Symmetries from Uniform Space Covering in Stochastic Discrimination

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Abstract. Studies on ensemble methods for classification suffer from the difficulty of modeling the complementary strengths of the components. Kleinberg's theory of stochastic discrimination (SD) addresses this rigorously via mathematical notions of enrichment, uniformity, and projectability of a model ensemble. We explain these concepts via a very simple numerical example that captures the basic principles of the SD theory and method. We focus on a fundamental symmetry in point set covering that is the key observation leading to the foundation of the theory. We believe a better understanding of the SD method will lead to developments of better tools for analyzing other ensemble methods.

1 Introduction

Methods for classifier combination, or ensemble learning, can be divided into two categories: 1) *decision optimization* methods that try to obtain *consensus* among a *given* set of classifiers to make the best decision; 2) *coverage optimization* methods that try to *create* a set of classifiers that can do well for all possible cases under a *fixed* decision combination function.

Decision optimization methods rely on the assumption that the given set of classifiers, typically of a small size, contain sufficient expert knowledge about the application domain, and each of them excels in a subset of all possible input. A decision combination function is chosen or trained to exploit the individual strengths while avoiding their weaknesses. Popular combination functions include majority/plurality votes[19], sum/product rules[14], rank/confidence score combination[12], and probabilistic methods[13]. These methods are known to be useful in many applications where reasonably good component classifiers can be developed. However, the joint capability of the classifiers sets an intrinsic limitation that a decision combination function cannot overcome. A challenge in this approach is to find out the "blind spots" of the ensemble and to obtain an additional classifier that covers them.

Coverage optimization methods use an automatic and systematic mechanism to generate new classifiers with the hope of covering all possible cases. A fixed function, typically simple in form, is used for decision combination. This can be training set subsampling, such as stacking[22], bagging[2], and boosting[5], feature subspace projection[10], superclass/subclass decomposition[4], or other

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methods for randomly perturbing the classifier training procedures[6]. Open questions in these methods are 1) how many classifiers are enough? 2) what kind of differences among the component classifiers yields the best combined accuracy? 3) how much limitation is set by the form of the component classifiers?

Apparently both categories of ensemble methods run into some dilemma. Should the component classifiers be weakened in order to achieve a stronger whole? Should some accuracy be sacrificed for the known samples to obtain better generalization for the unseen cases? Do we seek agreement, or differences among the component classifiers?

A central difficulty in studying the performance of these ensembles is how to model the complementary strengths among the classifiers. Many proofs rely on an assumption of statistical independence of component classifiers' decisions. But rarely is there any attempt to match this assumption with observations of the decisions. Often, global estimates of the component classifiers' accuracies are used in their selection, while in an ensemble what matter more are the local estimates, plus the relationship between the local accuracy estimates on samples that are close neighbors in the feature space.[1]

Deeper investigation of these issues leads back to three major concerns in choosing classifiers: discriminative power, use of complementary information, and generalization power. A complete theory on ensembles must address these three issues simultaneously. Many current theories rely, either explicitly or implicitly, on ideal assumptions on one or two of these issues, or have them omitted entirely, and are therefore incomplete.

Kleinberg's theory and method of stochastic discrimination (SD)[15][16] is the first attempt to explicitly address these issues simultaneously from a mathematical point of view. In this theory, rigorous notions are made for discriminative power, complementary information, and generalization power of an ensemble. A fundamental symmetry is observed between the probability of a fixed model covering a point in a given set and the probability of a fixed point being covered by a model in a given ensemble. The theory establishes that, these three conditions are sufficient for an ensemble to converge, with increases in its size, to the most accurate classifier for the application.

Kleinberg's analysis uses a set-theoretic abstraction to remove from consideration algorithmic details of classifiers, feature extraction processes, and training procedures. It considers only the classifiers' decision regions in the form of point sets, called *weak models*, in the feature space. A collection of classifiers is thus just a sample from the power set of the feature space. If the sample satisfies a uniformity condition, i.e., if its coverage is unbiased for any local region of the feature space, then a symmetry is observed between two probabilities (w.r.t. the feature space and w.r.t. the power set, respectively) of the same event that a point of a particular class is covered by a component of the sample. Discrimination between classes is achieved by requiring some minimum difference in each component's inclusion of points of different classes, which is trivial to satisfy. By

[1] There is more discussion on these difficulties in a recent review[8].

way of this symmetry, it is shown that if the sample of weak models is large, the discriminant function, defined on the coverage of the models on a single point and the class-specific differences within each model, converges to poles distinct by class with diminishing variance.

We believe that this symmetry is the key to the discussions on classifier combination. However, since the theory was developed from a fresh, original, and independent perspective on the problem of learning, there have not been many direct links made to the existing theories. As the concepts are new, the claims are high, the published algorithms appear simple, and the details of more sophisticated implementations are not known, the method has been poorly understood and is sometimes referred to as mysterious.

It is the goal of this lecture to illustrate the basic concepts in this theory and remove the apparent mystery. We present the principles of stochastic discrimination with a very simple numerical example. The example is so chosen that all computations can be easily traced step-by-step by hand or with very simple programs. We use Kleinberg's notation wherever possible to make it easier for the interested readers to follow up on the full theory in the original papers. Our emphasis is on explaining the concepts of uniformity and enrichment, and the behavior of the discriminant when both conditions are fulfilled. For the details of the mathematical theory and outlines of practical algorithms, please refer to Kleinberg's original publications[15][16][17][18].

2 Symmetry of Probabilities Induced by Uniform Space Covering

The SD method is based on a fundamental symmetry in point set covering. To illustrate this symmetry, we begin with a simple observation. Consider a set $S = \{a, b, c\}$ and all the subsets with two elements $s_1 = \{a, b\}$, $s_2 = \{a, c\}$, and $s_3 = \{b, c\}$. By our choice, each of these subsets has captured $2/3$ of the elements of S. We call this ratio r. Let us now look at each member of S, and check how many of these three subsets have included that member. For example, a is in two of them, so we say a is captured by $2/3$ of these subsets. We will obtain the same value $2/3$ for all elements of S. This value is the same as r. This is a consequence of the fact that we have used all such 2-member subsets and we have not biased this collection towards any element of S. With this observation, we begin a larger example.

Consider a set of 10 points in a one-dimensional feature space F. Let this set be called A. Assume that F contains only points in A and nothing else. Let each point in A be identified as q_0, q_1, ..., q_9 as follows.

. . . . . .

$q_0\ q_1\ q_2\ q_3\ q_4\ q_5\ q_6\ q_7\ q_8\ q_9$

Now consider the subsets of F. Let the collection of all such subsets be $\mathcal{M}$, which is the power set of F. We call each member m of $\mathcal{M}$ a *model*, and we restrict our consideration to only those models that contain 5 points in A, therefore each

Table 1. Models m_t in $M_{0.5,A}$ in the order of $M = m_1, m_2, ..., m_{252}$. Each model is shown with its elements denoted by the indices i of q_i in A. For example, $m_1 = \{q_3, q_5, q_6, q_8, q_9\}$.

m_t	elements	m_t	elements	m_t	elements	m_t	elements	m_t	elements	m_t	elements
m_1	35689	m_{43}	12689	m_{85}	24578	m_{127}	01469	m_{169}	02468	m_{211}	02458
m_2	01268	m_{44}	04569	m_{86}	23568	m_{128}	03679	m_{170}	35678	m_{212}	13457
m_3	04789	m_{45}	01245	m_{87}	01267	m_{129}	04579	m_{171}	03589	m_{213}	24689
m_4	25689	m_{46}	01458	m_{88}	01257	m_{130}	01237	m_{172}	34679	m_{214}	03478
m_5	02679	m_{47}	15679	m_{89}	05679	m_{131}	24789	m_{173}	12346	m_{215}	23589
m_6	34578	m_{48}	12457	m_{90}	24589	m_{132}	45689	m_{174}	12458	m_{216}	24679
m_7	13459	m_{49}	02379	m_{91}	04589	m_{133}	16789	m_{175}	35789	m_{217}	02456
m_8	01238	m_{50}	02568	m_{92}	12467	m_{134}	13479	m_{176}	02358	m_{218}	05689
m_9	12347	m_{51}	12357	m_{93}	13578	m_{135}	02349	m_{177}	35679	m_{219}	12789
m_{10}	01579	m_{52}	14678	m_{94}	02369	m_{136}	13469	m_{178}	13458	m_{220}	02346
m_{11}	34589	m_{53}	12678	m_{95}	12469	m_{137}	03678	m_{179}	01459	m_{221}	23489
m_{12}	03459	m_{54}	23567	m_{96}	04567	m_{138}	23679	m_{180}	03479	m_{222}	23467
m_{13}	23459	m_{55}	02789	m_{97}	14679	m_{139}	46789	m_{181}	14789	m_{223}	12489
m_{14}	02457	m_{56}	24567	m_{98}	13467	m_{140}	01468	m_{182}	23678	m_{224}	14589
m_{15}	02368	m_{57}	13569	m_{99}	45678	m_{141}	03689	m_{183}	03456	m_{225}	25678
m_{16}	02689	m_{58}	01259	m_{100}	03469	m_{142}	02478	m_{184}	13456	m_{226}	12579
m_{17}	01368	m_{59}	23479	m_{101}	34789	m_{143}	23457	m_{185}	01568	m_{227}	03458
m_{18}	13589	m_{60}	03579	m_{102}	45679	m_{144}	02347	m_{186}	01578	m_{228}	01569
m_{19}	14579	m_{61}	12368	m_{103}	01358	m_{145}	01289	m_{187}	01678	m_{229}	45789
m_{20}	23468	m_{62}	23578	m_{104}	01379	m_{146}	01369	m_{188}	12367	m_{230}	12358
m_{21}	26789	m_{63}	02345	m_{105}	01236	m_{147}	01356	m_{189}	12345	m_{231}	02579
m_{22}	15678	m_{64}	01479	m_{106}	01679	m_{148}	12379	m_{190}	25679	m_{232}	01457
m_{23}	04578	m_{65}	03569	m_{107}	13689	m_{149}	02569	m_{191}	02367	m_{233}	05789
m_{24}	04679	m_{66}	01346	m_{108}	12479	m_{150}	34678	m_{192}	01256	m_{234}	01247
m_{25}	02459	m_{67}	24568	m_{109}	14568	m_{151}	24569	m_{193}	13679	m_{235}	03467
m_{26}	12569	m_{68}	01359	m_{110}	15689	m_{152}	03578	m_{194}	04689	m_{236}	12359
m_{27}	01269	m_{69}	12459	m_{111}	01258	m_{153}	02359	m_{195}	04568	m_{237}	02567
m_{28}	06789	m_{70}	01239	m_{112}	12389	m_{154}	01234	m_{196}	12578	m_{238}	12356
m_{29}	01689	m_{71}	24678	m_{113}	03568	m_{155}	01345	m_{197}	12468	m_{239}	02469
m_{30}	01248	m_{72}	01347	m_{114}	23689	m_{156}	02348	m_{198}	03468	m_{240}	13468
m_{31}	12456	m_{73}	01467	m_{115}	23478	m_{157}	03457	m_{199}	34569	m_{241}	02479
m_{32}	13579	m_{74}	04678	m_{116}	34568	m_{158}	02357	m_{200}	12369	m_{242}	36789
m_{33}	34689	m_{75}	12589	m_{117}	23569	m_{159}	01235	m_{201}	13489	m_{243}	13568
m_{34}	12679	m_{76}	01348	m_{118}	14689	m_{160}	01378	m_{202}	12567	m_{244}	02467
m_{35}	12568	m_{77}	14569	m_{119}	23789	m_{161}	14567	m_{203}	02489	m_{245}	01589
m_{36}	34579	m_{78}	01789	m_{120}	01246	m_{162}	23458	m_{204}	02678	m_{246}	01478
m_{37}	01389	m_{79}	01367	m_{121}	23579	m_{163}	56789	m_{205}	13567	m_{247}	15789
m_{38}	23469	m_{80}	12478	m_{122}	01456	m_{164}	34567	m_{206}	01357	m_{248}	01349
m_{39}	24579	m_{81}	25789	m_{123}	23456	m_{165}	01249	m_{207}	01278	m_{249}	02356
m_{40}	02589	m_{82}	01489	m_{124}	03789	m_{166}	03489	m_{208}	02578	m_{250}	14578
m_{41}	01567	m_{83}	03567	m_{125}	05678	m_{167}	02389	m_{209}	12348	m_{251}	13789
m_{42}	13478	m_{84}	12349	m_{126}	13678	m_{168}	12378	m_{210}	01279	m_{252}	02378

model has a size that is 0.5 of the size of A. Let this set of models be called $M_{0.5,A}$. Some members of $M_{0.5,A}$ are as follows.

$$\{q_0, q_1, q_2, q_3, q_4\}$$
$$\{q_0, q_1, q_2, q_3, q_5\}$$
$$\{q_0, q_1, q_2, q_3, q_6\}$$
$$...$$

There are $C(10,5) = 252$ members in $M_{0.5,A}$. Let M be a pseudo-random permutation of members in $M_{0.5,A}$ as listed in Table 1. We identify models in this sequence by a single subscript such that $M = m_1, m_2, ..., m_{252}$. We expand a collection M_t by including more and more members of $M_{0.5,A}$ in the order of the sequence M as follows. $M_1 = \{m_1\}$, $M_2 = \{m_1, m_2\}$, ..., $M_t = \{m_1, m_2, ...m_t\}$.

Since each model covers some points in A, for each member q in A, we can count the number of models in M_t that include q, call this count $N(q, M_t)$, and calculate the ratio of this count over the size of M_t, call it $Y(q, M_t)$. That is, $Y(q, M_t) = Prob_{\mathcal{M}}(q \in m | m \in M_t)$. As M_t expands, this ratio changes and we show these changes for each q in Table 2. The values of $Y(q, M_t)$ are plotted in Figure 1. As is clearly visible in the Figure, the values of $Y(q, M_t)$ converge to

Table 2. Ratio of coverage of each point q by members of M_t as M_t expands

	$N(M_t, q)$										$Y(M_t, q)$									
M_t	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
M_1	0	0	0	1	0	1	1	0	1	1	0.00	0.00	0.00	1.00	0.00	1.00	1.00	0.00	1.00	1.00
M_2	1	1	1	1	0	1	2	0	2	1	0.50	0.50	0.50	0.50	0.00	0.50	1.00	0.00	1.00	0.50
M_3	2	1	1	1	1	1	2	1	3	2	0.67	0.33	0.33	0.33	0.33	0.33	0.67	0.33	1.00	0.67
M_4	2	1	2	1	1	2	3	1	4	3	0.50	0.25	0.50	0.25	0.25	0.50	0.75	0.25	1.00	0.75
M_5	3	1	3	1	1	2	4	2	4	4	0.60	0.20	0.60	0.20	0.20	0.40	0.80	0.40	0.80	0.80
M_6	3	1	3	2	2	3	4	3	5	4	0.50	0.17	0.50	0.33	0.33	0.50	0.67	0.50	0.83	0.67
M_7	3	2	3	3	3	4	4	3	5	5	0.43	0.29	0.43	0.43	0.43	0.57	0.57	0.43	0.71	0.71
M_8	4	3	4	4	3	4	4	3	6	5	0.50	0.38	0.50	0.50	0.38	0.50	0.50	0.38	0.75	0.62
M_9	4	4	5	5	4	4	4	4	6	5	0.44	0.44	0.56	0.56	0.44	0.44	0.44	0.44	0.67	0.56
M_{10}	5	5	5	5	4	5	4	5	6	6	0.50	0.50	0.50	0.50	0.40	0.50	0.40	0.50	0.60	0.60
...					...										...					
M_{159}	81	80	79	79	79	77	82	78	74	86	0.51	0.50	0.50	0.50	0.50	0.48	0.52	0.49	0.47	0.54
M_{160}	82	81	79	80	79	77	82	79	75	86	0.51	0.51	0.49	0.50	0.49	0.48	0.51	0.49	0.47	0.54
M_{161}	82	82	79	80	80	78	83	80	75	86	0.51	0.51	0.49	0.50	0.50	0.48	0.52	0.50	0.47	0.53
M_{162}	82	82	80	81	81	79	83	80	76	86	0.51	0.51	0.49	0.50	0.50	0.49	0.51	0.49	0.47	0.53
M_{163}	82	82	80	81	81	80	84	81	77	87	0.50	0.50	0.49	0.50	0.50	0.49	0.52	0.50	0.47	0.53
M_{164}	82	82	80	82	82	81	85	82	77	87	0.50	0.50	0.49	0.50	0.50	0.49	0.52	0.50	0.47	0.53
M_{165}	83	83	81	82	83	81	85	82	77	88	0.50	0.50	0.49	0.50	0.50	0.49	0.52	0.50	0.47	0.53
M_{166}	84	83	81	83	84	81	85	82	78	89	0.51	0.50	0.49	0.50	0.51	0.49	0.51	0.49	0.47	0.54
M_{167}	85	83	82	84	84	81	85	82	79	90	0.51	0.50	0.49	0.50	0.50	0.49	0.51	0.49	0.47	0.54
M_{168}	85	84	83	85	84	81	85	83	80	90	0.51	0.50	0.49	0.51	0.50	0.48	0.51	0.49	0.48	0.54
...					...										...					
M_{243}	120	120	123	122	122	122	124	120	120	122	0.49	0.49	0.51	0.50	0.50	0.50	0.51	0.49	0.49	0.50
M_{244}	121	120	124	122	123	122	125	121	120	122	0.50	0.49	0.51	0.50	0.50	0.50	0.51	0.50	0.49	0.50
M_{245}	122	121	124	122	123	123	125	121	121	123	0.50	0.49	0.51	0.50	0.50	0.50	0.51	0.49	0.49	0.50
M_{246}	123	122	124	122	124	123	125	122	122	123	0.50	0.50	0.50	0.50	0.50	0.50	0.51	0.50	0.50	0.50
M_{247}	123	123	124	122	124	124	125	123	123	124	0.50	0.50	0.50	0.49	0.50	0.50	0.51	0.50	0.50	0.50
M_{248}	124	124	124	123	125	124	125	123	123	125	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
M_{249}	125	124	125	124	125	125	126	123	123	125	0.50	0.50	0.50	0.50	0.50	0.50	0.51	0.49	0.49	0.50
M_{250}	125	125	125	124	126	126	126	124	124	125	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
M_{251}	125	126	125	125	126	126	126	125	125	126	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
M_{252}	126	126	126	126	126	126	126	126	126	126	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50

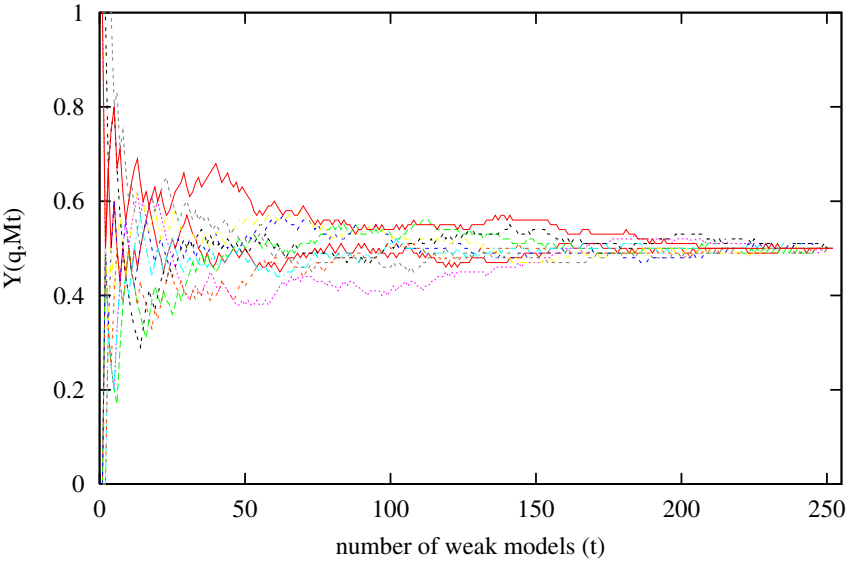


Fig. 1. Plot of $Y(q, M_t)$ versus t. Each line represents the trace of $Y(q, M_t)$ for a particular q as M_t expands.

0.5 for each q. Also notice that because of the randomization, we have expanded M_t in a way that M_t is not biased towards any particular q, therefore the values of $Y(q, M_t)$ are similar after M_t has acquired a certain size (say, when $t = 80$). When M_t=$M_{0.5,A}$, every point q is covered by the same number of models in

M_t, and their values of $Y(q, M_t)$ are identical and is equal to 0.5, which is the ratio of the size of each m relative to A (recall that we always include 5 points from A in each m).

Formally, when $t = 252$, $M_t = M_{0.5,A}$, from the perspective of a fixed q, the probability of it being contained in a model m from M_t is

$$Prob_{\mathcal{M}}(q \in m | m \in M_{0.5,A}) = 0.5.$$

We emphasize that this probability is a measure in the space $\mathcal{M}$ by writing the probability as $Prob_{\mathcal{M}}$. On the other hand, by the way each m is constructed, we know that from the perspective of a fixed m,

$$Prob_F(q \in m | q \in A) \ = \ 0.5.$$

Note that this probability is a measure in the space F. We have shown that these two probabilities, w.r.t. two different spaces, have identical values. In other words, let the membership function of m be $C_m(q)$, i.e., $C_m(q) = 1$ iff $q \in m$, the random variables $\lambda q C_m(q)$ and $\lambda m C_m(q)$ have the same probability distribution, when q is restricted to A and m is restricted to $M_{0.5,A}$. This is because both variables can have values that are either 1 or 0, and they have the value 1 with the same probability (0.5 in this case). This symmetry arises from the fact that the collection of models $M_{0.5,A}$ covers the set A uniformly, i.e., since we have used all members of $M_{0.5,A}$, each point q have the same chance to be included in one of these models. If any two points in a set S have the same chance to be included in a collection of models, we say that this collection is S-uniform. It can be shown, by a simple counting argument, that uniformity leads to the symmetry of $Prob_{\mathcal{M}}(q \in m | m \in M_{0.5,A})$ and $Prob_F(q \in m | q \in A)$, and hence distributions of $\lambda q C_m(q)$ and $\lambda m C_m(q)$.

The observation and utilization of this duality are central to the theory of stochastic discrimination. A critical point of the SD method is to enforce such a uniform cover on a set of points. That is, to construct a collection of models in a balanced way so that the uniformity (hence the duality) is achieved without exhausting all possible models from the space.

3 Two-Class Discrimination

Let us now label each point q in A by one of two classes c_1 (marked by "x") and c_2 (marked by "o") as follows.

x	x	x	o	o	o	o	x	x	o
q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9

This gives a training set TR_i for each class c_i. In particular,

$$TR_1 = \{q_0, q_1, q_2, q_7, q_8\},$$

and

$$TR_2 = \{q_3, q_4, q_5, q_6, q_9\}.$$

How can we build a classifier for c_1 and c_2 using models from $M_{0.5,A}$? First, we evaluate each model m by how well it has captured the members of each class. Define ratings r_i $(i = 1, 2)$ for each m as

$$r_i(m) = Prob_F(q \in m | q \in TR_i).$$

For example, consider model $m_1 = \{q_3, q_5, q_6, q_8, q_9\}$, where q_8 is in TR_1 and the rest are in TR_2. TR_1 has 5 members and 1 is in m_1, therefore $r_1(m_1) = 1/5 = 0.2$. TR_2 has (incidentally, also) 5 members and 4 of them are in m_1, therefore $r_2(m_1) = 4/5 = 0.8$. Thus these ratings represent the quality of the models as a description of each class. A model with a rating 1.0 for a class is a perfect model for that class. We call the difference between r_1 and r_2 the *degree of enrichment* of m with respect to classes $(1, 2)$, i.e., $d_{12} = r_1 - r_2$. A model m is *enriched* if $d_{12} \neq 0$. Now we define, for all enriched models m,

$$X_{12}(q, m) = \frac{C_m(q) - r_2(m)}{r_1(m) - r_2(m)},$$

and let $X_{12}(q, m)$ be 0 if $d_{12}(m) = 0$. For a given m, r_1 and r_2 are fixed, and the value of $X(q, m)$ for each q in A can have one of two values depending on whether q is in m. For example, for m_1, $r_1 = 0.2$ and $r_2 = 0.8$, so $X(q, m) = -1/3$ for points q_3, q_5, q_6, q_8, q_9, and $X(q, m) = 4/3$ for points q_0, q_1, q_2, q_4, q_7. Next, for each set $M_t = \{m_1, m_2, ..., m_t\}$, we define a discriminant

$$Y_{12}(q, M_t) = \frac{1}{t} \sum_{k=1}^{t} X_{12}(q, m_k).$$

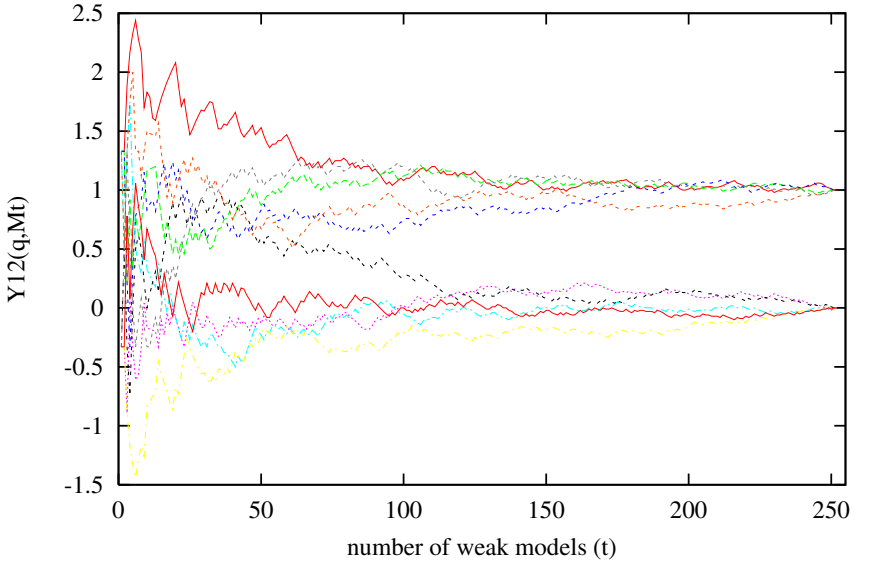


Fig. 2. Plot of $Y_{12}(q, M_t)$ versus t. Each line represents the trace of $Y_{12}(q, M_t)$ for a particular q as M_t expands.

Table 3. Changes of $Y_{12}(q, M_t)$ as M_t expands. For each t, we show the ratings for each new member of M_t, the values X_{12} for this new member, and Y_{12} for the collection M_t up to the inclusion of this new member.

					$X_{12}(q, m_t)$ if		$Y_{12}(q, M_t)$									
M_t	m_t	r_1	r_2	$r_1 - r_2$	$q \in m_t$	$q \notin m_t$	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
M_1	m_1	0.20	0.80	-0.60	-0.33	1.33	1.33	1.33	1.33	-0.33	1.33	-0.33	-0.33	1.33	-0.33	-0.33
M_2	m_2	0.80	0.20	0.60	1.33	-0.33	1.33	1.33	1.33	-0.33	0.50	-0.33	0.50	0.50	0.50	-0.33
M_3	m_3	0.60	0.40	0.20	3.00	-2.00	1.89	0.22	0.22	-0.89	1.33	-0.89	-0.33	1.33	1.33	0.78
M_4	m_4	0.40	0.60	-0.20	-2.00	3.00	2.17	0.92	-0.33	0.08	1.75	-1.17	-0.75	1.75	0.50	0.08
M_5	m_5	0.60	0.40	0.20	3.00	-2.00	2.33	0.33	0.33	-0.33	1.00	-1.33	0.00	2.00	0.00	0.67
M_6	m_6	0.40	0.60	-0.20	-2.00	3.00	2.44	0.78	0.78	-0.61	0.50	-1.44	0.50	1.33	-0.33	1.06
M_7	m_7	0.20	0.80	-0.60	-0.33	1.33	2.28	0.62	0.86	-0.57	0.38	-1.28	0.62	1.33	-0.10	0.86
M_8	m_8	0.80	0.20	0.60	1.33	-0.33	2.17	0.71	0.92	-0.33	0.29	-1.17	0.50	1.12	0.08	0.71
M_9	m_9	0.60	0.40	0.20	3.00	-2.00	1.70	0.96	1.15	0.04	0.59	-1.26	0.22	1.33	-0.15	0.41
M_{10}	m_{10}	0.60	0.40	0.20	3.00	-2.00	1.83	1.17	0.83	-0.17	0.33	-0.83	0.00	1.50	-0.33	0.67
...																
M_{159}	m_{159}	0.60	0.40	0.20	3.00	-2.00	1.02	1.05	0.89	0.18	-0.01	-0.18	0.07	0.95	1.09	-0.06
M_{160}	m_{160}	0.80	0.20	0.60	1.33	-0.33	1.02	1.05	0.89	0.19	-0.01	-0.18	0.06	0.95	1.09	-0.06
M_{161}	m_{161}	0.40	0.60	-0.20	-2.00	3.00	1.03	1.03	0.90	0.21	-0.02	-0.19	0.05	0.93	1.11	-0.04
M_{162}	m_{162}	0.40	0.60	-0.20	-2.00	3.00	1.04	1.04	0.88	0.19	-0.03	-0.20	0.07	0.94	1.09	-0.02
M_{163}	m_{163}	0.40	0.60	-0.20	-2.00	3.00	1.06	1.06	0.89	0.21	-0.02	-0.21	0.06	0.92	1.07	-0.04
M_{164}	m_{164}	0.20	0.80	-0.60	-0.33	1.33	1.06	1.06	0.90	0.21	-0.02	-0.21	0.05	0.92	1.07	-0.03
M_{165}	m_{165}	0.60	0.40	0.20	3.00	-2.00	1.07	1.07	0.91	0.19	0.00	-0.22	0.04	0.90	1.05	-0.01
M_{166}	m_{166}	0.40	0.60	-0.20	-2.00	3.00	1.05	1.08	0.92	0.18	-0.01	-0.20	0.06	0.91	1.03	-0.02
M_{167}	m_{167}	0.60	0.40	0.20	3.00	-2.00	1.06	1.06	0.93	0.20	-0.02	-0.21	0.05	0.89	1.04	0.00
M_{168}	m_{168}	0.80	0.20	0.60	1.33	-0.33	1.06	1.07	0.94	0.20	-0.03	-0.21	0.04	0.90	1.05	-0.01
...																
M_{243}	m_{243}	0.40	0.60	-0.20	-2.00	3.00	1.04	0.99	1.05	0.03	-0.01	-0.02	0.02	0.96	0.96	-0.02
M_{244}	m_{244}	0.60	0.40	0.20	3.00	-2.00	1.05	0.98	1.06	0.03	0.00	-0.03	0.03	0.97	0.95	-0.03
M_{245}	m_{245}	0.60	0.40	0.20	3.00	-2.00	1.06	0.98	1.04	0.02	0.00	-0.02	0.02	0.96	0.96	-0.02
M_{246}	m_{246}	0.80	0.20	0.60	1.33	-0.33	1.06	0.98	1.04	0.02	0.00	-0.02	0.02	0.96	0.96	-0.02
M_{247}	m_{247}	0.60	0.40	0.20	3.00	-2.00	1.05	0.99	1.03	0.01	-0.01	-0.01	0.01	0.97	0.97	-0.01
M_{248}	m_{248}	0.40	0.60	-0.20	-2.00	3.00	1.03	0.98	1.03	0.00	-0.01	0.01	0.03	0.97	0.97	-0.01
M_{249}	m_{249}	0.40	0.60	-0.20	-2.00	3.00	1.02	0.99	1.02	-0.01	0.00	0.00	0.02	0.98	0.98	0.00
M_{250}	m_{250}	0.60	0.40	0.20	3.00	-2.00	1.01	1.00	1.01	-0.02	0.01	0.01	0.01	0.99	0.99	-0.01
M_{251}	m_{251}	0.60	0.40	0.20	3.00	-2.00	1.00	1.00	1.00	0.00	0.00	0.00	0.00	1.00	1.00	0.00
M_{252}	m_{252}	0.80	0.20	0.60	1.33	-0.33	1.00	1.00	1.00	0.00	0.00	0.00	0.00	1.00	1.00	0.00

As the set M_t expands, the value of Y_{12} changes for each q. We show, in Table 3, the values of Y_{12} for each M_t and each q, and for each new member m_t of M_t, r_1, r_2, and the two values of X_{12}. The values of Y_{12} for each q are plotted in Figure 2.

In Figure 2 we see two separate trends. All those points that belong to class c_1 have their Y_{12} values converging to 1.0, and all those in c_2 converging to 0.0. Thus Y_{12} can be used with a threshold to classify an arbitrary point q. We can assign q to class c_1 if $Y_{12}(q, M_t) > 0.5$, and to class c_2 if $Y_{12}(q, M_t) < 0.5$, and remain undecided when $Y_{12}(q, M_t) = 0.5$. Observe that this classifier is fairly accurate far before M_t has expanded to the full set $M_{0.5,A}$. We can also change the two poles of Y_{12} to 1.0 and -1.0 respectively by simply rescaling and shifting X_{12}:

$$X_{12}(q, m) = 2(\frac{C_m(q) - r_2(m)}{r_1(m) - r_2(m)}) - 1.$$

How did this separation of trends happen? Let us now take a closer look at the models in each M_t and see how many of them cover each point q. For a given M_t, among its members, there can be different values of r_1 and r_2. But because of our choices of the sizes of TR_1, TR_2, and m, we have only a small set of distinct values that r_1 and r_2 can have. Namely, since each model has 5 points, there are only six possibilities as follows.

no. of points from TR_1	0	1	2	3	4	5
no. of points from TR_2	5	4	3	2	1	0
r_1	0.0	0.2	0.4	0.6	0.8	1.0
r_2	1.0	0.8	0.6	0.4	0.2	0.0

Note that in a general setting r_1 and r_2 do not have to sum up to 1. If we included models of a larger size, say, one with 10 points, we can have both r_1 and r_2 equal to 1.0. We have simplified matters by using models of a fixed size and training sets of the same size. According to the values of r_1 and r_2, in this case we have only 6 different kinds of models.

Now we take a detailed look at the coverage of each point q by each kind of models, i.e., models of a particular rating (quality) for each class. Let us count how many of the models of each value of r_1 and r_2 cover each point q, and call this $N_{M_t,r_1,TR_1}(q)$ and $N_{M_t,r_2,TR_2}(q)$ respectively. We can normalize this count by the number of models having each value of r_1 or r_2, and obtain a ratio $f_{M_t,r_1,TR_1}(q)$ and $f_{M_t,r_2,TR_2}(q)$ respectively. Thus, for each point q, we have "a profile of coverage" by models of each value of ratings r_1 and r_2 that is described by these ratios. For example, point q_0 at $t = 10$ is only covered by 5 models $(m_2, m_3, m_5, m_8, m_{10})$ in M_{10}, and from Table 3 we know that M_{10} has various numbers of models in each rating as summarized in the following table.

r_1	0.0	0.2	0.4	0.6	0.8	1.0
no. of models in M_{10} with r_1	0	2	2	4	2	0
$N_{M_{10},r_1,TR_1}(q_0)$	0	0	0	3	2	0
$f_{M_{10},r_1,TR_1}(q_0)$	0	0	0	0.75	1.0	0

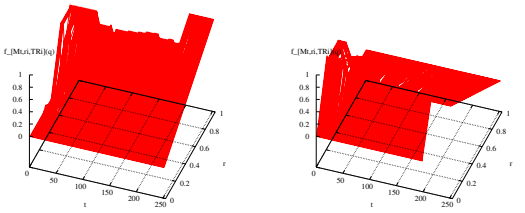
r_2	0.0	0.2	0.4	0.6	0.8	1.0
no. of models in M_{10} with r_2	0	2	4	2	2	0
$N_{M_{10},r_2,TR_2}(q_0)$	0	2	3	0	0	0
$f_{M_{10},r_2,TR_2}(q_0)$	0	1.0	0.75	0	0	0

We show such profiles for each point q and each set M_t in Figure 3 (as a function of r_1) and Figure 4 (as a function of r_2) respectively.

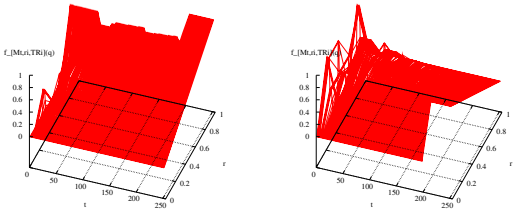
Observe that as t increases, the profiles of coverage for each point q converge to two distinct patterns. In Figure 3, the profiles for points in TR_1 converge to a diagonal $f_{M_t,r_1,TR_1} = r_1$, and in Figure 4, those for points in TR_2 also converge to a diagonal $f_{M_t,r_2,TR_2} = r_2$. That is, when $M_t = M_{0.5,A}$, we have for all q in TR_1 and for all r_1, $Prob_{\mathcal{M}}(q \in m | m \in M_{r_1,TR_1}) = r_1$, and for all q in TR_2 and for all r_2, $Prob_{\mathcal{M}}(q \in m | m \in M_{r_2,TR_2}) = r_2$. Thus we have the symmetry in place for both TR_1 and TR_2. This is a consequence of M_t being both TR_1-uniform and TR_2-uniform.

The discriminant $Y_{12}(q, M_t)$ is a summation over all models m in M_t, which can be decomposed into the sums of terms corresponding to different ratings r_i for either $i = 1$ or $i = 2$. To understand what happens with the points in TR_1, we can decompose their Y_{12} by values of r_1. Assume that there are t_x models in M_t that have $r_1 = x$. Since we have only 6 distinct values for x, M_t is a union of 6 disjoint sets, and Y_{12} can be decomposed as

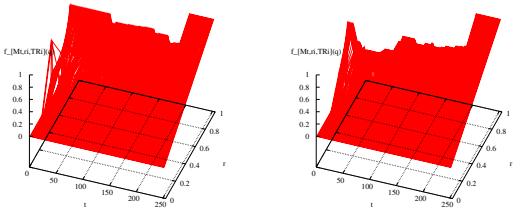
$$\begin{aligned} Y_{12}(q, M_t) = & \tfrac{t_{0.0}}{t}[\tfrac{1}{t_{0.0}} \textstyle\sum_{k_{0.0}=1}^{t_{0.0}} X_{12}(q, m_{k_{0.0}})] + \tfrac{t_{0.2}}{t}[\tfrac{1}{t_{0.2}} \textstyle\sum_{k_{0.2}=1}^{t_{0.2}} X_{12}(q, m_{k_{0.2}})] + \\ & \tfrac{t_{0.4}}{t}[\tfrac{1}{t_{0.4}} \textstyle\sum_{k_{0.4}=1}^{t_{0.4}} X_{12}(q, m_{k_{0.4}})] + \tfrac{t_{0.6}}{t}[\tfrac{1}{t_{0.6}} \textstyle\sum_{k_{0.6}=1}^{t_{0.6}} X_{12}(q, m_{k_{0.6}})] + \\ & \tfrac{t_{0.8}}{t}[\tfrac{1}{t_{0.8}} \textstyle\sum_{k_{0.8}=1}^{t_{0.8}} X_{12}(q, m_{k_{0.8}})] + \tfrac{t_{1.0}}{t}[\tfrac{1}{t_{1.0}} \textstyle\sum_{k_{1.0}=1}^{t_{1.0}} X_{12}(q, m_{k_{1.0}})]. \end{aligned}$$



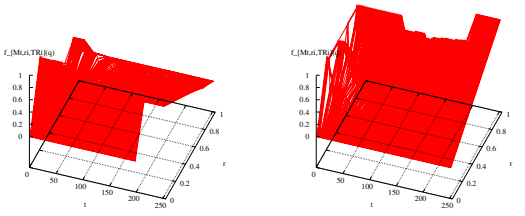
$q_0 \in TR_1$ $q_5 \in TR_2$



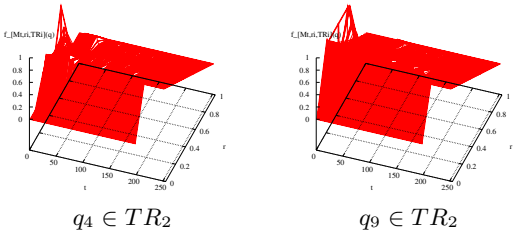
$q_1 \in TR_1$ $q_6 \in TR_2$



$q_2 \in TR_1$ $q_7 \in TR_1$



$q_3 \in TR_2$ $q_8 \in TR_1$



$q_4 \in TR_2$ $q_9 \in TR_2$

Fig. 3. $f_{M_t,r_1,TR_1}(q)$ for each point q and set M_t. In each plot, the x axis is t that ranges from 0 to 252, the y axis is r that ranges from 0 to 1, and the z axis is f_{M_t,r_1,TR_1}.

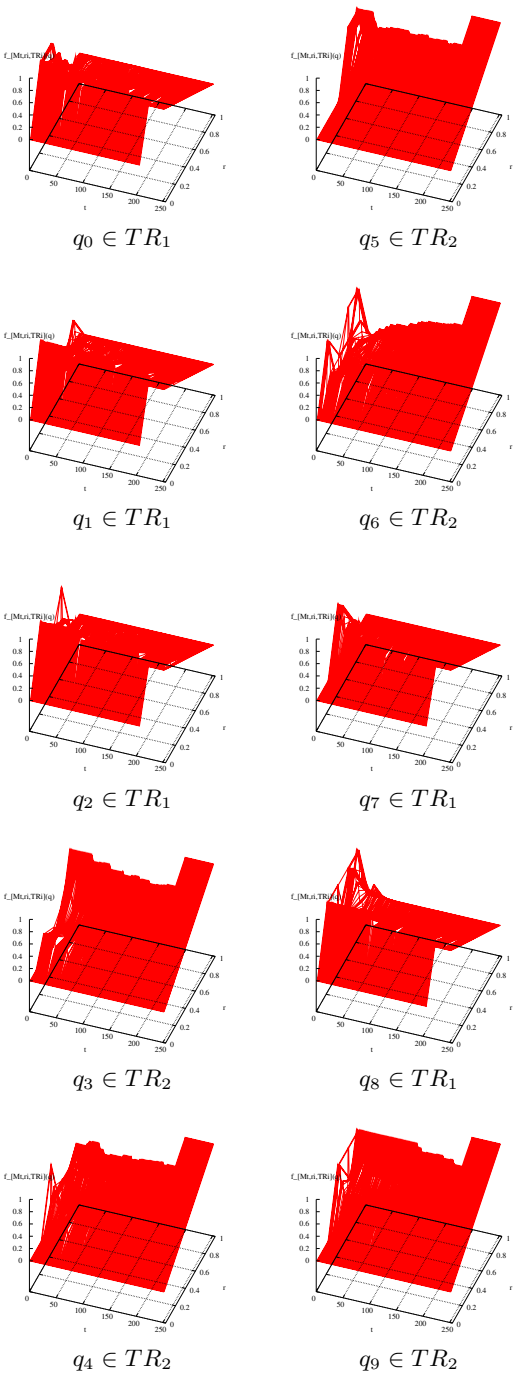


Fig. 4. $f_{M_t,r_2,TR_2}(q)$ for each point q and set M_t. In each plot, the x axis is t that ranges from 0 to 252, the y axis is r that ranges from 0 to 1, and the z axis is f_{M_t,r_2,TR_2}.

The factor in the square bracket of each term is the expectation of values of X_{12} corresponding to that particular rating $r_1 = x$. Since r_1 is the same for all m contributing to that term, by our choice of sizes of TR_1, TR_2, and the models, r_2 is also the same for all those m relevant to that term. Let that value of r_2 be y, we have, for each (fixed) q, each value of x and the associated value y,

$$E(X_{12}(q, m_x)) = E(\frac{C_{m_x}(q) - y}{x - y}) = \frac{E(C_{m_x}(q)) - y}{x - y} = \frac{x - y}{x - y} = 1.$$

The second to the last equality is a consequence of the uniformity of M_t: because the collection M_t (when $t = 252$) covers TR_1 uniformly, we have for each value x, $Prob_{\mathcal{M}}(q \in m | m \in M_{x,TR_1}) = x$, and since $C_{m_x}(q)$ has only two values (0 or 1), and $C_{m_x}(q) = 1$ iff $q \in m$, we have the expected value of $C_{m_x}(q)$ equal to x. Therefore

$$Y_{12}(q, M_t) = \frac{t_{0.0} + t_{0.2} + t_{0.4} + t_{0.6} + t_{0.8} + t_{1.0}}{t} = 1.$$

In a more general case, the values of r_2 are not necessarily equal for all models with the same value for r_1, so we cannot take y and $x - y$ out as constants. But then we can further split the term by the values of r_2, and proceed with the same argument.

A similar decomposition of Y_{12} into terms corresponding to different values of r_2 will show that $Y_{12}(q, M_t) = 0$ for those points in TR_2.

4 Projectability of Models

We have built a classifier and shown that it works for TR_1 and TR_2. How can this classifier work for an arbitrary point that is not in TR_1 or TR_2? Suppose that the feature space F contains other points p (marked by ","), and that each p is close to some training point q (marked by ".") as follows.

$$\begin{array}{cccccccccccccccccc} ., & & ., & & ., & & ., & & ., & & ., & & ., & & ., & & ., & ., \\ q_0, & p_0 & q_1, & p_1 & q_2, & p_2 & q_3, & p_3 & q_4, & p_4 & q_5, & p_5 & q_6, & p_6 & q_7, & p_7 & q_8, & p_8 & q_9, & p_9 \end{array}$$

We can take the models m as regions in the space that cover the points q in the same manner as before. Say, if each point q_i has a particular value of the feature v (in our one-dimensional feature space) that is $v(q_i)$. We can define a model by ranges of values for this feature, e.g., in our example m_1 covers q_3, q_5, q_6, q_8, q_9, so we take

$$\begin{aligned} m_1 = &\{q | \tfrac{v(q_2)+v(q_3)}{2} < v(q) < \tfrac{v(q_3)+v(q_4)}{2}\} \cup \\ &\{q | \tfrac{v(q_4)+v(q_5)}{2} < v(q) < \tfrac{v(q_6)+v(q_7)}{2}\} \cup \\ &\{q | \tfrac{v(q_7)+v(q_8)}{2} < v(q)\}. \end{aligned}$$

Thus we can tell if an arbitrary point p with value $v(p)$ for this feature is inside or outside this model.

We can calculate the model's ratings in exactly the same way as before, using only the points q. But now the same classifier works for the new points p, since we can use the new definitions of models to determine if p is inside or outside each model. Given the proximity relationship as above, those points will be assigned to the same class as their closest neighboring q. If these are indeed the true classes for the points p, the classifier is perfect for this new set. In the SD terminology, if we call the two subsets of points p that should be labeled as two different classes TE_1 and TE_2, i.e., $TE_1 = \{p_0, p_1, p_2, p_7, p_8\}$, $TE_2 = \{p_3, p_4, p_5, p_6, p_9\}$, we say that TR_1 and TE_1 are M_t-indiscernible, and similarly TR_2 and TE_2 are also M_t-indiscernible. This is to say, from the perspective of M_t, there is no difference between TR_1 and TE_1, or TR_2 and TE_2, therefore all the properties of M_t that are observed using TR_1 and TR_2 can be projected to TE_1 and TE_2. The central challenge of an SD method is to maintain projectability, uniformity, and enrichment of the collection of models at the same time.

5 Developments of SD Theory and Algorithms

5.1 Algorithmic Implementations

The method of stochastic discrimination constructs a classifier by combining a large number of simple discriminators that are called *weak models*. A weak model is simply a subset of the feature space. In summary, the classifier is constructed by a three-step process: (1) weak model generation, (2) weak model evaluation, and (3) weak model combination. The generator enumerates weak models in an arbitrary order and passes them on to the evaluator. The evaluator has access to the training set. It rates and filters the weak models according to their capability in capturing points of each class, and their contribution to satisfying the uniformity condition. The combiner then produces a discriminant function that depends on a point's membership in each model, and the models' ratings. At classification, a point is assigned to the class for which this discriminant has the highest value. Informally, the method captures the intuition of gaining wisdom from random guesses with feedback.

Weak model generation. Two guidelines should be observed in generating the weak models:

(1) *projectability*: A weak model should be able to capture enough points both inside and outside the training set so that the solution can be projectable to points not included in the training set. Geometrically, this means that a useful model must be of certain minimum size, and it should be able to capture points that are considered *neighbors* of one another. To guarantee similar accuracies of the classifier (based on similar ratings of the weak models) on both training and testing data, one also needs an assumption that the training data are *representative*. Data representativeness and model projectability are two sides of the same question. More discussions of this can be found in [1]. A weak model defines a *neighborhood* in the space, and we need a training sample in a neighborhood of every unseen sample. Otherwise, since our only knowledge of the class

boundaries is derived from the given training set, there is no basis for inference concerning regions of the feature space where no training samples are given.

(2) *simplicity of representation*: A weak model should have a simple representation. That means, the membership of an arbitrary point with respect to a model must be cheaply computable. To illustrate this, consider representing a model as a listing of all the points it contains. This is practically useless since the resultant solution could be as expensive as an exhaustive template matching using all the points in the feature space. An example of a model with a simple representation is a half-plane in a two-dimensional feature space.

Conditions (1) and (2) restrict the type of weak models yet by no means reduce the number of candidates to any tangible limit. To obtain an unbiased collection of the candidates with minimum effort, random sampling with replacement is useful. The training of the method thus relies on a stochastic process which, at each iteration, generates a weak model that satisfies the above conditions.

A convenient way to generate weak models randomly is to use a type of models that can be described by a small number of parameters. Then a stream of models can be created by pseudo-random choices on the values of the parameters. Some example types of models that can be generated this way include (1) half-spaces bounded by a threshold on a randomly selected feature dimension; (2) half-spaces bounded by a hyperplane of equi-distance to two randomly selected points; (3) regions bounded by two parallel hyperplanes perpendicular to a randomly selected axis; (4) hypercubes centered at randomly selected points with edges of varying lengths; and (5) balls (based on the city-block metric, Euclidean distance, or other dissimilarity measures) centered at randomly selected points with randomly selected radii. A model can also be a union or intersection of several regions of these types. An implementation of SD using hyper-rectangular boxes as weak models is described in [9].

A number of heuristics may be used in creating these models. These heuristics specify the way random points are chosen from the space, or set limits on the maximum and minimum sizes of the models. By this we mean restricting the choices of random points to, for instance, points in the space whose coordinates fall inside the range of those of the training samples, or restricting the radii of the balls to, for instance, a fraction of the range of values in a particular feature dimension. The purpose of these heuristics is to speed up the search for acceptable models by confining the search within the most interesting regions, or to guarantee a minimum model size.

Enrichment enforcement. The enrichment condition is relatively easy to enforce, as models biased towards one class are most common. But since the strength of the biases ($|d_{ij}(m)|$) determines the rate at which accuracy increases, we tend to prefer to use models with an enrichment degree further away from zero.

One way to implement this is to use a threshold on the enrichment degree to select weak models from the random stream so that they are of some minimum quality. In this way, one will be able to use a smaller collection of models to yield a classifier of the same level of accuracy. However, there are tradeoffs involved in doing this. For one thing, models of higher rating are less likely to appear in

the stream, therefore more random models have to be explored in order to find a sufficient number of higher quality weak models. And once the type of model is fixed and the value of the threshold is set, there is a risk that such models may never be found.

Alternatively, one can use the most enriched model found in a pre-determined number of trials. This also makes the time needed for training more predictable, and it permits a tradeoff between training time and quality of the weak models.

In enriching the model stream, it is important to remember that if the quality of weak models selected is allowed to get too high, there is a risk that they will become training set specific, that is, less likely to be projectable to unseen samples. This could present a problem since the projectability of the final classifier depends on the projectability of its component weak models.

Uniformity promotion. The uniformity condition is much more difficult to satisfy. Strict uniformity requires that every point be covered by the same number of weak models of every combination of per-class ratings. This is rather infeasible for continuous and unconstrained ratings.

One useful strategy is to use only weak models of a particular rating. In such cases, the ratings $r_i(m)$ and $r_j(m)$ are the same for all models m enriched for the discrimination between classes i and j, so we need only to make sure that each point is included in the same number of models. To enforce this, models can be created in groups such that each group partitions the entire space into a set of non-overlapping regions. An example is to use the leaves of a fully-split decision tree, where each leaf is perfectly enriched for one class, and each point is covered by exactly one leaf of each tree. For any pairwise discrimination between classes i and j, we can use only those leaves of the trees that contain only points of class i. In other words, $r_i(m)$ is always 1 and $r_j(m)$ is always 0. Constraints are put in the tree-construction process to guarantee some minimum projectability.

With other types of models, a first step to promote uniformity is to use models that are unions of small regions with simple boundaries. The component regions may be scattered throughout the space. These models have simple representations but can describe complicated class boundaries. They can have some minimum size and hence good projectability. At the same time, the scattered locations of component regions do not tend to cover large areas repeatedly.

A more sophisticated way to promote uniformity involves defining a measure of the lack of uniformity and an algorithm to minimize such a measure. The goal is to create or retain more models located in areas where the coverage is thinner. An example of such a measure is the count of those points that are covered by a less-than-average number of previously retained models. For each point x in the class c_0 to be positively enriched, we calculate, out of all previous models used for that class, how many of them have covered x. If the coverage is less than the average for class c_0, we call x a weak point. When a new model is created, we check how many such weak points are covered by the new model. The ratio of the set of covered weak points to the set of all the weak points is used as a merit score of how well this model improves uniformity. We can accept only those models with a score over a pre-set threshold, or take the model with the

best score found in a pre-set number of trials. One can go further to introduce a bias to the model generator so that models covering the weak points are more likely to be created. The later turns out to be a very effective strategy that led to good results in our experiments.

5.2 Alternative Discriminants and Approximate Uniformity

The method outlined above allows for rich possibilities of variations in SD algorithms. The variations may be in the design of the weak model generator, or in ways to enforce the enrichment and uniformity conditions. It is also possible to change the definition of the discriminant, or to use different kinds of ratings.

A variant of the discriminating function is studied in detail in [1]. In this variant, the ratings are defined as

$$r'_i(m) = \frac{|m \cap TR_i|}{|m \cap TR|},$$

for all i. It is an estimate of the posterior probability that a point belongs to class i given the condition that it is included in model m. The discriminant for class i is defined to be:

$$W_i(q) = \frac{\sum_{k=1,\ldots,p_i} C_m(q) r'_i(m)}{\sum_{k=1,\ldots,p_i} C_m(q)}.$$

where p_i is the number of models accumulated for class i.

It turns out that, with this discriminant, the classifier also approaches perfection asymptotically provided that an additional *symmetry* condition is satisfied. The symmetry condition requires that the ensemble includes the same number of models for all permutations of $(r'_1, r'_2, ..., r'_n)$. It prevents biases created by using more (i, j)-enriched models than (j, i)-enriched models for all pairs (i, j)[1]. Again, this condition may be enforced by using only certain particular permutations of the r' ratings, which is the basis of the *random decision forest* method[7][10]. This alternative discriminant is convenient for multi-class discrimination problems.

The SD theory establishes the mathematical concepts of enrichment, uniformity, and projectability of a weak model ensemble. Bounds on classification accuracy are developed based on strict requirements on these conditions, which is a mathematical idealization. In practice, there are often difficult tradeoffs among the three conditions. Thus it is important to understand how much of the classification performance is affected when these conditions are weakened. This is the subject of study in [3], where notions of near uniformity and weak indiscernibility are introduced and their implications are studied.

5.3 Structured Collections of Weak Models

As a constructive procedure, the method of stochastic discrimination depends on a detailed control of the uniformity of model coverage, which is outlined

but not fully published in the literature[17]. The method of random subspaces followed these ideas but attempted a different approach. Instead of obtaining weak discrimination and projectability through simplicity of the model form, and forcing uniformity by sophisticated algorithms, the method uses complete, locally pure partitions given by fully split decision trees[7][10] or nearest neighbor classifiers[11] to achieve strong discrimination and uniformity, and then explicitly forces different generalization patterns on the component classifiers. This is done by training large capacity component classifiers such as nearest neighbors and decision trees to fully fit the data, but restricting the training of each classifier to a coordinate subspace of the feature space where all the data points are projected, so that classifications remain invariant in the complement subspace. If there is no ambiguity in the subspaces, the individual classifiers maintain maximum accuracy on the training data, with no cases deliberately chosen to be sacrificed, and thus the method does not run into the paradox of sacrificing some training points in the hope for better generalization accuracy. This is to create a collection of weak models in a structured way.

However the tension among the three factors persists. There is another difficult tradeoff in how much discriminating power to retain for the component classifiers. Can every one use only a single feature dimension so as to maximize invariance in the complement dimensions? Also, projection to coordinate subspaces sets parts of the decision boundaries parallel to the coordinate axes. Augmenting the raw features by simple transformations[10] introduces more flexibility, but it may still be insufficient for an arbitrary problem. Optimization of generalization performance will continue to depend on a detailed control of the projections to suit a particular problem.

6 Conclusions

The theory of stochastic discrimination identifies three and only three sufficient conditions for a classifier to achieve maximum accuracy for a problem. These are just the three elements long believed to be important in pattern recognition: discrimination power, complementary information, and generalization ability. It sets a foundation for theories of ensemble learning. Many current questions on classifier combination can have an answer in the arguments of the SD theory: What is good about building the classifier on weak models instead of strong models? Because weak models are easier to obtain, and their smaller capacity renders them less sensitive to sampling errors in small training sets[20][21], thus they are more likely to have similar coverage on the unseen points from the same problem. Why are many models needed? Because the method relies on the law of large numbers to reduce the variance of the discriminant on each single point. How should these models complement each other? The uniformity condition specifies exactly what kind of correlation is needed among the individual models.

Finally, we emphasize that the accuracy of SD methods is not achieved by intentionally limiting the VC dimension[20] of the complete system; the com-

bination of many weak models can have a very large VC dimension. It is a consequence of the symmetry relating probabilities in the two spaces, and the law of large numbers. It is a structural property of the topological space given by the points and their combinations. The observation of this symmetry and its relationship to ensemble learning is a deep insight of Kleinberg's that we believe can lead to a better understanding of other ensemble methods.

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References

1. R. Berlind, *An Alternative Method of Stochastic Discrimination with Applications to Pattern Recognition*, Doctoral Dissertation, Department of Mathematics, State University of New York at Buffalo, 1994.
2. L. Breiman, "Bagging predictors," *Machine Learning*, **24**, 1996, 123-140.
3. D. Chen, *Estimates of Classification Accuracies for Kleinberg's Method of Stochastic Discrimination in Pattern Recognition*, Doctoral Dissertation, Department of Mathematics, State University of New York at Buffalo, 1998.
4. T.G. Dietterich, G. Bakiri, "Solving multiclass learning problems via error-correcting output codes," *Journal of Artificial Intelligence Research*, **2**, 1995, 263-286.
5. Y. Freund, R.E. Schapire, "Experiments with a New Boosting Algorithm," *Proceedings of the Thirteenth International Conference on Machine Learning*, Bari, Italy, July 3-6, 1996, 148-156.
6. L.K. Hansen, P. Salamon, "Neural network ensembles," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **PAMI-12**, 10, October 1990, 993-1001.
7. T.K. Ho, "Random Decision Forests," *Proceedings of the 3rd International Conference on Document Analysis and Recognition*, Montreal, Canada, August 14-18, 1995, 278-282.
8. T.K. Ho, "Multiple classifier combination: Lessons and next steps," in A. Kandel, H. Bunke, (eds.), *Hybrid Methods in Pattern Recognition*, World Scientific, 2002.
9. T.K. Ho, E.M. Kleinberg, "Building Projectable Classifiers of Arbitrary Complexity," *Proceedings of the 13th International Conference on Pattern Recognition*, Vienna, Austria, August 25-30, 1996, 880-885.
10. T.K. Ho, "The random subspace method for constructing decision forests," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **20**, 8, August 1998, 832-844.
11. T.K. Ho, "Nearest Neighbors in Random Subspaces," *Proceedings of the Second International Workshop on Statistical Techniques in Pattern Recognition*, Sydney, Australia, August 11-13, 1998, 640-648.
12. T.K. Ho, J. J. Hull, S.N. Srihari, "Decision Combination in Multiple Classifier Systems," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **PAMI-16**, 1, January 1994, 66-75.

13. Y.S. Huang, C.Y. Suen, "A method of combining multiple experts for the recognition of unconstrained handwritten numerals," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **PAMI-17**, 1, January 1995, 90-94.
14. J. Kittler, M. Hatef, R.P.W. Duin, J. Matas, "On combining classifiers," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **PAMI-20**, 3, March 1998, 226-239.
15. E.M. Kleinberg, "Stochastic Discrimination," *Annals of Mathematics and Artificial Intelligence*, **1**, 1990, 207-239.
16. E.M. Kleinberg, "An overtraining-resistant stochastic modeling method for pattern recognition," *Annals of Statistics*, **4**, 6, December 1996, 2319-2349.
17. E.M. Kleinberg, "On the algorithmic implementation of stochastic discrimination," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **PAMI-22**, 5, May 2000, 473-490.
18. E.M. Kleinberg, "A mathematically rigorous foundation for supervised learning," in J. Kittler, F. Roli, (eds.), *Multiple Classifier Systems*, Lecture Notes in Computer Science 1857, Springer, 2000, 67-76.
19. L. Lam, C.Y. Suen, "Application of majority voting to pattern recognition," *IEEE Transactions on Systems, Man, and Cybernetics*, **SMC-27**, 5, September/October 1997, 553-568.
20. V. Vapnik, *Estimation of Dependences Based on Empirical Data*, Springer-Verlag, 1982.
21. V. Vapnik, *Statistical Learning Theory*, John Wiley & Sons, 1998.
22. D.H. Wolpert, "Stacked generalization," *Neural Networks*, **5**, 1992, 241-259.