the corresponding optimisation problem. Large deviations results may also be obtained in other cases by working with other urn models, but not with the approach taken here which relies on the Gartner-Ellis theorem. Such an analysis would allow assumptions such as converters with a fixed output or other routing assumptions to be included in the model.

A Establishing the Large Deviations Principle

Theorem A.1 *A large deviations principle exists for the conditional occupancy problem.*

Proof

We will establish the conditions needed for the Gartner-Ellis theorem [1] to apply. Given a sequence b_n such that $b_n/n \to \xi$, define the sequence of random variables

$$R_n \equiv \sum_{j=1}^{b_n} B_j^n, \tag{3}$$

where B_j^n is the random number of balls needed to occupy the jth cell when there are n cells and $j-1$ cells are already occupied. Piling up corresponds to making more throws than usual and so we will be interested in the probability,

$$\mathbb{P}\{\sum_{j=1}^{b_n} B_j^n \geq n\theta\} \quad . \tag{4}$$

which is a large deviations event provided $\xi < 1 - e^{-\theta}$. Setting,

$$\varphi_n(t) \equiv \frac{1}{n} \log \mathbb{E}\left[e^{tR_n}\right] \tag{5}$$

we find that

$$\begin{aligned} \varphi_n(t) &= \frac{1}{n}\sum_{j=1}^{b_n} \log(p_j e^t) - \frac{1}{n}\sum_{j=1}^{b_n} \log(1 - q_j e^t), \\ &= t\frac{b_n}{n} + \frac{1}{n}\sum_{j=1}^{b_n} \log(p_j) - \frac{1}{n}\sum_{j=1}^{b_n} \log(1 - q_j e^t) \end{aligned} \tag{6}$$

which is well defined provided $t < -\log \xi$, otherwise,

$$\mathbb{E}\left[e^{tR_n}\right] = \infty \quad . \tag{7}$$

As argued earlier,

$$\begin{aligned} \frac{1}{n}\sum \log p_j &\to \int_0^\xi \log(1-x)dx \\ \frac{1}{n}\sum \log(1 - q_j e^t) &\to \int_0^\xi \log(1 - e^t x)dx \ , \end{aligned}$$

so that

$$\lim_n \varphi_n(t) = \varphi(t) = t\xi - (1-\xi)\log(1-\xi) + \frac{(1-e^t\xi)}{e^t}\log(1-e^t\xi) \tag{8}$$

It follows that the *effective domain* $\mathcal{D}_\varphi$ is $(-\infty, -\log\xi)$ and $0 \in \mathcal{D}_\varphi$. It follows that Assumption 2.3.2 of [1] applies to φ.

φ is clearly lower semi-continuous. Differentiating, we find that for $t \in \mathcal{D}_\varphi$

$$\varphi'(t) = \xi - \log(1 - \xi e^t) - \xi e^t \tag{9}$$

and exists. Moreover $\varphi'(t) \to \infty$ as $t \to -\log \xi$. φ is therefore steep. It follows that φ is *essentially smooth*, as defined in 2.3.5 of [1]. We have thus validated the conditions for the Gartner-Ellis theorem to apply and it follows that the sequence of random variables $\{R_n\}$ satisfies a large deviations principle with good rate function,

$$\begin{aligned} J(\xi,\theta) &\equiv \sup_t t\theta - \varphi(t) \\ &= (\theta-\xi)\log\rho + (1-\xi)\log(1-\xi) - \frac{1-\rho\xi}{\rho}\log(1-\rho\xi) \end{aligned} \tag{10}$$

where ρ satisfies

$$\theta\rho = -\log(1-\rho\xi) \quad . \tag{11}$$

This completes the proof. □

We now establish some useful properties of $J(\beta-\alpha,\beta)$ for $\beta > \alpha$.

Lemma A.1 *Fix* $\alpha \in (0,1)$ *then* J *is a non-negative decreasing function in the interval* $(\alpha,\beta_0]$*, where* $\beta_0 = \min(1,\beta_\alpha)$ *and* β_α *is the root of the equation* $e^{-\beta}+\beta-1=\alpha$*. Moreover* $J \uparrow \infty$ *as* $\beta \downarrow \alpha$ *and if* $\beta_0 < 1$ *then* $J(\beta_0-\alpha,\beta_0) = 0$*. and* $dJ/d\beta \to 0$ *as* $\beta \uparrow \beta_\alpha$*. In addition* J *is strictly convex in the interval* $(\alpha,\beta_0]$.

Proof

It is convenient to suppose that $\beta_\alpha \leq 1$, a similar argument applies if this is not the case. We set $\theta = \beta, \xi = \beta - \alpha$ and take derivatives,

$$\begin{aligned} \frac{dJ}{d\beta} &= -\log(1-\xi) - 1 + \log(1-\xi\rho) + 1 \\ &+ \frac{\theta-\xi}{\rho}\rho_\beta + \frac{\rho_\beta}{\rho^2}\log(1-\xi\rho) + \frac{\xi}{\rho}\rho_\beta \\ &= \log\frac{(1-\xi\rho)}{(1-\xi)} \end{aligned} \tag{12}$$

where $\rho_\beta \equiv d\rho/d\beta$, since the coefficients of ρ_β cancel. We thus see that J is decreasing provided $\rho > 1$. Also $\rho = 1$ implies that

$$e^{-\beta} + \beta - 1 = \alpha \tag{13}$$

which implies that $\beta = \beta_\alpha$. Differentiating both sides of the equation $\theta\rho = -\log(1-\xi\rho)$ with respect to β gives,

$$\rho + \theta\rho_\beta = \frac{\xi\rho_\beta + \rho}{1-\xi\rho}, \tag{14}$$

so that

$$\left(\theta - \frac{\xi}{1-\xi\rho}\right)\rho_\beta = \frac{\xi\rho^2}{1-\xi\rho}. \tag{15}$$

The RHS is positive and since

$$\begin{aligned}\theta\rho - \frac{\xi\rho}{1-\xi\rho} &= \theta\rho - e^{\theta\rho} + 1 \\ &< 0\end{aligned}$$

as $\rho > 0$, we may deduce that $\rho_\beta < 0$. Thus as

$$\beta \uparrow \beta_\alpha, \quad \rho \downarrow 1, \quad J \downarrow 0 \quad . \tag{16}$$

Since ρ is determined by $1 - e^{-\theta\rho} = \xi\rho$ it can also be seen that $\rho \uparrow \infty$ as $\beta \downarrow \alpha$ and thus $J \uparrow \infty$.

To show convexity of J it is sufficient to show that $\log(1-\xi\rho)$ is increasing, since $-\log(1-\xi)$ is an increasing function of β. Alternatively we wish to show that $\xi\rho$ is decreasing. However,

$$\begin{aligned}\frac{d(\xi\rho)}{d\beta} &= \rho + \xi\rho_\beta \\ &= (1-\xi\rho)\,(\rho + \theta\rho_\beta)\end{aligned}$$

from (14) and

$$\rho + \theta\rho_\beta = \frac{(\theta - \xi)\rho}{(\theta(1-\xi\rho) - \xi)} \quad .$$

from (15) after rearrangement.

The numerator is positive and ρ times the denominator is

$$\theta\rho(1-\xi\rho) - \xi\rho = e^{-\theta\rho}\left(\theta\rho + 1 - e^{\theta\rho}\right) < 0 \quad .$$

Hence

$$\frac{d(\xi\rho)}{d\beta} < 0$$

as required. □

Since both l, J are strictly convex it follows that $H(\beta) = J(\beta, \beta - \alpha) + l(\beta, a)$ is also convex and has a unique minimum. Since H is decreasing in the interval (α, a) it follows that $\overline{\beta} \in (a, \beta_0]$. In the case $\beta_0 < 1$ $\overline{\beta}$ is determined by $H'(\overline{\beta}) = 0$.

This leads to,

Lemma A.2 *Provided that $\beta_\alpha < 1$ the exponent is given by*

$$\mathcal{E}(\alpha) = l(\overline{\beta}, a) + \alpha \log \rho(\overline{\beta}) + (1 + \alpha - \overline{\beta}) \log(1 + \alpha - \overline{\beta}) + \overline{\beta}\left(1 - (\overline{\beta} - \alpha)\rho(\overline{\beta})\right) \quad (17)$$

where

$$\rho(\beta) = \frac{(\beta - a)}{(1-a)\beta(\beta - \alpha)} + \frac{(1-\beta)a}{(1-a)\beta} \quad (18)$$

and $\overline{\beta}$ is the unique root in (a, β_α) of

$$\beta\rho(\beta) = -\log(1 - (\beta - \alpha)\rho(\beta)) \quad (19)$$

such that $\rho(\beta) \geq 1$.

Proof

Making the substitution $\theta = \beta, \xi = \beta - \alpha$ and then differentiating H and setting it to 0 yields,

$$\log \frac{\beta}{a} - \log \frac{1-\beta}{1-a} + \log \frac{(1 - \xi\rho)}{(1-\xi)} = 0\ . \quad (20)$$

Straightforward manipulation then yields $\rho(\beta)$. If we are given $\beta \in (a, \beta_\alpha)$ such that $\rho(\beta) \geq 1$ and β is a root of (19) then β determines a minimum of H which has already been established to be unique. □

Corollary A.1 *$\mathcal{E}$ is non-negative, monotone increasing.*

Proof

□

We now consider the asymptotic exponent for the case where there is a binomial number of packets R_n with parameter a. We will prove

Theorem A.2 *Let w_n be a sequence of converter requirements such that $w_n/n \to \alpha$. We then have that*

$$\lim_n -\frac{1}{n}\mathbb{P}\{W_n \geq w_n\} = \mathcal{E}(\alpha) = \inf_\beta J(\beta, \beta - \alpha) + l(\beta, a)\ , \quad (21)$$

where W_n is the number of converters actually used per use of the switch.

Proof

The idea of the proof is that the total probability of this event is the sum of n exponentially small quantities in n and this sum is therefore dominated by the term with the smallest exponent. Let R_n be the number of packets arriving at the switch for a given frequency. Observe that we cannot require w converters if $R_n \leq w$. For convenience we suppose that $\beta_0 < 1$. The case when $\beta_0 = 1$ requires only minor modifications.

Define the entropy function as

$$h(p) \equiv -\left(p \log p + (1-p)\log(1-p)\right) \quad .$$

Given r such that $r \geq 1, n-r \geq 1, r = n\beta$, we have that

$$\sqrt{\frac{1}{8n\beta(1-\beta)}} \leq \binom{n}{r} e^{-nh(\beta)} \leq \sqrt{\frac{1}{2\pi n\beta(1-\beta)}} \tag{22}$$

see for example [7].

We then have the following approximation to the binomial probabilities.

$$\mathbb{P}\{R_n = r\} = \binom{n}{r} a^r (1-a)^{n-r} = e^{n(\varepsilon_n(\beta)+h(\beta)+f(a,\beta))} = e^{n(\varepsilon_n(\beta)+l(\beta,a))} \tag{23}$$

where $\varepsilon = O(-n^{-1}\log\sqrt{n})$ from (22) and $f(x,y) \equiv y\log x + (1-y)\log(1-x)$.

To obtain an upper bound we use Chernoff's theorem on the conditional probabilities. This will be an exponential bound for β provided

$$e^{-\beta} + \beta - 1 < \alpha \quad . \tag{24}$$

Indeed

$$\mathbb{P}\{\sum_{j=1}^{r-w} B_j^n \geq r | R_n = r\} \leq e^{-nJ(\beta-\alpha,\beta)+\Delta_n} \tag{25}$$

where Δ_n is an error term. This error corresponds to replacing the sums in (7) by their corresponding integrals and by approximating w by $n\alpha$. We may take a uniform bound $\Delta_n < \Delta$ for β in the range $(\alpha+\epsilon, \beta_0 - \epsilon]$, with $\beta^* < \beta_0 - \epsilon$. For $\beta < \alpha + \epsilon$ it will be convenient to bound,

$$J(\beta-\alpha,\beta) > J(\epsilon, \alpha+\epsilon)$$

and for $\beta > \beta_0 - \epsilon$ the conditional probability will be bounded by 1. Now $\beta^* \in (a, \beta_0)$ so that putting together (23) and (25) we obtain

$$\begin{aligned} \mathbb{P}\{W_n \geq w_n\} &= \mathbb{P}\{\sum_{j=1}^{R_n-w_n} B_j^n \geq R_n\} \\ &\leq n \times e^{-nH(\beta^*)} \quad . \end{aligned} \tag{26}$$

for ϵ sufficiently small. From this it immediately follows that

$$\limsup_{n} \frac{1}{n}\mathbb{P}\{W_n \geq w_n\} \leq -H(\beta^*) \tag{27}$$

To obtain a lower bound take a sequence r_n such that $r_n/n \to \beta^*$.

$$\mathbb{P}\{\sum_{j=1}^{R_n-w_n} B_j^n \geq R_n\} \geq \mathbb{P}\{\sum_{j=1}^{r_n-w_n} B_j^n \geq r_n\}\mathbb{P}\{R_n = r_n\} \ . \tag{28}$$

Using theorem A.1, we have that

$$\begin{aligned} \liminf_{n} \frac{1}{n} \log \mathbb{P}\{\sum_{j=1}^{r_n-w_n} B_j^n \geq r_n\} &\geq -J(\beta^*-\alpha,\beta^*) \\ \lim_{n} \frac{1}{n} \log \mathbb{P}\{R_n = r_n\} &= -l(\beta^*,a) \ . \end{aligned} \tag{29}$$

Putting these together we obtain,

$$\begin{aligned} \liminf_{n} \frac{1}{n} \log \mathbb{P}\{W_n \geq w_n\} &= \liminf_{n} \frac{1}{n} \log \mathbb{P}\{\sum_{j=1}^{R_n-w_n} B_j^n \geq R_n\} \\ &\geq -H(\beta^*) \end{aligned} \tag{30}$$

which establishes the lower bound.

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