Contraction Approach to Power Control, with Non-Monotonic Applications

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***Abstract*—In wireless networks, monotonic, strictly subhomogeneous functions have been used to analyze power control algorithms. We provide an alternative analysis based on the observation that such functions are shrinking with respect to certain metrics. These metrics are then used to analyze problems involving two classes of non-monotonic functions. The first class consists of normalized interference functions that can be used to compute the maximum achievable signal-to-interference ratio under power constraints. The second class consists of absolutely subhomogeneous functions that are used to approximate power control dynamics with outages. The eigensolutions of monotonic, strictly subhomogeneous functions are characterized as part of this development.**

I. Introduction

For communication systems with interference between users, power control is used to ensure that users receive sufficiently strong signals and sufficiently low interference. The effectiveness of power control often has a crucial effect on the performance of systems such as wireless cellular networks, mobile ad hoc networks, and digital subscriber line bundles. This paper develops a mathematical framework that is useful for analyzing power control systems, including some systems with non-monotonic behavior.

A wireless code-division multiple access uplink provides a canonical example. Here, geographically distributed mobiles transmit signals that are received at a number of base stations. The goal of the uplink power control problem is to choose minimal transmission powers such that all mobiles achieve a satisfactory signal to interference plus noise ratio (SINR). Reference [1] showed that under two simple conditions, this minimum power solution is well-defined and can be obtained by simple distributed synchronous and asynchronous algorithms. The synchronous algorithms are based on repeated iterations of an *interference function* $\mathbf{f}$. The two required conditions, monotonicity and scalability of $\mathbf{f}$, are satisfied by a wide variety of system models. Further variations on the algorithms were explored in [2] and subsequent work. In keeping with some related literature (e.g. [3]), we use the term *strictly subhomogeneous* in place of *scalable*.

In Section II of this paper, we re-examine the properties of monotonic, strictly subhomogeneous functions from a contraction mapping point of view, showing that such functions are *shrinking* with respect to certain metrics. The theory provides alternative proofs for results in [1], but more importantly, can be extended to certain non-monotonic functions. In Section III, the results are used to compute a maximal achievable SINR via iterations of a non-monotonic function. In Section IV, non-monotonicity is required to model a threshold effect in which calls with very low SINR are dropped.

The general approach in all cases is to show that the function in question is shrinking with respect to a metric, and that the range of the function is contained in a compact set. Global convergence to a unique fixed point then follows by a version of the Banach fixed point theorem. The metrics, based on componentwise log ratios of vectors, are related to Hilbert's projective pseudo-metric, which plays a key role in some treatments of the Perron-Frobenius theory of non-negative matrices [4], [5]. Some of the results are based on an analysis of the eigensystems of monotonic, strictly subhomogeneous functions. This analysis, which has connections to non-linear generalizations of Perron-Frobenius theory such as [6], [7], is of some independent interest.

Throughout the paper, we indicate vector quantities in bold ($\mathbf{x}$ rather than x). The positive and non-negative orthant in m dimensions are denoted $\mathbb{R}_+^m$ and $\bar{\mathbb{R}}_+^m$, respectively. The relation $\mathbf{x} \leq \mathbf{y}$ denotes $x_i \leq y_i$ for each component, and similarly $\mathbf{x} < \mathbf{y}$ is defined componentwise. The notation $\mathbf{x} \lneq \mathbf{y}$ indicates that $\mathbf{x} \leq \mathbf{y}$ with $\mathbf{x} \neq \mathbf{y}$. The rectangle $\mathbf{a} \leq \mathbf{x} \leq \mathbf{b}$ is denoted $R(\mathbf{a}, \mathbf{b})$, and for a function $\mathbf{f}$, $\mathbf{f}^n$ denotes the n-fold composition $\mathbf{f}^n = \mathbf{f} \circ \mathbf{f}^{n-1}$.

II. Wireless Uplink Power Control

For a system with M active mobiles, represent the uplink transmission powers by the vector $\mathbf{p} \in \mathbb{R}_+^M$. For the mobile m, one can define an *interference* function $f_m : \bar{\mathbb{R}}_+^M \to \bar{\mathbb{R}}_+$; given that all (other) mobiles transmit with powers specified by $\mathbf{p}$, $f_m(\mathbf{p})$ is the minimum power at which mobile m must transmit to achieve its target SINR. As a concrete example, consider a system with K base stations, where mobile m homes to a fixed base station $k(m)$, and where $\mathcal{M}_k$ is the set of mobiles that home to base station k. Denote by g_{km} the gain from mobile m to base station k, and by η_k the ambient noise power at base station k. Then the SINR for mobile $m \in \mathcal{M}_k$ is $g_{km}p_m / \left(\sum_{n \neq m} g_{kn}p_n + \eta_k\right)$. If the desired SINR is γ, then the interference function is

$$f_m(\mathbf{p}) = \frac{\gamma}{g_{k(m)m}} \left(\sum_{n \neq m} g_{k(m)n}p_n + \eta_{k(m)} \right). \qquad (1)$$

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One can combine the individual interference functions into the vector function $\mathbf{f} : \mathbb{R}_+^M \to \mathbb{R}_+^M$. The power control problem is then simply to find the minimal vector $\mathbf{p}$ such that $\mathbf{p} \geq \mathbf{f}(\mathbf{p})$. One can define a wide variety of interference functions under different operational models. Following [1], we will only require that the function have the following two properties:

- *Monotonicity*: $\mathbf{x} \leq \mathbf{y}$ implies $\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(\mathbf{y})$.
- *Strict subhomogeneity*: for each $\alpha > 1$, $\mathbf{f}(\alpha\mathbf{x}) < \alpha\mathbf{f}(\mathbf{x})$.

The original definition also required positivity, ($\mathbf{f}(\mathbf{0}) > \mathbf{0}$) but [2] points out that this is a consequence of the other two properties. We will refer to a function satisfying these two properties as monotonic and strictly subhomogeneous (MSS). When the strict inequality above is relaxed to weak inequality, we say that a function is monotonic and subhomogeneous (MS). Reference [1] showed that an MSS function has at most one fixed point $\mathbf{v} = \mathbf{f}(\mathbf{v})$. If it has at least one feasible point $\mathbf{x} \geq \mathbf{f}(\mathbf{x})$, then there is a fixed point and this point is minimal in the sense that $\mathbf{v} \leq \mathbf{x}$ for any feasible $\mathbf{x}$. Iterations of the form $\mathbf{p}[n+1] = \mathbf{f}(\mathbf{p}[n])$ converge to $\mathbf{v}$, and the iterations themselves have nice monotonicity properties; for example, if one starts at a feasible point $\mathbf{p}[0] \geq \mathbf{f}(\mathbf{p}[0])$, then the sequence $\mathbf{p}[n]$ decreases monotonically to $\mathbf{v}$.

In the original paper, these and other properties were proved with elegant, direct arguments. In this paper, we will reformulate the proof in terms of non-expansive and shrinking mappings. We will then show how this approach allows us to generalize to related applications featuring non-monotonic functions.

A mapping $\mathbf{f}$ from a metric space (X, μ) to itself is said to be

- *nonexpansive* if $\mu(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq \mu(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in X$;
- *shrinking* if $\mu(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x})) < \mu(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \neq \mathbf{y} \in X$;
- a *contraction* if there is $c < 1$ such that $\mu(\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{y})) \leq c\ \mu(\mathbf{x}, \mathbf{y})$ for $\mathbf{x} \neq \mathbf{y} \in X$.

As is well-known, the Banach fixed point theorem states that a contraction on a complete metric space converges globally to a unique fixed point. The shrinking property yields the same result if the metric space is known to be compact [8].

Theorem 2.1: Suppose that $\mathbf{f} : X \to X$ is shrinking on a compact metric space (X, μ). Then $\mathbf{f}$ has exactly one fixed point $\mathbf{v}$, and $\mathbf{f}^n(\mathbf{x})$ converges to $\mathbf{v}$ for each $\mathbf{x} \in X$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^m$, define

$$\begin{aligned} \mu_a(\mathbf{x}, \mathbf{y}) &= \max_{i=1}^{m} |\log x_i/y_i| \\ \mu_s(\mathbf{x}, \mathbf{y}) &= \max_{i=1}^{m} (\log x_i/y_i)^+ + \max_{i=1}^{m} (\log y_i/x_i)^+ \end{aligned}$$

where $(x)^+$ denotes $\max(x, 0)$. The functions μ_a and μ_s are metrics on $\mathbb{R}_+^m$. To see this, consider the metrics on $\mathbb{R}^m$ defined by

$$\begin{aligned} \rho_a(\mathbf{x}, \mathbf{y}) &= \max_{i=1}^{m} |x_i - y_i| \\ \rho_s(\mathbf{x}, \mathbf{y}) &= \max_{i=1}^{m} (x_i - y_i)^+ + \max_{i=1}^{m} (y_i - x_i)^+ \end{aligned}$$

and note that the component-wise logarithm defines an isomorphism from $\left(\mathbb{R}_+^m, \mu\right)$ to $(\mathbb{R}^m, \rho)$ in each case. For later reference, note that a finite closed rectangle in $\mathbb{R}_+^m$ bounded away from the origin is compact under μ_a (or μ_s), since it corresponds to a finite closed rectangle in $\mathbb{R}^m$ under ρ_a (or ρ_s).

While an MSS function is always strictly positive, components of an MS function $\mathbf{f}$ may be zero for certain inputs. However, it is not difficult to show that if $f_i(\mathbf{x}) = 0$ for a strictly positive input $\mathbf{x} > \mathbf{0}$, then f_i is identically zero for any input. From now on, we will use the term MS to refer to functions from which any such degenerate components have been eliminated, so that $\mathbf{f}(\mathbf{x}) > \mathbf{0}$ when $\mathbf{x} > \mathbf{0}$.

Lemma 2.2: Suppose that $\mathbf{f} : \mathbb{R}_+^m \to \mathbb{R}_+^n$ is MS. Then $\mathbf{f}$ is non-expansive with respect to μ_a and μ_s. If $\mathbf{f}$ is MSS, then it is shrinking with respect to these metrics.

Proof: Suppose first that strict subhomogeneity holds. Take $\mathbf{x}, \mathbf{y} > 0$ with $\mu_a(\mathbf{x}, \mathbf{y}) = \delta > 0$. Then $\mathbf{x} \leq e^{\delta}\mathbf{y}$. Applying monotonicity and then strict subhomogeneity, one obtains

$$\mathbf{f}(\mathbf{x}) \leq \mathbf{f}(e^{\delta}\mathbf{y}) < e^{\delta}\mathbf{f}(\mathbf{y}),$$

so that $\log f_i(\mathbf{x})/f_i(\mathbf{y}) < \delta$ for each i. A symmetric argument upper bounds $f_i(\mathbf{y})/f_i(\mathbf{x})$. Hence $\mathbf{f}$ is shrinking with respect to μ_a. When the strict inequalities are relaxed, $\mathbf{f}$ is non-expansive. A similar argument applies for μ_s. ■

The metrics μ_a and μ_s are closely related. Metric μ_s turns out to be useful in studying normalized iterations in Section III, while μ_a is used to study absolutely subhomogeneous functions in Section IV. Either one suffices for the following result.

Theorem 2.3: Suppose that $\mathbf{f} : \bar{\mathbb{R}}_+^m \to \mathbb{R}_+^m$ is MSS. Then $\mathbf{f}$ has at most one fixed point. If there is $\mathbf{y}$ with $\mathbf{f}(\mathbf{y}) \leq \mathbf{y}$, then there is a fixed point $\mathbf{v} > \mathbf{0}$, and $\mathbf{f}^n(\mathbf{x})$ converges to $\mathbf{v}$ for any $\mathbf{x} \geq \mathbf{0}$.

Proof: Since $\mathbf{f}(\mathbf{0}) > \mathbf{0}$, any fixed point is positive. The shrinking property of $\mathbf{f}$ on $(\mathbb{R}_+^m, \mu_a)$ then ensures that there cannot be more than one fixed point. Finally, if $\mathbf{f}(\mathbf{y}) \leq \mathbf{y}$, then also $\mathbf{f}(\alpha\mathbf{y}) \leq \alpha\mathbf{y}$ for any $\alpha \geq 1$ so that $\mathbf{f}$ maps $R(\mathbf{f}(\mathbf{0}), \alpha\mathbf{y})$ to itself. Given $\mathbf{x}$, there is a rectangle of this form containing $\mathbf{f}(\mathbf{x})$, and the result follows from Theorem 2.1. ■

III. Computing Feasible SINR

For given SINR targets, the methods of the previous section can be used to compute the minimal feasible power vector. This method converges globally whenever the problem is feasible. A related problem of interest is to compute the maximum target SINR that can be achieved for all users, for given power constraints. Let $\bar{\mathbf{f}}$ represent the interference function for the SINR target $\gamma = 1$ (e.g. the result of setting $\gamma = 1$ in (1)). Then we want to compute the largest γ such that $\mathbf{p} \geq \gamma\ \bar{\mathbf{f}}(\mathbf{p})$ has a solution. If the interference function is linear, say $\bar{\mathbf{f}}(\mathbf{p}) = A\mathbf{p}$, with A non-negative and primitive, then Perron-Frobenius theory indicates that the maximal γ is the inverse of the Perron-eigenvalue r. This value can be computed via the power method; for any $\mathbf{p} \gneq 0$, the sequence $A^n\mathbf{p}/\left\|A^n\mathbf{p}\right\|$ converges to the Perron eigenvector of A, and r is the corresponding eigenvalue. We will show that for any MSS interference function, an analogous method can be used to compute an eigensolution $r\mathbf{v} = \bar{\mathbf{f}}(\mathbf{v})$ such that $1/r$ is the maximal feasible SINR. In the linear case, the same eigenvalue is obtained for any scaling of the eigenvector. By contrast, with an MSS interference function, the maximal SINR increases (eigenvalue decreases) as the size of the eigenvector is allowed to increase. We show that repeated iterations of the function $\mathbf{g}(\mathbf{p}) = \theta\, \mathbf{f}(\mathbf{p}) / \|\mathbf{f}(\mathbf{p})\|$ converge to the solution obtaining the maximal SINR possible under the constraint $\|\mathbf{p}\| \le \theta$. These results follow immediately from structural properties of the eigensystem of MSS functions, which we derive next.

For an MSS function $\mathbf{f} : \bar{\mathbb{R}}^m_+ \to \bar{\mathbb{R}}^m_+$, we are interested in all solutions of the equation $r\mathbf{v} = \mathbf{f}(\mathbf{v})$. Since $\mathbf{f}(\mathbf{x}) \ge \mathbf{f}(\mathbf{0}) > \mathbf{0}$, we must have $r > 0$ and $\mathbf{v} > 0$. A number of properties follow immediately from Theorem 2.3.

Lemma 3.1: Let $\mathbf{f}$ be MSS. Then

1) There is at most one eigenvector $\mathbf{v}$ associated with a given $r > 0$.
2) If r is an eigenvalue, then so is s for every $s > r$.
3) If $s > r$, $r\mathbf{v} = \mathbf{f}(\mathbf{v})$ and $s\mathbf{u} = \mathbf{f}(\mathbf{u})$, then $\mathbf{u} < \mathbf{v}$.

Proof:

1) The function $\mathbf{f}_{[r]}(\mathbf{x}) := \mathbf{f}(\mathbf{x})/r$ is MSS, and hence has at most one fixed point. Fixed points of $\mathbf{f}_{[r]}$ are in one-to-one correspondence with eigenvectors of $\mathbf{f}$ with eigenvalue r.
2) Let $r\mathbf{v} = \mathbf{f}(\mathbf{v})$. Then $\mathbf{f}_{[s]}(\mathbf{v}) < \mathbf{v}$, and so $\mathbf{f}_{[s]}$ has a fixed point.
3) Continuing the argument above, the sequence $\mathbf{f}^n_{[s]}(\mathbf{v})$ decreases monotonically to $\mathbf{u}$. Then $\mathbf{u} \le \mathbf{f}_{[s]}(\mathbf{v}) < \mathbf{v}$.

■

Although there are always an infinite number of eigenvectors, each can be uniquely identified by its norm. We say that a norm is *monotonic* (on the orthant $\bar{\mathbb{R}}^m_+$) if $\mathbf{x} \le \mathbf{y}$ implies $\|x\| \le \|y\|$. Examples of monotonic vector norms include the max norm, Euclidean norm, and L_1 norm.

Theorem 3.2: Fix any monotonic norm $\|\cdot\|$ and MSS function $\mathbf{f}$. For each $\theta > 0$, there is exactly one eigenvector $\mathbf{v}$ and associated eigenvalue r of $\mathbf{f}$ such that $\|\mathbf{v}\| = \theta$.

Proof: Consider the function

$$\mathbf{g}(\mathbf{x}) = \frac{\theta\, \mathbf{f}(\mathbf{x})}{\|\mathbf{f}(\mathbf{x})\|},$$

and observe that the fixed points of $\mathbf{g}$ are in one-to-one correspondence with the eigenvectors of $\mathbf{f}$ having norm θ. Since a monotonic norm is an MS function from $\mathbb{R}^m_+$ to $\mathbb{R}_+$, Lemma 3.3 below shows that $\mathbf{h}(\mathbf{x}) = \mathbf{x} / \|\mathbf{x}\|$ is non-expansive with respect to μ_s. Then by virtue of Lemma 2.2, $\mathbf{g} = \theta\, \mathbf{h} \circ \mathbf{f}$ is shrinking with respect to μ_s.

For any input, $\mathbf{g}$ satisfies $\|\mathbf{g}\| = \theta$. Since a monotonic vector norm has bounded level sets, we have $\mathbf{g} \le \mathbf{b}$ for some finite $\mathbf{b}$. A lower bound for the range of $\mathbf{g}^2$ is therefore

$$\mathbf{g}^2(\mathbf{x}) \ge \frac{\theta \mathbf{f}(\mathbf{0})}{\|\mathbf{f}(\mathbf{b})\|} = \mathbf{a} > \mathbf{0}.$$

and we see that the range of $\mathbf{g}^n$ falls inside the finite positive rectangle $R(\mathbf{a}, \mathbf{b})$ for $n \ge 2$. By Theorem 2.1, $\mathbf{g}^n$ converges globally to a unique fixed point $\mathbf{v}$. Hence there is a unique eigenvector of $\mathbf{f}$ having norm θ. Statement (3) of Lemma 3.1 shows that only one eigenvalue can be associated with this eigenvector. ■

The proof of Lemma 3.3 is postponed to the end of this section. By virtue of Lemma 3.1 and Theorem 3.2, we can define a strictly decreasing function r_θ and strictly increasing vector function $\mathbf{v}_\theta$ which together describe the complete set of eigensolutions of $\mathbf{f}$.

Returning to the problem of computing a maximal SINR for a uplink system modeled by an MSS interference function $\bar{\mathbf{f}}$, we have a family of solutions $\mathbf{v}_\theta$ such that the SINR $\gamma_\theta = 1/r_\theta$ is achieved, and γ_θ is the maximal SINR achievable within the constraint $\|\mathbf{p}\| \le \theta$. Repeated iteration of the function $\mathbf{g}$ defined in the proof of Theorem 3.2 provides a constructive method for computing γ_θ. Note that the power constraint can be expressed by *any* monotonic norm; for example one can use $\|\mathbf{p}\| = \max_i p_i$ or $\|\mathbf{p}\| = \sum_i p_i$ to compute the maximum SINR under maximum or total power constraints, respectively.

It remains to prove that normalization by a MS function is non-expansive with respect to μ_s.

Lemma 3.3: Suppose that $\mathbf{h}(\mathbf{x}) = \mathbf{x}/\nu(\mathbf{x})$, where $\nu : \mathbb{R}^m_+ \to \mathbb{R}_+$ is MS. Then $\mathbf{h}$ is non-expansive on $(\mathbb{R}^m_+, \mu_s)$.

Proof: Consider $\mathbf{x}, \mathbf{y} > 0$ with $\mu_s(\mathbf{x}, \mathbf{y}) = \delta > 0$.

When $\mathbf{h}(\mathbf{x}) \not> \mathbf{h}(\mathbf{y})$ and $\mathbf{h}(\mathbf{y}) \not> \mathbf{h}(\mathbf{x})$, we have

$$\begin{aligned} \mu_s(\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{y})) &= \max_i \log \frac{x_i \nu(\mathbf{y})}{y_i \nu(\mathbf{x})} + \max_i \log \frac{y_i \nu(\mathbf{x})}{x_i \nu(\mathbf{y})} \\ &= \max_i \log \frac{x_i}{y_i} + \max_i \log \frac{y_i}{x_i} \\ &\le \mu_s(\mathbf{x}, \mathbf{y}) \end{aligned}$$

So it remains to consider cases in which $\mathbf{h}(\mathbf{x}) > \mathbf{h}(\mathbf{y})$ or $\mathbf{h}(\mathbf{y}) > \mathbf{h}(\mathbf{x})$.

- Suppose $\mathbf{x} \ge \mathbf{y}$. One has $\mathbf{y} \le \mathbf{x} \le e^\delta \mathbf{y}$ and $\nu(\mathbf{y}) \le \nu(\mathbf{x}) \le e^\delta \nu(\mathbf{y})$. Moreover,

$$\alpha := \max_i \left(\frac{h_i(\mathbf{x})}{h_i(\mathbf{y})} \right) = \frac{\nu(\mathbf{y})}{\nu(\mathbf{x})} \max_i \frac{x_i}{y_i} = \frac{\nu(\mathbf{y})}{\nu(\mathbf{x})} e^\delta.$$

  The bounds on ν then show that $1 \le \alpha$ so that $\mathbf{h}(\mathbf{x}) \not< \mathbf{h}(\mathbf{y})$. On the other hand, if $\mathbf{h}(\mathbf{x}) > \mathbf{h}(\mathbf{y})$, then $\mu_s = \log \alpha \le \delta$.
- The case $\mathbf{y} \ge \mathbf{x}$ follows by a symmetric argument.
- Suppose that $\mathbf{x} \not> \mathbf{y}$, $\mathbf{y} \not> \mathbf{x}$. Then $\delta = \alpha + \beta$ where $e^{-\beta}\mathbf{y} \le \mathbf{x} \le e^\alpha \mathbf{y}$ and $e^{-\beta}\nu(\mathbf{y}) \le \nu(\mathbf{x}) \le e^\alpha \nu(\mathbf{y})$. Then

$$\max_i \left(\frac{h_i(\mathbf{x})}{h_i(\mathbf{y})} \right) = \frac{\nu(\mathbf{y})}{\nu(\mathbf{x})} \max_i \frac{x_i}{y_i} = \frac{\nu(\mathbf{y})}{\nu(\mathbf{x})} e^\alpha \ge 1$$

  ensuring that $\mathbf{h}(\mathbf{y}) \not> \mathbf{h}(\mathbf{x})$. Similarly, considering $\max_i h_i(\mathbf{y})/h_i(\mathbf{x})$ shows that $\mathbf{h}(\mathbf{x}) \not> \mathbf{h}(\mathbf{y})$.

■

IV. Uplink Power Control with Outages

The model and much of the discussion in this section is based on [9], which considers power control in a system with outages. The resulting interference function is non-monotonic, reflecting the fact that excessive interference can cause a mobile to become inactive. In practice, there is usually a maximum power, say $\hat{p}$, at which a mobile is allowed to transmit. A number of choices are possible when the required power is higher than allowed, i.e. when $f_m(\mathbf{p}) > \hat{p}$. One option is to transmit at power $\min(\hat{p}, f_m(\mathbf{p}))$, effectively getting as close to the target SINR as possible under the constraint. The convergence properties of such a scheme can be studied using the results above, because for any fixed $\mathbf{b} > 0$ and MSS function $\mathbf{f}$, the function $\min(\mathbf{b}, \mathbf{f}(\cdot))$ is again MSS [1].

Another possibility is that a mobile that is not able to achieve its target SINR will block its communication session and become inactive, so that its transmitted power goes to zero. Define the function $\mathbf{h} : \bar{\mathbb{R}}_+^m \to \bar{\mathbb{R}}_+^m$ by

$$h_i(\mathbf{x}) = \begin{cases} x_i & x_i \le b_i \\ 0 & x_i > b_i \end{cases} .$$

For a given interference function $\mathbf{f}$, one obtains a dynamical system $\mathbf{g} = \mathbf{h} \circ \mathbf{f}$. It can readily be shown that such a system need not have a fixed point (because of the discontinuities), or it can have multiple fixed points.

In designing and optimizing cellular networks, it is useful to simulate the performance of a given configuration under realistic conditions, including realistic power control. In this application, one is not exactly interested in the dynamics of a power control scheme, but in its expected behavior, where the expectation is taken over random quantities including mobile positions and fading coefficients. One is interested in computing equilibria of a smoothed dynamical system involving expected power and interference levels. The smoothing implied by randomization typically ensures that fixed points exist (via Brouwer's theorem), but when outages are modeled, the interference function is non-monotonic, and fixed points no longer need to be unique in general.

In a model with random fading, let $x_m = \sum_{n \ne m} g_{kn} p_n + \eta_k$ represent *expected* noise plus interference for for mobile $m \in \mathcal{M}_k$. If the random gain from m to k is denoted $F_{km} g_{km}$, where F_{km} is a random fading coefficient, then under a simple outage model, mobile m would respond by choosing transmit power $P_m = h(\gamma x_m / (F_{km} g_{km}))$. The expectation of the transmit power would be

$$\bar{p}_m = E[P_m] = E\left[h\left(\frac{\gamma x_m}{F_{km} g_{km}}\right)\right] := \phi(\gamma x_m / g_{km}).$$

The function ϕ is a smoothed version of h, which we will refer to as the *smoothed mobile behavior function*. The solid curve in Figure 1 depicts the graph of h (with $b_i = 2$), and the dashed curves represent the smoothed function ϕ under various fading models. Here the log of the fading coefficient was taken to be a double exponential random variable with pdf $\lambda/2e^{-\lambda|x|}$, for $\lambda = \{2, 4, 8, 16\}$. For this and other fading models such as log-normal, the smoothed function increases monotonically (and subhomogeneously) before saturating and decreasing to zero at some rate. As λ increases and the variance of the fading decreases, the smoothed curves approximate h more and more closely. In the network design application, one is interested in

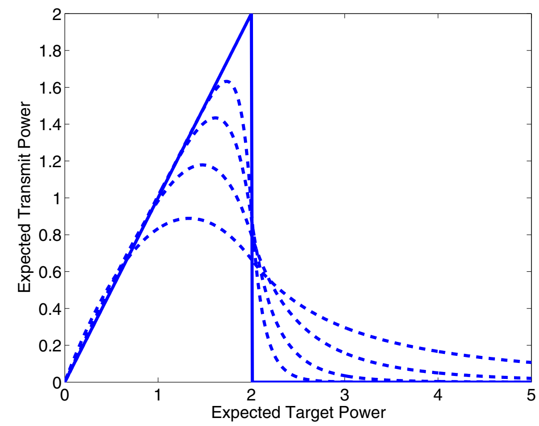

Fig. 1. Outage function h (solid curve) and smoothed mobile behavior functions ϕ (dashed curves).

finding equilibria (fixed points) of the smoothed function

$$\mathbf{g}(\mathbf{p}) = \boldsymbol{\phi}(\mathbf{f}(\mathbf{p})), \tag{2}$$

where $\boldsymbol{\phi}$ represents the componentwise application of the smoothed mobile behavior function ϕ.

Our contribution to this problem is to observe that if ϕ does not decrease too quickly, then the contraction approach can still be used to demonstrate the existence of a unique fixed point. Say that a function $\mathbf{f}$ is *absolutely subhomogeneous* if

$$e^{-|a|}\mathbf{f}(\mathbf{x}) \le \mathbf{f}(e^a \mathbf{x}) \le e^{|a|}\mathbf{f}(\mathbf{x})$$

for every $\mathbf{x} \gneq \mathbf{0}$ and scalar a. A subhomogeneous function can not increase by more than a factor α when the input is scaled by a factor α. For an absolutely subhomogeneous function, relative changes to the output, up or down, cannot be larger than relative changes to the input. Such a function cannot grow more than linearly, nor decay faster than an inverse law.

Lemma 4.1: Suppose that $\boldsymbol{\phi} : \mathbb{R}_+^N \to \mathbb{R}_+^M$ is such that

1) Each component function ϕ_i of $\boldsymbol{\phi}$ is a scalar function, depending on at most one input variable.
2) Each component function ϕ_i is absolutely subhomogeneous.

Then $\boldsymbol{\phi}$ is non-expansive with respect to μ_a.

Proof: Take $\mathbf{x}, \mathbf{y} > 0$ with $\mu_a(\mathbf{x}, \mathbf{y}) = \delta > 0$ so that the values $\lambda_i = \log x_i / y_i$ for $i = 1, \ldots, n$ satisfy $|\lambda_i| \le \delta$. Suppose that the component function ϕ_j depends on input i. Since $x_i = e^{\lambda_i} y_i$, we have

$$e^{-|\lambda_j|}\phi_j(y_i) \le \phi_j(x_i) \le e^{|\lambda_j|}\phi_j(y_i)$$

and $\mu_a(\boldsymbol{\phi}(\mathbf{x}), \boldsymbol{\phi}(\mathbf{y})) \le \delta$. ■

Theorem 4.2: Suppose that $\mathbf{g} = \boldsymbol{\phi} \circ \mathbf{f}$, where the component functions ϕ_m are bounded, absolutely subhomogeneous, and univariate, and where $\mathbf{f}$ is MSS. Then $\mathbf{g}$ has a unique fixed point $\mathbf{v}$, and $\mathbf{g}^n(\mathbf{p}) \to \mathbf{v}$ globally on $\bar{\mathbb{R}}_+^M$.

Proof: Since f is shrinking with respect to μ_a, and $\boldsymbol{\phi}$ is non-expansive, $\mathbf{g}$ is shrinking. It only remains to show that $\mathbf{g}^n$ eventually enters a finite, positive rectangle that is closed under $\mathbf{g}$, so that Theorem 2.1 can be applied.

Let $\mathcal{R}_0 = \bar{\mathbb{R}}_+^M$, and for $n \geq 1$ let $\mathcal{R}_n = \mathbf{g}(\mathcal{R}_{n-1})$ be the range of $\mathbf{g}^n$. If $\mathbf{b}$ denotes the vector of componentwise bounds on $\boldsymbol{\phi}$, then $\mathcal{R}_1 \subset R(\mathbf{0}, \mathbf{b})$, and by monotonicity, $\mathbf{f}(\mathcal{R}_1) \subset R(\mathbf{f}(\mathbf{0}), \mathbf{f}(\mathbf{b}))$. This last rectangle is bounded away from the origin. If any smoothed mobile behavior function ϕ_m is identically zero, the corresponding user can be ignored and the system dimension reduced. Otherwise, $\phi_m(\alpha) = \beta > 0$ for some $\alpha > 0$, and absolute subhomogeneity implies

$$\phi_m(t) \geq \beta \min(t/\alpha, \alpha/t)$$

for all $t \geq 0$. Thus $\phi_m(t)$ is bounded away from zero for t in any finite interval away from zero, and we may find a positive vector $\mathbf{a}$ such that $\mathcal{R}_2 = \boldsymbol{\phi}(\mathbf{f}(\mathcal{R}_1)) \subset R(\mathbf{a}, \mathbf{b})$. Since $\mathcal{R}_1 \subset \mathcal{R}_0$, the sets $\mathcal{R}_n$ form a nonincreasing sequence, and $\mathcal{R}_n \subset R(\mathbf{a}, \mathbf{b})$ for $n \geq 2$. ■

While subhomogeneity is a natural condition, it is not expected that absolutely subhomogeneous mobile behavior functions will be encountered in practice. As suggested by Figure 1, under an outage model with typical random fading, the mobile transmit power eventually falls off faster than hyperbolically with increasing interference. However, absolutely subhomogeneous functions are interesting in that they provide models with unique fixed points which are qualitatively similar to more realistic behavior functions. Of particular interest is the *least absolutely subhomogeneous upper bound*. For a given non-negative scalar function $\phi : [a, b] \to \bar{\mathbb{R}}_+$, there is a well-defined absolutely subhomogeneous least upper bound $q : [a, b] \to \bar{\mathbb{R}}_+$ given by

$$q(x) = \sup_{t \in [a,b]} \phi(t) \min(x/t, t/x).$$

The reader can verify that q is absolutely subhomogeneous, satisfies $q(x) \geq \phi(x)$, and is dominated by every other absolutely subhomogeneous upper bound on ϕ. Thus $q(x)$ (or another absolutely subhomogeneous approximation) could be used in computations where it is desired to guarantee a unique fixed point.

This approach can be more quantitatively accurate in the reduced dimension power control model focused on in [9]. Here, it is the signal power to total received power ratio (SRPR) that is tracked, rather than the SINR. If r_k is the expected total received power at station k, then the target power for mobile $m \in \mathcal{M}_k$ is $\gamma r_k / g_{km}$, and the expected transmit power is $\phi(\gamma r_k / g_{km})$. Then the total received power in turn can be expressed

$$r_k = \sum_j \left[\sum_{n \in \mathcal{M}_j} g_{kn} \phi\left(\frac{\gamma r_j}{g_{jn}}\right) \right] + \eta_k = \sum_j A_{kj}(r_j) + \eta_k,$$

resulting in a fixed point equation in the K-dimensional vector $\mathbf{r}$. Each *interaction function* A_{kj} is the result of adding together multiple scaled copies of the mobile behavior function ϕ. Due to averaging effects, A_{kj} tends to be much smoother than the constituent pieces, and in realistic scenarios such a function is approximated reasonably well by its absolutely subhomogeneous least upper bound. Figure 2 depicts a representative interaction function A_{kj} and its least absolutely subhomogeneous upper bound, taken from a simulation with thirty base stations and 400 active mobiles, with log-exponential fading with parameter $\lambda = 10$. The upper bound is a reasonable approximation to the function, particularly in the lower ranges of r_j where significant numbers of mobiles are not in outage. Replacing each interaction function by an absolutely subhomogeneous approximation results in a model that is guaranteed to converge to a unique fixed point. In [9], there are a large number of "mobiles", used to represent expected mobile density at points on a geographic grid, and the functions A_{kj} are correspondingly smoother and better approximated by absolutely subhomogeneous functions.

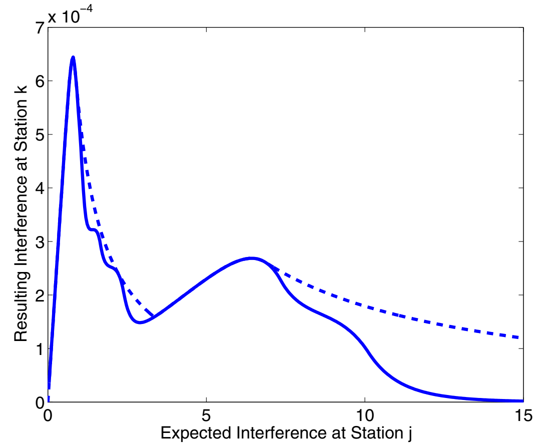

Fig. 2. A representative interaction function A_{kj} (solid curve) and its least absolutely subhomogeneous upper bound (dashed curve).

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