Differential Unitary Space-Time Modulation

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Abstract

We present a framework for differential modulation with multiple antennas across a continuously fading channel, where neither the transmitter nor the receiver knows the fading coefficients. The framework can be seen as a natural extension of standard differential phase shift keying (DPSK) commonly used in single-antenna unknown-channel systems. We show how our differential framework links the unknown-channel system with a known-channel system, and we develop performance design criteria. As a special case, we introduce a class of diagonal signals where only one antenna is active at any time, and demonstrate how these signals may be used to achieve full transmitter diversity and low probability of error.

Index Terms—Multi-element antenna arrays, wireless communications, fading channels, transmitter diversity, receiver diversity

1 Introduction

Recent advances in communicating across multiple-antenna wireless communication links show that these links can support very high data rates with low error probabilities, especially when the wireless channel response is known at the receiver [1, 2]. However, the assumption that the channel is known is questionable in a rapidly changing mobile environment, or when multiple transmitter antennas are employed. In [3], a new class of signals called *unitary space-time* signals is proposed that is well-tailored for Rayleigh flat-fading channels where neither the transmitter nor the receiver knows the fading coefficients. In [4], a systematic approach to designing unitary space-time signals is presented. The unitary space-time signals are suited particularly well to piecewise-constant fading models. In this note, we show how to modify these signals to work when the fading changes continuously. The modified signals, which we denote *differential unitary space-time modulation*, are easily implemented and achieve full-antenna diversity.

Differential phase-shift keying (DPSK) has long been used in single-antenna unknown-channel links when the channel has a phase response that is approximately constant from one time sample to the next. Differential modulation encodes the transmitted information into phase differences from symbol to symbol. The receiver decodes the information in the current symbol by comparing its phase to the phase of the previous symbol. DPSK is widely used because many continuously fading channels change little between successive time samples. In fact, many continuously fading channels are approximately constant for a time interval T often much larger than two samples.

Suppose that we transmit signals in blocks of T time samples. We think of standard DPSK as employing blocks of $T = 2$ time samples, since information is essentially transmitted by first providing a reference symbol and then a differentially phase-shifted symbol. Of course, after the starting symbol, each symbol acts as a reference for the next symbol, so we really have signals that occupy two symbols but overlap by one symbol. We wish to employ such an overlapping differential scheme with $M > 1$ transmitter antennas.

As our starting point, we use constellations of $T \times M$ unitary space-time signals proposed in [3] for piecewise-constant fading. The mth column of any signal contains the signal transmitted on antenna m as a function of time. Intuitive and theoretical arguments in [5] and [3] show that unitary space-time signals are not only simple to demodulate, but also attain capacity when used in conjunction with coding in a multiple-antenna Rayleigh fading channel when either $T \gg M$ or the signal-to-noise ratio is reasonably large and $T > M$.

As an extension of single-antenna DPSK, we show that there is a simple and general framework to differentially overlap the multiple-antenna unitary space-time signals that allows them to be used for continuous fading. For M transmitter antennas, we assume that $T = 2M$ and design the $T \times M$ matrix signals so that they may be overlapped in time by M symbols. For example, if the fading is constant in blocks of, say, ten symbols (often a reasonable assumption), this allows us to use differential modulation for at least five transmitter antennas.

We also show how our differential framework allows us to intimately connect signal design for unknown channels to design for channels that are known at the receiver [6, 7]. Using a few simple assumptions, we are led naturally to constellations of matrices that form groups, and eventually to constellations of so-called *diagonal* signals, where at any given time only one antenna is active. The diagonal signals fully utilize the transmitter antenna diversity and can be optimized to achieve low error probability across a Rayleigh flat-fading channel. Several examples and performance simulations are given.

2 Multiple antennas in unknown Rayleigh flat fading

In this section we present the channel model and summarize some known results for a multiple-antenna communication link in Rayleigh flat fading. We first need to set some notation.

2.1 Notation

I_M is a $M \times M$ identity matrix, $\mathcal{CN}(0,1)$ is the complex-normal zero-mean unit-variance distribution where the real and imaginary components of each random variable are independent and each have variance $1/2$, and † denotes complex conjugate transpose of a vector or matrix. The Frobenius norm of a $T \times M$ matrix $A = \{a_{tm}\}$ is given by

$$\|A\|^2 = \mathrm{tr}\,(A^\dagger A) = \mathrm{tr}\,(AA^\dagger) = \sum_{t=1}^{T}\sum_{m=1}^{M} |a_{tm}|^2 = \sum_{m=1}^{\min(M,T)} \sigma_m(A)^2, \tag{1}$$

where $\sigma_m(A)$ is the mth singular value of A.

2.2 Rayleigh flat-fading channel model

Consider a communication link comprising M transmitter antennas and N receiver antennas that operates in a Rayleigh flat-fading environment. Each receiver antenna responds to each transmitter antenna through a statistically independent fading coefficient. The received signals are corrupted by additive noise that is statistically independent among the N receiver antennas and the symbol periods. We use complex baseband notation: at time t we transmit the complex symbols s_{tm} on antennas $m = 1, \ldots, M$, and we receive x_{tn}

on receiver antennas $n = 1, \ldots, N$. The action of the channel is modeled by

$$x_{tn} = \sqrt{\rho} \sum_{m=1}^{M} h_{tmn} s_{tm} + w_{tn}, \qquad t = 0, 1, \ldots, \quad n = 1 \ldots N. \tag{2}$$

Here h_{tmn} is the complex-valued fading coefficient between the mth transmitter antenna and the nth receiver antenna at time t. The fading coefficients are assumed to be independent with respect to m and n (but not t), and are $\mathcal{CN}(0,1)$ distributed (Rayleigh amplitude, uniform phase). The additive noise at time t and receiver antenna n is denoted w_{tn}, and is independent, with respect to both t and n, identically distributed $\mathcal{CN}(0,1)$. The realizations of h_{tmn}, $m = 1, \ldots, M$, $n = 1, \ldots, N$ are known neither to the transmitter nor the receiver. The transmitted symbols are normalized to obey

$$\mathbf{E} \sum_{m=1}^{M} |s_{tm}|^2 = 1, \tag{3}$$

where $\mathbf{E}$ denotes expectation. Equations (2) and (3) ensure that ρ is the expected signal-to-noise ratio (SNR) at each receiver antenna, independently of the number of transmitter antennas M. Equivalently, the total transmitted power does not depend on M.

We assume that the fading coefficients change continuously according to a model such as Jakes' [8]. While the exact model for the continuous fading is unimportant, we require the fading coefficients to be approximately constant for overlapping blocks of $T \geqslant 2$ symbol periods. We have some freedom to choose T, but it generally can be no larger than the approximate coherence time (in symbols) of the fading process.

In one block of T successive symbols the time index of the fading coefficients can be dropped, and sent and received signals can be combined into T-vectors. Equation (2) can then be written compactly as

$$X = \sqrt{\rho} S H + W \tag{4}$$

where X is the $T \times N$ complex matrix of received signals x_{tn}, S is the $T \times M$ matrix of transmitted signals s_{tm}, H is the $M \times N$ matrix of Rayleigh fading coefficients h_{mn} (assumed time-invariant within the block), and W is the $T \times N$ matrix of additive receiver noise w_{tn}. In this notation, the M columns of S represent the signals sent on the M transmitter antennas as functions of time.

2.3 Unitary Space Time Modulation

We now consider how to choose a constellation of L signals $S_0, \ldots, S_{L-1}$, each a $T \times M$ matrix, to transmit data across this multi-antenna wireless channel. We use unitary space-time signals $S_0 = \sqrt{T/M}\Phi_0, \ldots, S_{L-1} =$

$\sqrt{T/M}\Phi_{L-1}$, where the $T \times M$ matrices Φ_ℓ obey $\Phi_0^\dagger \Phi_0 = \ldots = \Phi_{L-1}^\dagger \Phi_{L-1} = I_M$. The normalization $\sqrt{T/M}$ ensures that the matrix-signals satisfy the energy constraint (3).

For a piecewise-constant fading channel, it is argued in [5] and [3] that the capacity-achieving distribution for reasonably large T or ρ is $S = \sqrt{T/M}\Phi$, where $\Phi^\dagger \Phi = I_M$ and Φ is isotropically distributed. Because $\Phi_\ell^\dagger \Phi_\ell = I_M$, we are implicitly assuming that $M \leqslant T$; as shown in [5], this assumption is not restrictive because there is no gain in capacity by making $M > T$.

It is also shown in [3] that the maximum likelihood demodulator for a constellation of unitary space-time signals is a matrix noncoherent correlation receiver

$$\Phi_{\rm ml} = \arg \max_{\Phi_\ell = \Phi_0, \ldots, \Phi_{L-1}} \left\| X^\dagger \Phi_\ell \right\|, \tag{5}$$

and that the two-signal probability of mistaking S_ℓ for $S_{\ell'}$ or vice-versa is (see Appendix B of [3])

$$P_e = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega^2 + 1/4} \prod_{m=1}^{M} \left[1 + \frac{(\rho T/M)^2 (1 - d_{\ell\ell' m}^2)(\omega^2 + 1/4)}{1 + \rho T/M} \right]^{-N} \tag{6}$$

where $1 \geqslant d_{\ell\ell' 1} \geqslant \ldots \geqslant d_{\ell\ell' M} \geqslant 0$ are the singular values of the $M \times M$ correlation matrix $\Phi_\ell^\dagger \Phi_{\ell'}$ ($d_{\ell\ell' m} = \sigma_m(\Phi_\ell^\dagger \Phi_{\ell'})$). The pairwise probability of error P_e decreases as any $d_{\ell\ell' m}$ decreases, and has Chernoff upper bound

$$P_e \leqslant \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{(\rho T/M)^2 (1 - d_{\ell\ell' m}^2)}{4(1 + \rho T/M)} \right]^{-N}. \tag{7}$$

For the noncoherent receiver, the pairwise probability of error is lowest when the two matrix-valued signals are as orthogonal as possible, and is highest when the signals are as parallel as possible. Hence, the probability of error is lowest when $d_{\ell\ell' 1} = \ldots = d_{\ell\ell' M} = 0$ and highest when $d_{\ell\ell' 1} = \ldots = d_{\ell\ell' M} = 1$. We obtain $d_{\ell\ell' 1} = \ldots = d_{\ell\ell' M} = 0$ when the columns of S_ℓ are all orthogonal to all the columns of $S_{\ell'}$. The ideal constellation $S_0, \ldots, S_{L-1}$ therefore has all the columns of S_ℓ orthogonal to all the columns of $S_{\ell'}$ for $\ell' \neq \ell = 0, \ldots, L-1$. However, because the columns of each S_ℓ are within themselves orthogonal to one another, all the pairwise $d_{\ell\ell' 1}, \ldots, d_{\ell\ell' M}$ cannot all be made zero if $L > T/M$.

In general, we strive to build constellations which make the pairwise probability of error P_e between any two signals S_ℓ and $S_{\ell'}$ as small as possible. Optimizing the exact probability of error (6) or its Chernoff upper bound (7) is awkward because they depend on the SNR ρ. Rather than picking a particular ρ, we design constellations that work well for all sufficiently large ρ, where the Chernoff upper bound on S_ℓ and $S_{\ell'}$ depends dominantly on the product

$$\prod_{m=1}^{M} (1 - d_{\ell\ell' m}^2).$$

As shown in [9, Sec. 12.4.3] one can think of $d_{\ell\ell' m}$ as the cosine of the principal angle $\theta_{\ell\ell' m}$ between the subspaces spanned by the columns of Φ_ℓ and $\Phi_{\ell'}$. The above expression can therefore be interpreted as the product of the squares of the sines of the m principal angles. To obtain a quantity that can be compared for different M, we define $\zeta_{\ell\ell'}$ as the geometric mean of the sines of the m principal angles[1]

$$\zeta_{\ell\ell'} = \prod_{m=1}^{M} \sin(\theta_{\ell\ell' m})^{\frac{1}{M}} = \left[\prod_{m=1}^{M} (1 - d_{\ell\ell' m}^2)\right]^{\frac{1}{2M}}. \tag{8}$$

Because $0 \leqslant d_{\ell\ell' m} \leqslant 1$, we have $0 \leqslant \zeta_{\ell\ell'} \leqslant 1$, and, in particular, if $\zeta_{\ell\ell'}$ is small the pairwise probability of error is large, and if $\zeta_{\ell\ell'}$ is large the probability of error is small. Define now the *diversity product* ζ as

$$\zeta = \min_{0 \leqslant \ell < \ell' \leqslant L-1} \zeta_{\ell\ell'}, \tag{9}$$

In this paper, we choose constellations that maximize the diversity product ζ. In particular, any constellation with non zero diversity product is said to have full transmitter diversity.

In [4], constellations are chosen that minimize δ where

$$\delta = \max_{0 \leqslant \ell < \ell' \leqslant L-1} \frac{1}{\sqrt{M}} \left\| \Phi_\ell^\dagger \Phi_{\ell'} \right\| = \sqrt{\frac{1}{M} \sum_{m=1}^{M} d_{\ell\ell' m}^2}. \tag{10}$$

We have $0 \leqslant \delta \leqslant 1$ and, by (7), small δ implies small probability of error. For small $d_{\ell\ell' m}$,

$$\zeta_{\ell\ell'}^2 = 1 - \frac{1}{M} \sum_{m=1}^{M} d_{\ell\ell' m}^2 + O(d_{\ell\ell' m}^4) = 1 - \frac{1}{M} \left\| \Phi_{\ell'}^\dagger \Phi_\ell \right\|^2 + O(d_{\ell\ell' m}^4).$$

Thus $\zeta^2 \approx 1 - \delta^2$ and small δ in general implies large ζ. We find that maximizing the diversity product ζ to be more useful than minimizing δ because minimizing δ does not guarantee full diversity. Note that maximizing ζ is fundamentally different from maximizing Euclidean distance; two signals that have large Euclidean distance can have small diversity product ζ. In fact, diametrically opposite signals $S_{\ell'} = -S_\ell$ have $\zeta_{\ell\ell'} = 0$.

In constructing a constellation of signals, we note that the probability of error of the entire constellation (not just the pairwise error) is invariant to two types of signal transformations: 1) left-multiplication by a common $T \times T$ unitary matrix, $\Phi_\ell \rightarrow \Psi \Phi_\ell,\ \ell = 0, \cdots, L-1$; 2) right-multiplication by individual $M \times M$ unitary matrices, $\Phi_\ell \rightarrow \Phi_\ell \Upsilon_\ell,\ \ell = 0, \cdots, L-1$. We can intuitively understand these transformations by

[1] The original March 1999 version of this paper defined $\zeta_{\ell\ell'}$ as the square of its current definition. However, the current definition is more amenable to interpretation, especially when the channel is known.

viewing the left-multiplication as a simultaneous permutation in time of all the signals, and the individual right-multiplications as permutations of the antennas. The ordering of the antennas is immaterial because all of the antennas are statistically equivalent; see [3, Section 6.2].

3 Standard Single-Antenna Differential Modulation

In this section we give a short review of standard single-antenna differential phase-shift keying (DPSK) [10, 11]. While we do not offer any new material here, we present DPSK in an unusual framework that ultimately makes our transition to multiple antennas easier.

DPSK is traditionally used when the channel changes the phase of the symbol in an unknown, but consistent or slowly varying way. The data information is sent in the difference of the phases of two consecutive symbols. For a data rate of R bits per channel use, we need $L = 2^R$ symbols; the most common techniques use symbols that are Lth roots of unity

$$\varphi_\ell = e^{2\pi i\ell/L}, \qquad \ell = 0, \ldots, L-1. \tag{11}$$

Suppose we want to send a data sequence of integers $z_1, z_2, \ldots$ with $z_t \in \{0, \ldots, L-1\}$. The transmitter sends the symbol stream $s_1, s_2, \ldots$

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \end{bmatrix} \quad \text{where} \quad s_t = \varphi_{z_t} s_{t-1}, \quad t = 1, 2, \ldots \quad (s_0 = 1).$$

(In the matrices and sequences shown in this paper we always represent the time axis vertically.) The initial symbol $s_0 = 1$ does not carry any information and can be thought of as a training symbol. The received data $x_1, x_2, \ldots$, are processed by computing the differential phases

$$\theta_t = \arg x_{t-1}^* x_t, \qquad t = 1, 2, \ldots$$

which are quantized to form an estimate of the integer sequence

$$\hat{z}_t = \lfloor \theta_t L/(2\pi) + 1/2 \rfloor \mod L, \qquad t = 1, 2, \ldots. \tag{12}$$

The received and transmitted symbols are related by the equation

$$x_t = \sqrt{\rho} h_t s_t + w_t, \qquad t = 0, 1, \ldots .$$

This is the single-antenna version of the model (2), where h_t is the complex valued fading coefficient which is either constant or varies slowly with t. There are two sources of possible errors ($\hat{z}_t \neq z_t$): the additive noise, and time-variations in the phase of the fading coefficient. The demodulation rule (12) does not depend on earlier demodulation decisions, but only on the received symbols x_{t-1} and x_t; demodulation errors therefore do not propagate.

There is a slightly different way to look at DPSK modulation and demodulation that fits into our multiple-antenna model (4) with $M = N = 1$. Since DPSK demodulation requires two successive symbols, we consider the transmitted signals as occupying overlapping intervals of length $T = 2$ and consider modulation and demodulation using the maximum likelihood receiver given in Section 2. One can view the signal constellation as containing two-dimensional vectors of the type

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_{\ell(1)} \\ \varphi_{\ell(2)} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{2\pi i \ell(1)/L} \\ e^{2\pi i \ell(2)/L} \end{bmatrix} \qquad \ell = 0, \ldots, L-1. \tag{13}$$

(Recall in Section 2.3 that the transmitted signal S_ℓ is Φ_ℓ multiplied by $\sqrt{T/M} = \sqrt{2}$.) The signals form an equivalence class invariant under phase shifts; i.e., Φ_ℓ and $e^{i\theta}\Phi_\ell$ are indistinguishable to the receiver for all θ. A phase shift can be seen as a right-multiplication by the 1×1 unitary matrix $e^{i\theta}$, which does not change the constellation (see the last paragraph of Section 2.3). Therefore, one has a canonical representation

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \varphi_\ell \end{bmatrix}, \tag{14}$$

where φ_ℓ is given in (11).

Effectively, to generate a DPSK signal, the transmitter preprocesses the signal vector Φ_ℓ by rotating Φ_ℓ until its first symbol equals the symbol previously sent. The transmitter then sends only the (normalized) second symbol of the rotated Φ_ℓ, thus representing Φ_ℓ by only one sent symbol. The receiver is aware of this preprocessing and demodulates the current received symbol by combining it with the previous received symbol to form a two-symbol vector again. More formally, the transmitter computes the cumulative sum

$$y_t = (y_{t-1} + z_t) \bmod L, \qquad t = 1, 2, \ldots \qquad \text{with} \quad y_0 = 0.$$

The very first signal sent is $[1 \;\; \varphi_{z_1}]^T = [\varphi_{y_0} \;\; \varphi_{y_1}]^T$, and we now wish to send the signal $[1 \;\; \varphi_{z_2}]^T$. Instead of sending both components of this signal, we rotate this signal to another element of its equivalence class, obtained by multiplying by the scalar $\varphi_{y_1} = \varphi_{z_1}$, namely $[\varphi_{z_1} \;\; \varphi_{z_1}\varphi_{z_2}]^T = [\varphi_{y_1} \;\; \varphi_{y_2}]^T$. The transmitter then sends only φ_{y_2}. Figure 1 schematically displays differential modulation.

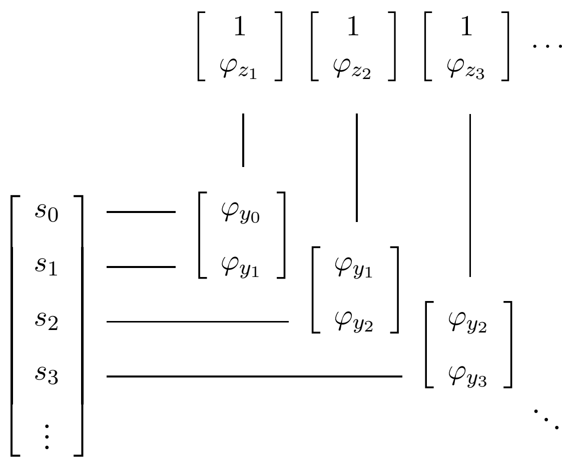

Figure 1: Schematic representation of differential phase modulation. Along the top, from left to right, are the symbols $[1 \;\; \varphi_{z_t}]^T$ one wants to send. These are multiplied by $\varphi_{y_{t-1}}$ so that they can overlap, as shown diagonally downward. The overlapped signals are then transmitted ($s_t = \varphi_{y_\tau}$) on the channel.

The receiver now groups received symbols in (overlapping) vectors of length two

$$X = \begin{bmatrix} x_{t-1} \\ x_t \end{bmatrix},$$

and computes the noncoherent maximum likelihood demodulation according to (5)

$$(\hat{z}_t)_{\mathrm{ml}} = \arg \max_{\ell=0,\ldots,L-1} |\Phi_\ell^* X|$$

This corresponds to DPSK demodulation given in (12) because

$$\arg \max_\ell |\Phi_\ell^* X| = \arg \max_\ell |x_{t-1} + \varphi_\ell^* x_t| = \arg \max_\ell \left|\varphi_\ell + x_{t-1}^* x_t\right|.$$

The term $x_{t-1}^* x_t$ computes the phase difference between successive received symbols, and maximizing $\left|\varphi_\ell + x_{t-1}^* x_t\right|$ finds the φ_ℓ whose phase matches this difference most closely.

4 Multiple-Antenna Differential Modulation

The previous section shows that standard differential modulation effectively uses a block of length two that overlaps by one symbol, where one symbol acts as a reference for the next. Information is delivered in the phase difference between symbols. When we have M transmitter antennas we need a block of $M \times M$ space-time symbols to act as a reference for the next block. Hence, we consider signals of size $2M \times M$ that we overlap by M samples, and effectively deliver information in the matrix quotient of the two blocks.

4.1 Signal requirements for differential modulation

With multiple antennas, we accomplish differential modulation by overlapping the $T \times M$ matrix signals Φ_ℓ by $T/2$ symbols. We therefore choose $T = 2M$. We now explore the structure that $\Phi_0, \ldots, \Phi_{L-1}$ must have to permit overlapping. Using a notation similar to (13), we let each signal Φ_ℓ have the form

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} V_{\ell 1} \\ V_{\ell 2} \end{bmatrix}, \qquad \ell = 0, \ldots, L-1,$$

where $V_{\ell 1}$ and $V_{\ell 2}$ are, for the moment, arbitrary $M \times M$ complex matrices[2]. Because $\Phi_\ell^\dagger \Phi_\ell = I_M$, it follows that

$$V_{\ell 1}^\dagger V_{\ell 1} + V_{\ell 2}^\dagger V_{\ell 2} = 2 I_M. \tag{15}$$

In Section 2.3 it is shown that Φ_ℓ and $\Phi_\ell \Upsilon_\ell$, $\ell = 0, \ldots, L-1$ are indistinguishable for arbitrary unitary $M \times M$ matrices Υ_ℓ. To help overlap the signals in a fashion similar to Section 3, we therefore have the freedom to "preprocess" each signal Φ_ℓ to be sent by right-multiplying by a unitary matrix so that its first $M \times M$ matrix block equals the second matrix block of the previously (also possibly preprocessed) sent symbol, say $\Phi_{\ell'}$ (see the rules for signal manipulation at the end of Section 2.3). After Φ_ℓ is preprocessed, because its first block equals the second block of the signal already sent, we then need to send only its (normalized) second block. For this overlapping to succeed, we therefore require that a unitary transformation exist between the first block of Φ_ℓ and the second block of $\Phi_{\ell'}$; i.e., for any ℓ and ℓ', the equation

$$V_{\ell' 2} \Upsilon_{\ell' \ell} = V_{\ell 1}, \tag{16}$$

should have a solution for some unitary $\Upsilon_{\ell' \ell}$.

The most general set of $V_{\ell 1}$ and $V_{\ell 2}$ matrices that satisfy (15) and (16) is described by Peter Oswald in

[2] The $\sqrt{2}$ normalization may seem odd, but it ultimately allows us to choose the V matrices to be unitary while $\Phi_\ell^\dagger \Phi_\ell = I_M$.

[12], where he shows that the best diversity product ζ is, in general, obtained by choosing $V_{\ell 1}$ and $V_{\ell 2}$ to be unitary. We therefore restrict ourselves to this case. If $V_{\ell 1}$ and $V_{\ell 2}$ are unitary for all ℓ then (15) holds trivially, and (16) has the unitary solution $\Upsilon_{\ell' \ell} = V_{\ell' 2}^{\dagger} V_{\ell 1}$. With this choice, because Φ_ℓ and $\Phi_\ell V_{\ell 1}^{\dagger}$ are indistinguishable at the receiver, we have a canonical representation

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} I_M \\ V_\ell \end{bmatrix}, \tag{17}$$

where $V_\ell = V_{\ell 2} V_{\ell 1}^{\dagger}$ is unitary. Without loss of generality, we can thus assume the following:

Assumption 1 *The signals Φ_ℓ are of the form (17) where V_ℓ is a unitary matrix.*

Observe the formal similarity with (14).

4.2 Differential transmission

In standard single-antenna DPSK, Section 3 shows that the equivalent Φ_ℓ signals can be thought of as two-dimensional vectors whose first components are 1, and whose second components are used to form the transmitted signal. Similarly, in equation (17), the signals $\Phi_0, \ldots, \Phi_{L-1}$, are $T \times M$ matrices whose first halves are I_M, and whose second halves are used to form the transmission matrix in our M-antenna differential modulation scheme. Therefore the channel is used in blocks of $M = T/2$ symbols. Let us use τ to index blocks of M consecutive symbols; the running time index of channel uses is then $t = \tau M + m - 1$ with $m = 1, \ldots, M$. A transmission data rate of R bits per channel use requires a constellation with $L = 2^{RM}$ signals; thus L distinct V_ℓ matrices are needed. We again have an integer data sequence $z_1, z_2, \ldots$ with $z_\tau \in \{0, \ldots, L-1\}$.

Figure 2 schematically displays multiple-antenna differential modulation. Here the M columns of each $\mathcal{S}_\tau$ (which are $M \times M$ matrices) represent what is transmitted on the M antennas as functions of time for M symbols. The first transmission is $\sqrt{T/M}\Phi_{z_1} = \sqrt{2}\Phi_{z_1}$; that is, an identity matrix $\mathcal{S}_0 = I_M$ is sent, followed by $\mathcal{S}_1 = V_{z_1}$. Next, we wish to send $\sqrt{2}\Phi_{z_2}$. To make the identity block of Φ_{z_2} overlap with the last sent block V_{z_1}, we postmultiply Φ_{z_2} by V_{z_1}. The second block of Φ_{z_2} then becomes $V_{z_2} V_{z_1}$ and, hence $\mathcal{S}_2 = V_{z_2} V_{z_1} = V_{z_2} \mathcal{S}_1$. In general, the differential transmission scheme sends the matrices

$$\mathcal{S}_\tau = V_{z_\tau} \, \mathcal{S}_{\tau-1} \qquad \tau = 1, 2, \ldots. \tag{18}$$

This is the *fundamental differential transmission equation*. Clearly, all the transmitted matrices $\mathcal{S}_\tau$ will be unitary.

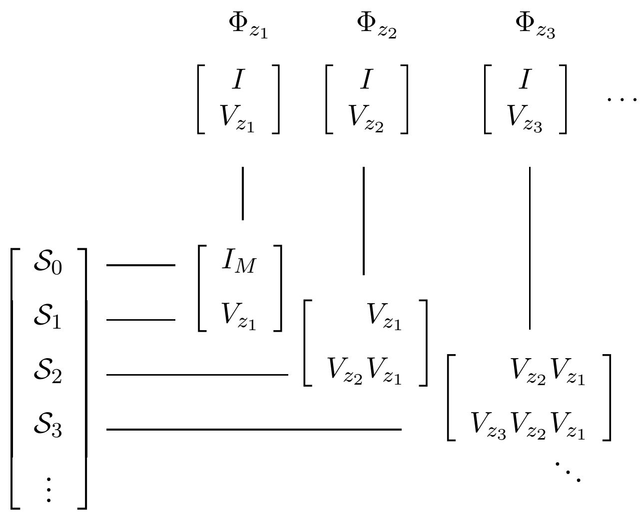

Figure 2: Schematic representation of M-antenna differential modulation. Along the top, from left to right, are the symbols Φ_{z_τ} one wants to send. These are right-multiplied by the previously transmitted block so that they can overlap, as shown diagonally downward. The overlapped signals, which obey $\mathcal{S}_\tau = V_{z_\tau}\mathcal{S}_{\tau-1}$, are then transmitted on the channel. Compare Figure 1, which shows the overlapping scheme for standard single-antenna differential phase modulation.

4.3 Differential reception

With N receiver antennas, the demodulator receives a stream

$$\begin{bmatrix} \mathcal{X}_0 \\ \mathcal{X}_1 \\ \mathcal{X}_2 \\ \vdots \end{bmatrix},$$

where $\mathcal{X}_\tau$ is an $M \times N$ matrix. Demodulation requires looking at two successive matrices to form a matrix with $T = 2M$ rows,

$$X = \begin{bmatrix} \mathcal{X}_{\tau-1} \\ \mathcal{X}_\tau \end{bmatrix}.$$

We assume that the fading coefficients are constant across the $T = 2M$ time samples represented in the rows of X. Then the relationship with the sent stream is

$$\mathcal{X}_{\tau-1} = \sqrt{\rho}\mathcal{S}_{\tau-1}H + \mathcal{W}_{\tau-1} \qquad (19)$$

$$\mathcal{X}_\tau = \sqrt{\rho}\mathcal{S}_\tau H + \mathcal{W}_\tau, \qquad (20)$$

where $\mathcal{W}_\tau$ is a $M \times N$ matrix of additive independent $\mathcal{CN}(0,1)$ receiver noise. The maximum likelihood demodulator (5) is

$$(\hat{z}_\tau)_{\rm ml} = \arg \max_{\ell=0,\ldots,L-1} \left\| \Phi_\ell^\dagger X \right\| = \arg \max_{\ell=0,\ldots,L-1} \left\| \mathcal{X}_{\tau-1} + V_\ell^\dagger \mathcal{X}_\tau \right\|, \tag{21}$$

where the norm is as defined in (1).

Substituting the fundamental differential transmitter equation $\mathcal{S}_\tau = V_{z_\tau} \mathcal{S}_{\tau-1}$ into (20) and applying (19) yield

$$\mathcal{X}_\tau = V_{z_\tau} \mathcal{X}_{\tau-1} + \mathcal{W}_\tau - V_{z_\tau} \mathcal{W}_{\tau-1}.$$

Because the noise matrices are independent and statistically invariant to multiplication by unitary matrices, we may write this as

$$\mathcal{X}_\tau = V_{z_\tau} \mathcal{X}_{\tau-1} + \sqrt{2} \mathcal{W}'_\tau, \tag{22}$$

where $\mathcal{W}'_\tau$ is a $M \times N$ matrix of additive independent $\mathcal{CN}(0,1)$ noise. This is the *fundamental differential receiver equation*.

Remarkably, the matrix of fading coefficients H does not appear in the fundamental differential receiver equation (22). In fact, formally, this equation shows that the signal V_{z_τ} appears to be transmitted through a channel with fading response $\mathcal{X}_{\tau-1}$, which is *known* to the receiver, and corrupted by noise with twice the variance. This corresponds to the well known result that standard single-antenna differential modulation suffers from approximately a 3 dB performance loss in effective SNR when the channel is unknown versus when it is known.

5 Connection between Unknown and Known Channel

Equation (22) demonstrates that our multiple-antenna differential setting appears to turn the original unknown-channel communication problem into a known channel problem. In this section we explore this connection further. We first review some facts about the known channel.

5.1 Known channel

We consider signals that are $M \times M$ matrices. The action of the channel is

$$\mathcal{X}_\tau = \sqrt{\rho} \mathcal{S}_\tau H + \mathcal{W}_\tau, \tag{23}$$

where H is known to the receiver. We assume that the constellation consists of $L = 2^{RM}$ signals Ψ_ℓ that are unitary. The transmission matrix is then

$$\mathcal{S}_\tau = \Psi_{z_\tau}.$$

Because H is known at the receiver, the maximum likelihood demodulator is the coherent receiver

$$(\hat{z}_\tau)_{\rm ml} = \arg \min_{\ell=0,\dots,L-1} \|\mathcal{X}_\tau - \sqrt{\rho}\Psi_\ell H\| . \tag{24}$$

and has pairwise probability of error Chernoff upper bound given by [6], [3]

$$P_e \leqslant \frac{1}{2} \prod_{m=1}^{M} \left[1 + \frac{\rho T}{4M} \sigma_m^2(\Psi_{\ell'} - \Psi_\ell)\right]^{-N} . \tag{25}$$

Hence, good constellations $\Psi_1, \dots, \Psi_L$ have singular values

$$\sigma_m(\Psi_{\ell'} - \Psi_\ell), \qquad m = 1, \dots, M$$

that are as large as possible for $\ell' \neq \ell$. For large SNR, the probability of error depends dominantly on the product

$$\prod_{m=1}^{M} \sigma_m(\Psi_\ell - \Psi_{\ell'}) = |\det(\Psi_\ell - \Psi_{\ell'})| . \tag{26}$$

In particular, a larger product equates to a smaller error probability.

5.2 Connection between signal designs

Recall in Section 2.3 that the unknown-channel signals Φ_ℓ are $T \times M$ matrices obeying $\Phi_\ell^\dagger \Phi_\ell = I$, and that a good constellation $\Phi_1, \dots, \Phi_L$ has singular values

$$d_{\ell\ell' m} = \sigma_m(\Phi_{\ell'}^\dagger \Phi_\ell), \qquad m = 1, \dots, M$$

that are as small as possible for $\ell' \neq \ell$. If we view the identity block of the differential unitary space-time signal construction of Φ_ℓ as training to learn the matrix channel H, we may build Φ_ℓ as

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} I_M \\ \Psi_\ell \end{bmatrix},$$

where Ψ_ℓ are unitary matrices taken from a constellation of known-channel signals. Then

$$\Phi_{\ell'}^{\dagger}\Phi_\ell = (I_M + \Psi_{\ell'}^{\dagger}\Psi_\ell)/2,$$

which implies that

$$\sigma_m^2(\Phi_{\ell'}^{\dagger}\Phi_\ell) = \frac{1}{4}\sigma_m^2(I_M + \Psi_{\ell'}^{\dagger}\Psi_\ell) = \frac{1}{4}\lambda_m(2I_M + \Psi_{\ell'}^{\dagger}\Psi_\ell + \Psi_\ell^{\dagger}\Psi_{\ell'}), \tag{27}$$

where $\lambda_m(\cdot)$ is the mth eigenvalue of the matrix $(\cdot)$. Hence

$$1 - \sigma_m^2(\Phi_{\ell'}^{\dagger}\Phi_\ell) = \frac{1}{4}\lambda_m(2I_M - \Psi_{\ell'}^{\dagger}\Psi_\ell - \Psi_\ell^{\dagger}\Psi_{\ell'}) = \frac{1}{4}\sigma_m^2(I_M - \Psi_{\ell'}^{\dagger}\Psi_\ell) = \frac{1}{4}\sigma_m^2(\Psi_{\ell'} - \Psi_\ell). \tag{28}$$

Equation (28) says that minimizing the singular values of the correlations of the unknown-channel signals is equivalent to maximizing the singular values of the differences of the known-channel signals. We can now write $\zeta_{\ell\ell'}$ in (8) as

$$\zeta_{\ell\ell'} = \prod_{m=1}^{M}(1 - \sigma_m^2(\Phi_{\ell'}^{\dagger}\Phi_\ell))^{\frac{1}{2M}} = \frac{1}{2}\prod_{m=1}^{M}\sigma_m(\Psi_{\ell'} - \Psi_\ell)^{\frac{1}{M}} = \frac{1}{2}\left|\det\left(\Psi_{\ell'} - \Psi_\ell\right)\right|^{\frac{1}{M}}. \tag{29}$$

As argued in Section 2.3, large $\zeta_{\ell\ell'}$ equates to small pairwise error probability when ρ is large and the channel is unknown. On the other hand, equation (26) states that large $|\det(\Psi_\ell' - \Psi_\ell)|$ also equates to small pairwise error probability when the channel is known. Thus, a constellation of good known-channel matrix signals can be augmented with an identity matrix block to form a constellation of good unknown-channel matrix signals. Conversely, a constellation of good unknown-channel signals of the form (17) has V_ℓ matrices that form a constellation of good known-channel signals. Intuitively, the identity block can be viewed as training from which the channel is learned before the second block carrying data is sent. Differential modulation, of course, lets the training and data blocks overlap. The diversity product for differential modulation can now be written as

$$\zeta = \frac{1}{2}\min_{0\leqslant\ell<\ell'\leqslant L-1}\left|\det\left(V_\ell - V_{\ell'}\right)\right|^{\frac{1}{M}}. \tag{30}$$

By comparing the Chernoff bounds (7) and (25), and using (26) we see from the factor $1/2$ in (29) that the performance advantage for knowing versus not knowing the channel is approximately 3 dB in SNR.

5.3 Connection between demodulation strategies

The fundamental differential receiver equation (22) is

$$\mathcal{X}_\tau = V_{z_\tau}\mathcal{X}_{\tau-1} + \sqrt{2}\mathcal{W}'_\tau.$$

As we have remarked, $\mathcal{X}_{\tau-1}$ can be viewed as a known channel through which the signal matrix V_{z_τ} is sent. We may demodulate z_τ using (24) to obtain

$$\begin{aligned}
\hat{z}_\tau &= \arg \min_{\ell=0,\ldots L-1} \|\mathcal{X}_\tau - \Psi_\ell \mathcal{X}_{\tau-1}\| \\
&= \arg \min_{\ell=0,\ldots,L-1} \operatorname{tr}\left(\mathcal{X}_\tau^\dagger \mathcal{X}_\tau + \mathcal{X}_{\tau-1}^\dagger \mathcal{X}_{\tau-1} - \mathcal{X}_{\tau-1}^\dagger \Psi_\ell^\dagger \mathcal{X}_\tau - \mathcal{X}_\tau^\dagger \Psi_\ell \mathcal{X}_{\tau-1}\right) \\
&= \arg \max_{\ell=0,\ldots,L-1} \operatorname{tr}\left(\mathcal{X}_{\tau-1}^\dagger \Psi_\ell^\dagger \mathcal{X}_\tau + \mathcal{X}_\tau^\dagger \Psi_\ell \mathcal{X}_{\tau-1}\right)
\end{aligned}$$

This estimate is exactly the maximum likelihood demodulator for the unknown channel (21):

$$(\hat{z}_\tau)_{\mathrm{ml}} = \arg \max_{\ell=0,\ldots,L-1} \left\|\mathcal{X}_{\tau-1} + \Psi_\ell^\dagger \mathcal{X}_\tau\right\| = \arg \max_{\ell=0,\ldots,L-1} \operatorname{tr}\left(\mathcal{X}_{\tau-1}^\dagger \Psi_\ell^\dagger \mathcal{X}_\tau + \mathcal{X}_\tau^\dagger \Psi_\ell \mathcal{X}_{\tau-1}\right).$$

These connections imply that the differential scheme can use existing constellations and demodulation methods from the known channel such as, for example, the orthogonal designs of [7].

6 Group constellations

Let $\mathcal{V}$ be the set of L distinct unitary matrices

$$\mathcal{V} = \{V_0, \ldots, V_{L-1}\}.$$

We have not yet imposed any structure on the set $\mathcal{V}$. In this section, we assume that $\mathcal{V}$ forms a group. We show how this assumption simplifies the transmission scheme and the constellation design.

6.1 Group conditions

In order for a set $\mathcal{V}$ to form a group under matrix multiplication, we need to impose four conditions: internal composition, associativity, existence of an identity element, and existence of an inverse element for each element. We briefly discuss these conditions and show that imposing internal composition essentially imposes the remaining three.

Internal composition: In standard single-antenna scalar DPSK with $T = 2$ (reviewed in Section 3), the product of any two symbols, φ_ℓ and $\varphi_{\ell'}$, is another symbol. In a similar fashion, we impose an internal composition rule on $\mathcal{V}$. For any $\ell,\ \ell' \in \{0, \dots, L-1\}$, it is required that

$$V_\ell V_{\ell'} = V_{\ell''} \tag{31}$$

for some $\ell'' \in \{0, \dots, L-1\}$. We may define an equivalent (isomorphic) additive operation on the indices as

$$\ell'' = \ell \oplus \ell'.$$

Associativity: Follows immediately from the associativity of matrix multiplication.

Identity element: In Section 2.3 it is mentioned that every signal Φ_ℓ in the constellation may be premultiplied by the same fixed $T \times T$ unitary matrix without changing the error performance of the constellation. The first element of the constellation is

$$\Phi_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_M \\ V_0 \end{bmatrix}.$$

We now premultiply every member of the constellation with the unitary matrix

$$\begin{bmatrix} I_M & 0_M \\ 0_M & V_0^\dagger \end{bmatrix}.$$

This gives an equivalent constellation whose first element has two identity matrices. Thus without loss of generality we can always assume a constellation with $V_0 = I_M \in \mathcal{V}$.

Inverse element: We show that because we impose internal composition, any element, say V_1, automatically has an inverse in $\mathcal{V}$. Since $\mathcal{V}$ comprises unitary matrices, the matrix products $V_1V_0,\ V_1V_1, \dots, V_1V_{L-1}$ are all distinct, and are all again in $\mathcal{V}$; they consequently form a permutation of the elements of $\mathcal{V}$. In particular, there is an index ℓ such that $V_1V_\ell = V_0 = I$. Hence, $V_1^{-1} = V_\ell$.

Of the four requirements that a group must satisfy, we have shown that imposing internal composition automatically imposes the remaining three.

Assumption 2 *The set of unitary matrices $\mathcal{V}$ forms a group.*

Note that since $\mathcal{V}$ is a finite group of size L, its elements must all be Lth roots of unity: $V_\ell^L = I_M$ for $\ell = 0, \ldots, L-1$.

6.2 Advantages of group constellations

Differential modulation as in Section 4.2 can now be written more succinctly by letting

$$y_\tau = z_\tau \oplus y_{\tau-1}, \qquad \tau = 1, 2, \ldots, \quad y_0 = 0 \tag{32}$$

so that

$$V_{y_\tau} = V_{z_\tau} V_{y_{\tau-1}}.$$

The transmitted matrix is

$$\mathcal{S}_\tau = V_{y_\tau} = V_{z_\tau} \mathcal{S}_{\tau-1} \quad t = 1, 2, \ldots.$$

Thus, unlike the general case, when $\mathcal{V}$ is a group each transmitted matrix is an element of $\mathcal{V}$.

One advantage of a group constellation is that the transmitter never has to explicitly multiply matrices, but only needs to compute (32) using a lookup table. Another advantage is simplified design. Good constellations are often found by searching over large candidate sets. Computing ζ for a general candidate constellation requires checking $(L-1)L/2$ correlations of the form

$$\Phi_{\ell'}^\dagger \Phi_\ell = \frac{1}{2}\left(I_M + V_{\ell'}^\dagger V_\ell\right). \tag{33}$$

However, when $\mathcal{V}$ is a group it suffices to check only $L-1$ correlations; in particular, one may check the singular values of $\Phi_0^\dagger \Phi_\ell = (1/2)(I_M + V_\ell)$. Figure 3 schematically displays multiple-antenna differential modulation when the constellation forms a group.

6.3 Abelian group constellations

We now impose the requirement that the product of any two matrices of $\mathcal{V}$ commutes.

Assumption 3 *The group $\mathcal{V}$ is Abelian.*

Imposing commutativity has some appealing consequences. Since $V_0, \ldots, V_{L-1}$ are unitary, they are normal matrices, and can be written as $V_\ell = P_\ell \Lambda_\ell P_\ell^\dagger$, where the matrix of eigenvectors P_ℓ obeys $P_\ell^\dagger P_\ell = P_\ell P_\ell^\dagger = I$, and Λ_ℓ is a matrix of eigenvalues of V_ℓ [13]. But because $V_0, \ldots, V_{L-1}$ commute, they share a common set of eigenvectors, $P \overset{\text{def}}{=} P_0 = P_1 = \ldots = P_{L-1}$; see [13, p. 420]. Consequently, this constellation of matrices can be diagonalized into a new constellation comprising diagonal matrices of eigenvalues using

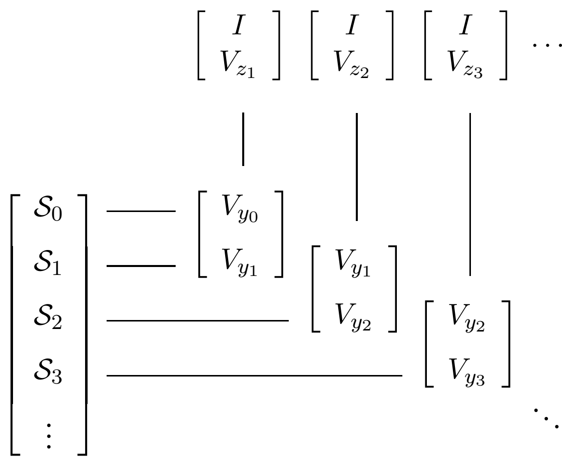

Figure 3: Schematic representation of M-antenna differential modulation when the constellation forms a group. Along the top, from left to right, are the symbols one wants to send. These are right-multiplied by the previously transmitted block $V_{y_{\tau-1}}$ so that they can overlap, as shown diagonally downward. Unlike in Figure 2, the transmitted signals are always members of the constellation, just as in standard scalar DPSK.

one fixed ℓ-independent similarity transform $V_\ell \to P^{-1}V_\ell P$. The similarity transform does not effect the error performance of the constellation because it is equivalent to postmultiplying every signal Φ_ℓ by the unitary $M \times M$ matrix P and premultiplying Φ_ℓ by the unitary $2M \times 2M$ matrix

$$\begin{bmatrix} P^{-1} & 0_M \\ 0_M & P^{-1} \end{bmatrix}.$$

Thus, assuming $\mathcal{V}$ is Abelian is equivalent to assuming that all of its elements are diagonal matrices. If all the V_ℓ are diagonal, then the signals Φ_ℓ consist of two diagonal blocks (the first of which is identity). This implies that at any given time only one antenna is active. We call these signals *diagonal*.

6.3.1 Cyclic construction

A simple way to build the commutative group $\mathcal{V}$ with L elements is to make it cyclic. Then V_ℓ is of the form

$$V_\ell = V_1^\ell, \qquad \ell = 0, \ldots, L-1$$

where the generator matrix V_1 is an Lth root of the unity. Addition on the indices

$$\ell'' = \ell \oplus \ell'$$

then becomes

$$\ell'' = \ell + \ell' \pmod{L}.$$

Hence, the transmitter does not even need a lookup table to compute the differential transmission scheme. The matrix V_1 is diagonal and can be written as

$$V_1 = \begin{bmatrix} e^{i\frac{2\pi}{L}u_1} & 0 & \cdots \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{i\frac{2\pi}{L}u_M} \end{bmatrix}, \qquad u_m \in \{0, \ldots, L-1\}, \quad m = 1, \ldots, M.$$

With this cyclic construction, the $2M \times M$ signals Φ_ℓ are given by

$$\Phi_\ell = \Theta^\ell \, \Phi_0, \tag{34}$$

where

$$\Theta = \begin{bmatrix} I_M & 0_M \\ 0_M & V_1 \end{bmatrix}, \quad \text{and} \quad \Phi_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_M \\ I_M \end{bmatrix}.$$

The ℓth signal in the constellation therefore has the form

$$\Phi_\ell = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & \cdots \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \\ e^{i\frac{2\pi}{L}u_1\ell} & 0 & \cdots \\ 0 & \ddots & 0 \\ 0 & \cdots & e^{i\frac{2\pi}{L}u_M\ell} \end{bmatrix}, \qquad \ell = 0, \ldots, L-1. \tag{35}$$

These signals have a very simple interpretation. At any time, only one transmitter antenna is active and transmitting either a reference symbol (which in differential modulation is actually the previously sent symbol) or a phase-shifted symbol. Thus, within the τth block, antenna m transmits at time $t = \tau M + m$ a symbol that is differentially phase shifted by $(2\pi/L)u_m\ell$ relative to its previous transmission. The value of ℓ is determined by the data. It is important to note that the phase shifts are potentially different for each antenna. When $M = 1$, the signals reduce to standard DPSK.

Signal matrices V_ℓ with low pairwise probability of demodulation error form correlations (33) with singular values that are small for all $\ell' \neq \ell$. The singular values of $(1/2)(I_M + V_1^\ell)$ are

$$d_{0\ell m} = (1/2)\left|1 + e^{i2\pi u_m \ell/L}\right| = \sqrt{1/2 + (1/2)\cos(2\pi u_m \ell/L)} = \left|\cos(\pi u_m \ell/L)\right|. \tag{36}$$

Thus

$$\zeta_{0\ell} = \left| \prod_{m=1}^{M} \sin(\pi u_m \ell / L) \right|^{\frac{1}{M}} . \tag{37}$$

Our maximin design requirement is to find $u_1, \ldots, u_M$ satisfying

$$\{u_1, \ldots, u_M\} = \arg \max_{0 \leqslant u_1, \ldots, u_M \leqslant L-1} \min_{\ell=1,\ldots,L-1} \left| \prod_{m=1}^{M} \sin(\pi u_m \ell / L) \right|^{\frac{1}{M}} .$$

One can see that if $u_1, \ldots, u_M$ and L share a common factor, then $V_0, \ldots, V_{L-1}$ are not distinct. Our maximin design requirement ensures that the signals are distinct.

6.3.2 Multi-cyclic construction

In general, if L is not prime, a finite Abelian group of size L may be written as a cross product of cyclic groups [14, p. 109]. A corresponding signal construction that is multi-index and systematic may be defined. Consider a factorization of L given by

$$L = \prod_{k=1}^{K} L_k.$$

Using a multi-index notation $\ell = (\ell_1, \ell_2, \cdots, \ell_K)$ with $0 \leqslant \ell_k < L_k$, the group elements are given by

$$V_\ell = \prod_{k=1}^{K} \Lambda_k^{\ell_k}.$$

Here Λ_k is a diagonal matrix with diagonal elements $\lambda_{km} = \exp(2\pi i u_{mk} / L_k)$. The diagonal elements of V_ℓ are thus $\exp(i\alpha_{\ell m})$ with

$$\alpha_{\ell m} = 2\pi \sum_{k=1}^{K} u_{mk} \ell_k / L_k, \qquad m = 1, \ldots, M.$$

and the singular values of the correlation matrices are $d_{\ell m} = |1 + v_{\ell m}| / 2 = |\cos(\alpha_{\ell m})|$.

When the L_k are pairwise relatively prime, the group is cyclic, otherwise it is multi-cyclic. For a multi-cyclic group at least two of the L_k share a factor; it therefore uses an alphabet with less then L elements. Thus, for any m there are two diagonal matrices with the same mth diagonal element. The difference between these two matrices therefore is zero in its mth column, its determinant is zero, and thus $\zeta = 0$. Multi-cyclic groups cannot have full diversity and we do not consider them any further.

7 Design and Performance of Constellations

7.1 Constellation design

In this section, we give the performance of constellations of diagonal signals designed for $M = 1, \ldots, 5$ transmitter antennas. In the search for good constellations, we may employ some simplifying rules that cause no loss of generality, regardless of the performance criterion used:

1) Because every antenna is statistically equivalent to every other, we may impose the ordering $u_1 \leqslant u_2 \leqslant \ldots \leqslant u_M$.

2) We may assume that $u_m > 0$, because if $u_m = 0$, then the mth antenna can only transmit the symbol 1 and is effectively rendered inoperative.

3) The constellations generated by $u_1, \ldots, u_M$ and $\alpha u_1, \ldots, \ldots, \alpha u_M$ are identical for all α relatively prime to L. From equation (34), we see that multiplication by α simply reorders the signals in increasing $\alpha\ell \pmod L$ instead of increasing ℓ.

7.2 Search method

In Section 4, we mention that constellations of differential unitary space-time signals can be designed with a maximin procedure: find the $u_1, \ldots, u_M \in \{0, \ldots, L-1\}$ that maximize the diversity product

$$\zeta = \min_{\ell \in \{1,\ldots,L-1\}} \left| \prod_{m=1}^{M} \sin(\pi u_m \ell / L) \right|^{\frac{1}{M}}. \tag{38}$$

We do not know of explicit solutions to this procedure, and we therefore resort to exhaustive computer searches. We consider only single-index cyclic constructions $K = 1$. Candidates for the best set of $u_1, \ldots, u_M \in \{0, \ldots, L-1\}$ are generated exhaustively, tested for performance by computing the diversity product, and kept if they exceed the previously best candidate.

The search space can be reduced using the following rules:

a) Equation (38) does not change if u_m is replaced by $L - u_m$. We may therefore restrict our search to $u_m \in \{0, \ldots, L/2\}$ (assuming L is even).

b) If u_m shares a factor with L then there is an $\ell \in \{1, \ldots, L-1\}$ for which $u_m \ell = 0 \pmod L$; this implies that the diversity product is zero. Thus, we can restrict the search to u_m that are relatively prime to L.

M	R	L	δ	ζ	$[u_1\, u_2\, \cdots\, u_M]$	P_e union bound ($\rho = 20$ dB)
1	1	2	0	1	[1] (standard DBPSK)	9.9e-3
2	1	4	0.7071	0.7071	[1 1]	1.7e-3
3	1	8	0.7860	0.5134	[1 1 3]	4.6e-4
4	1	16	0.7071	0.5453	[1 3 5 7]	6.7e-5
5	1	32	0.8179	0.4095	[1 5 7 9 11]	3.0e-5
1	2	4	0.7071	0.7071	[1] (standard DQPSK)	4.9e-2
2	2	16	0.9239	0.3826	[1 7]	3.4e-2
3	2	64	0.9389	0.2765	[1 11 27]	2.6e-2
4	2	256	0.9335	0.2208	[1 25 97 107]	1.7e-2
5	2	1024	0.9389	0.1999	[1 157 283 415 487]	9.1e-3

Table 1: Systematic antenna constellations for $M = 1, 2, 3, 4$ and 5 transmitter antennas and rate $R = 1, 2$ that maximize the diversity product ζ in (9). The number of signals in the constellation is $L = 2^{RM}$, and δ is defined in (10). The P_e upper bound is a union bound on block error rate obtained by summing over $\ell \neq \ell'$ the Chernoff bounds (7) with $\rho = 20$ dB.

c) By Rule b), we may assume that u_1 is relatively prime to L. But then there exists an α such that $\alpha u_1 = 1 \pmod L$. By multiplying $u_2, \ldots, u_M$ by this same α, and using Rule 3) above, we may assume that $u_1 = 1$.

d) In equation (38), the product for ℓ and $L - \ell$ is the same; it is 1 for $\ell = L/2$ (assuming L is even). Thus, the minimum may be taken over $\ell \in \{1, \ldots, L/2 - 1\}$.

Table 1 shows the results of our searches for constellations of $L = 2^{RM}$ that maximize ζ. For comparison, we also include the values of δ, but no attempt to minimize δ was made. Because L is a power of two, only odd u_m appear. For $M = 1$ transmitter antenna, the search naturally produces differential BPSK ($R = 1$) and differential QPSK ($R = 2$). Also included is an upper bound on the block error rate obtained by summing over $\ell \neq \ell'$ the Chernoff bounds (7) with $\rho = 20$ dB.

Comments:

1. We choose to maximize ζ in (38) rather than minimize δ in (10) because, for example, there are two $M = 2$, $R = 1$ constellations that have the same δ but very different ζ's and performances. The poorer performing constellation has $u = [1\ \ 2]$, for which $\delta = 0.7071$, $\zeta = 0$, and union bound $P_e \leqslant$1.1e-2 at $\rho = 20$ dB. The better performing constellation has $u = [1\ \ 1]$ (see also Table 1), for which $\delta = 0.7071$, $\zeta = 0.7071$, and union bound $P_e \leqslant$1.7e-3.

2. We did not search for constellations with more than 2^{RM} signals from which we would employ a subset.

7.3 Constellation performance

In our models, we assume that the channel remains approximately constant for $T = 2M$ symbols. In real communication systems, our model is therefore accurate when the coherence time of the fading process between the two terminals is at least this long. In our simulations, the fading is assumed to be independent between antennas but correlated in time according to Jakes' model [8]. A typical physical scenario where such a model is appropriate is a base station antenna array communicating with a mobile. If we assume that the mobile is traveling at approximately 25 m/s (55 mph) and operating at 900 MHz, the Doppler shift is approximately $f_D = 75$ Hz. The Jakes correlation between two fading coefficients t time samples apart is $J_0(2\pi f_D T_s t)$, where T_s is the sampling period and J_0 is the zero-order Bessel function of the first kind. We assume that $T_s = 1/30,000$ so $T_s f_D = 0.0025$. The Jakes correlation function has its first zero at $t \approx 153$. This means that fading samples separated by much less than 153 symbols, say $T = 15$ symbols, are approximately equal and our model is accurate for $T \leqslant 15$ or $M \leqslant 7$.

We suppose that binary data are to be transmitted, and we therefore have to assign the bits to the constellation signals. We do not yet know how to make an effective gray-code type of assignment, but we observe that, in our simulations, $L = 2^{RM}$ is always even. Therefore, $u_1, \ldots, u_M$ are all odd (see Rule 3), hence $V_1^{L/2} = -I_M$ and $\Phi_\ell^\dagger \Phi_{\ell+L/2} = 0$. Hence, signals offset by $L/2$ are maximally separated and are given complementary bit assignments.

Figures 4 and 5 show the bit error performance of $M = 1$, 2, 3, 4, and 5 transmitter antennas and one receiver antenna for $R = 1$ and $R = 2$. We see that the differential unitary space-time signals are especially effective at high SNR. This is not inconsistent with claims in [3] that unitary space-time signals are best suited for high SNR. We also note that the block error union bounds presented in Table 1 give rough indications of the bit error performances shown in the figures. Because the fading is continuous, the effects of variations in the fading coefficients should be more apparent with large blocklength T. Since $T = 2M$, the effects equivalently should be apparent for large M. This perhaps explains the limited gain in performance for $M = 5$ over $M = 4$ when $R = 1$, and the slight appearance of an error floor at very high SNR's.

8 Concluding remarks

An advantage of our diagonal signals (35) is their simplicity. Because only one antenna transmits at any given time, one power amplifier can be switched among the antennas. But this amplifier must deliver M-times the power it would otherwise deliver if there were an array of M amplifiers simultaneously driving the other antennas. Consequently, this amplifier needs to have a larger linear operating range than an amplifier

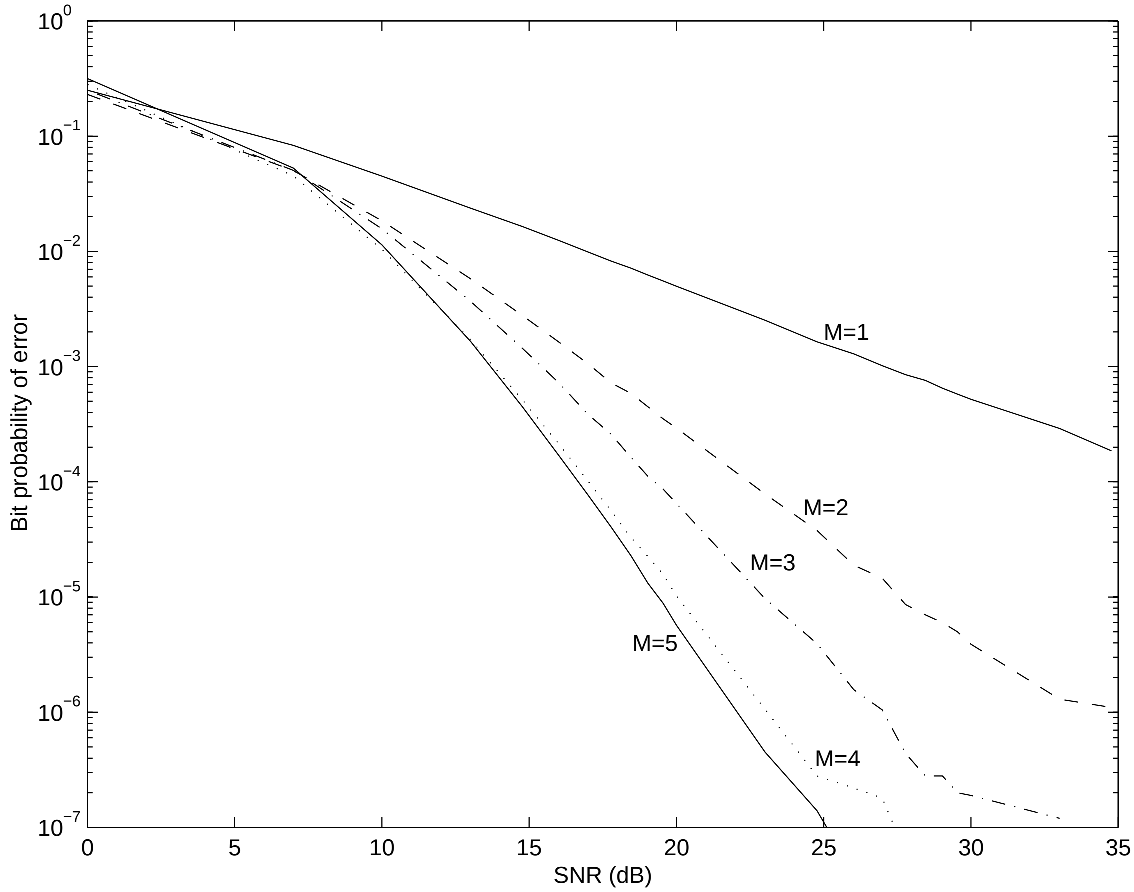

Figure 4: Performance of $M = 1$, 2, 3, 4, and 5 transmitter antennas and $N = 1$ receiver antenna as a function of SNR ρ. The channel has unknown Rayleigh fading that is changing continuously according to Jakes' model with parameter $f_D T_s = 0.0025$. The data rate is $R = 1$, and the signal constellations used are given in Table 1.

array would. Amplifiers with a large linear range are often expensive to design and build. It may therefore occasionally be desirable to have all M antennas transmitting simultaneously at lower power. In this case, we may modify the constellation by post-multiplying our signals by a fixed $M \times M$ unitary matrix such as a discrete Fourier transform matrix. This has the effect of smearing the transmitted symbol on any active antenna across all of the antennas. On the other hand, the entire constellation may be premultiplied by a common $T \times T$ unitary matrix, smearing the symbols in time. As is mentioned in Section 2.3, neither constellation modification affects its error performance in any way.

The diagonal signals are the natural consequence of three assumptions. The first assumption, which appears in Section 4.1, gives the block-unitary structure of Φ_ℓ and is essentially unrestrictive. The second

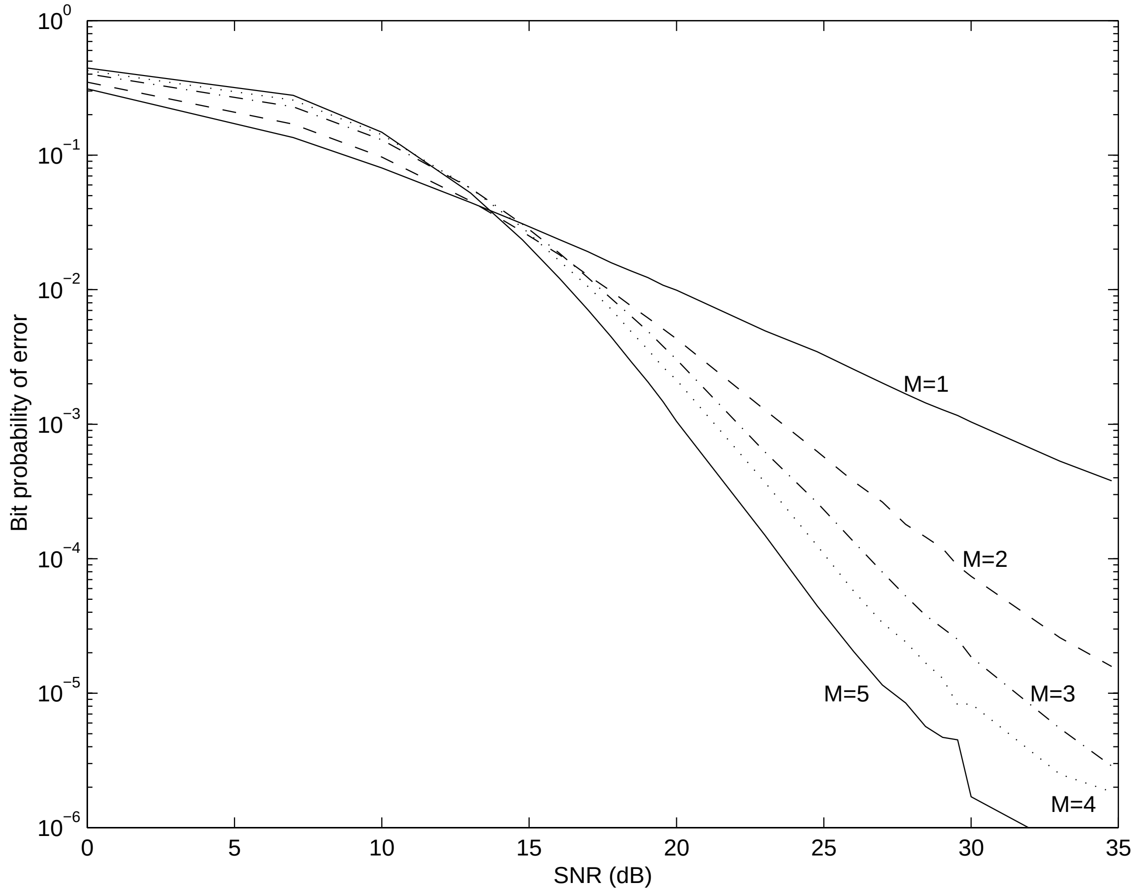

Figure 5: Performance of $M = 1$, 2, 3, 4, and 5 transmitter antennas and $N = 1$ receiver antenna as a function of SNR ρ. The channel has unknown Rayleigh fading that is changing continuously according to Jakes' model with parameter $f_D T_s = 0.0025$. The data rate is $R = 2$, and the signal constellations used are given in Table 1.

assumption, which appears in Section 6.1, requires the signal matrices to form a group, and is appealing because it simplifies signal design and generation. We do not know how restrictive this assumption is and how much constellation performance suffers by considering only groups. The final assumption, which appears in Section 6.3, requires the group to be Abelian. We have experimentally found this assumption to be fairly restrictive and the performance of diagonal signal to degrade significantly for rates $R > 2$.

The general differential framework we have described is a natural extension of standard DPSK to more than one transmitter antenna. It is flexible and can accommodate all rates and any number of antennas. The framework allows broad classes of unitary matrix-valued signals to be chained together differentially; a class of diagonal signals was given as a simple special case. Maximum likelihood decoding was shown to be a simple matrix noncoherent receiver, and pairwise error performance was measured with a diversity product. It remains a rich open problem to find other classes of group and non-group high-rate constellations with large diversity products.

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After finishing this work, we learned of an differential modulation scheme proposed by Tarokh and Jafarkhani [15]. While similar in its transmission of signal matrices that depend differentially on the input data, their approach is based specifically on orthogonal designs. We also learned of an approach by Hughes [16] who has a differential construction similar to the construction in our paper. Hughes focuses on group codes, and two-antenna codes with cyclic and quaternionic structures are explicitly designed.

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