Frames Everywhere

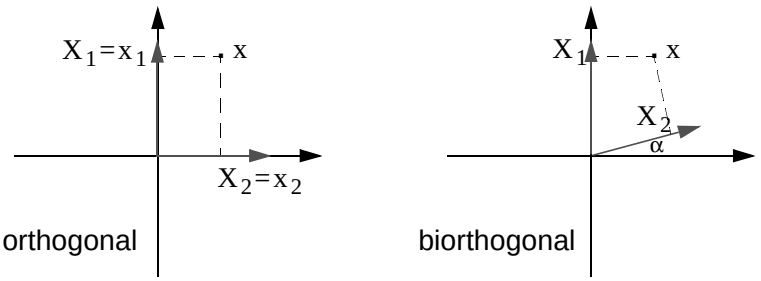
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A Simple Example

Transmit two real numbers x_1 and x_2 over the Internet

- make a real vector $x = (x_1, x_2)$

(1) Transmit projections



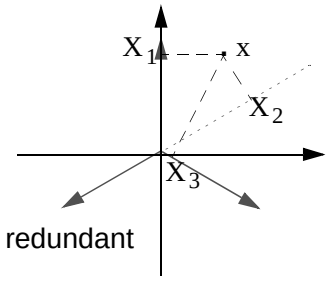
- in either case getting both leads to perfect recovery of x
- in either case losing one leads to the loss of one subspace, no recovery
- once we start quantizing, biorthogonal is worse
- not robust to losses
- however, no redundancy (no penalty in rate)

(2) Try something else

- send both projections twice x_1, x_2, x_1, x_2
- more robust, losing one ensures perfect recovery
- more expensive (100% penalty in rate)
- redundancy 100%

(3) Compromise

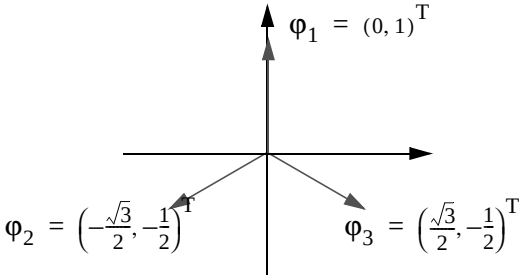
- choose more vectors "at random"
- redundancy built in the representation
- losing one still perfect recovery
- more robust then (1) but less than (2)
- more expensive than (1) but less than (2)
- redundancy 50%



Let us investigate this system further

- are the vectors linearly independent? NO

$$\varphi_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\varphi_2 - \varphi_3$$



- are they complete? can we write any vector $x \in \Re^2$ as a linear combination of φ_i? YES

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{2}{3} x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{2}{3}\left(-\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)\begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix} + \frac{2}{3}\left(\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)\begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

$\langle \varphi_1, x \rangle$ $\langle \varphi_2, x \rangle$ $\langle \varphi_3, x \rangle$

$$x = \frac{2}{3}\sum_{i=1}^{3} \langle \varphi_i, x \rangle \varphi_i = \sum_{i=1}^{3} \langle \varphi_i, x \rangle \tilde{\varphi}_i$$

- we normalized the expansion by the "redundancy" factor

Redundancy ratio: $A = \frac{3}{2}$

Conservation of energy?

$$\|X\|^2 = \sum_{i=1}^{3} |\langle \varphi_i, x \rangle|^2 = x_2^2 + \left(-\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)^2 + \left(\frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2\right)^2$$

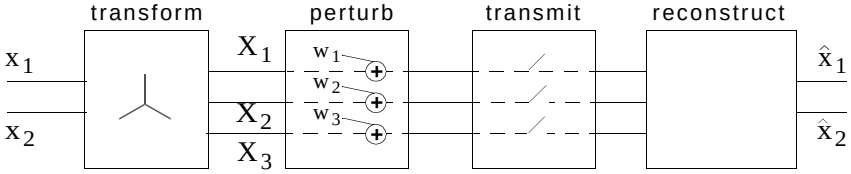
$$= \frac{3}{2}(x_1^2 + x_2^2) = \frac{3}{2}\|x\|^2$$

- intuitive: similar to Parseval, but with a factor 3/2
- similar to ONB, but for 3/2

A bit more particular than a random collection of 3 vectors

- it is a frame
- uniform tight frame
- "Mercedes-Benz" frame 

Make it more realistic...
...perturb the system



Perturbation

- additive white noise $E[w_i] = 0$ and $E[w_i w_j] = \delta_{ij}\sigma^2$
- error of the reconstruction

$$MSE_{MB} = \frac{1}{2}E\|x - \hat{x}\|^2 = \frac{1}{2}E\left\|\frac{2}{3}\sum_{i=1}^{3} w_i\varphi_i\right\|^2 = \frac{2}{3}\sigma^2$$

- with an ONB

$$MSE_{ONB} = \sigma^2 = \frac{3}{2}MSE_{MB}$$

- with a biorthogonal basis (minimum for α=0, ONB)

$$MSE_{BI} = \sigma^2\frac{2}{1 + \cos 2\alpha}$$

Did we really select this frame "randomly"?

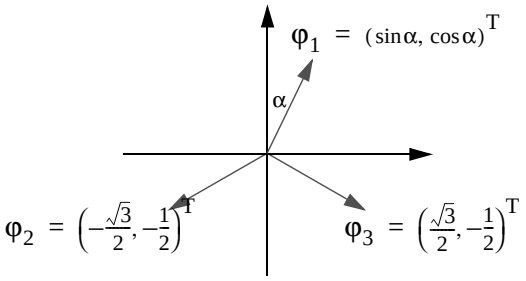
Choose another frame with norm-1 vectors

- perturb the first vector by α radians clockwise
- error of the reconstruction

$$\mathrm{MSE} = \frac{1}{2}\sigma^2 \frac{3}{9/4 - (\sin\alpha)^2}$$

- minimum achieved for

$$\alpha = 0!$$



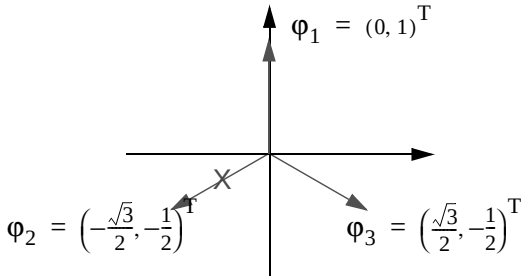
Theorem: MB frame minimizes the MSE among all other frames with norm-1 vectors

- robustness to quantization noise

...not that "random" after all

How about losses now?

- suppose φ_2 is lost



- we can still reconstruct (just not that elegantly)

$$x = \sum_{i=1,3} \langle \varphi_i, x \rangle \hat{\varphi}_i \quad \text{with} \quad \hat{\varphi}_1 = \left(\frac{1}{\sqrt{3}}, 1\right)^T \quad \text{and} \quad \hat{\varphi}_3 = \left(\frac{2}{\sqrt{3}}, 0\right)^T$$

- error of the reconstruction

$$\mathrm{MSE}_{\{2\}} = \frac{4}{3}\sigma^2 = 2\mathrm{MSE}_{\mathrm{MB}}$$

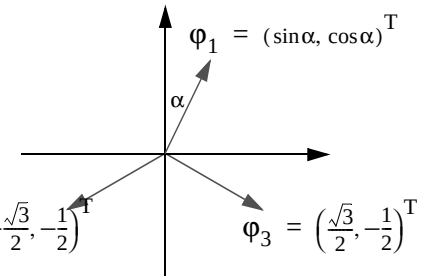
Is there another frame with a lower MSE?

- does it really matter what we start with if there are losses?
- perturb the first vector as before

$$MSE_{\{1\}} = \frac{4}{3}\sigma^2$$

$$MSE_{\{2\}} = \sigma^2 \frac{4}{2 + \cos 2\alpha + \sqrt{3}\sin 2\alpha}$$

$$MSE_{\{3\}} = \sigma^2 \frac{4}{2 + \cos 2\alpha - \sqrt{3}\sin 2\alpha}$$



$$MSE_1 = \frac{4}{9}\sigma^2 \frac{15 + 10\cos 2\alpha + 2\cos 4\alpha}{(1 + 2\cos 2\alpha)^2}$$

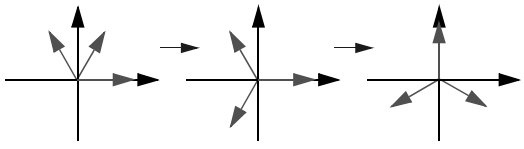
- minimum achieved again for $\alpha = 0$!
- so it does matter where we start from

Theorem: MB frame minimizes the MSE averaged over all possible erasures of one coefficient, among all other frames with norm-1 vectors

One more interesting tidbit

- take the following 3-dimensional basis $F_3 = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{3}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{3}} \end{bmatrix}$
- project it onto a 2-dimensional subspace using the following orthogonal projection $P^2 = P$ and $P^T = P$ $P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$
- what do we get? [NTh]

$$F_3 \cdot P = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & 0 \\ \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & 0 \end{bmatrix}$$



Outline

Frames

- what?
- why?
- where?

Tutorial on frames

- frames in $\Re^N$
- particular application: robustness to losses

Where?

- oversampled perfect reconstruction filter banks
- resilience to additive noise
- in multiple-antenna wireless systems
- frames and quantum measurement

Acknowledgments

Quantized frame expansions with erasures

- Vivek Goyal

- Martin Vetterli
- Pier Luigi Dragotti
- Jon Kelner
- Aurelie Lozano
- Mike Andrews

History

[Naimark, 1946]

- compression of bases to obtain frames

[Duffin & Schaeffer, 1952]

- abstract framework for the idea of time-frequency decomposition of Gabor
- frames for Hilbert spaces

[Gabor, 1946r]

- time-frequency atoms

[Daubechies, Grossman & Meyer, 1986]

- applications to wavelet and Gabor transforms

[Casazza, 2000]

- beautiful tutorial on the art of frame theory

[Heil & Walnut, 1989]

- Gabor (Weyl-Heisenberg frames)

[Daubechies, 1990, 1992]

- excellent introduction into frames with emphasis on Gabor and wavelet frames

Applications (by no means complete...)

[Cvetkovic & Vetterli, 1998]

- oversampled perfect reconstruction filter banks

[Daubechies, 1992]

- numerical stability of reconstruction

[Goyal, Vetterli & Thao, 1998]

- resilience to quantization

[Benedetto, Pfander, Colella, 1995, 1998]

- freedom to capture significant signal characteristics

[Hochwald, Marzetta, Richardson, Sweldens, Urbanke, 2000]

- design of unitary space-time constellations for multiple wireless antenna systems

[Goyal, Kovacevic, Kelner, Dragotti, Lozano, Vetterli, 1998-2001]

- robust transmission over the Internet and wireless

[Soljanin, 2000]

- frames and quantum information theory

[Eldar & Forney, 2001]

- frames and quantum measurements

Basics on frames

Def: A set of vectors $\{\varphi_i\}_{i \in I}$ is a frame in a Hilbert space H, if there exist $0 < A \le B < \infty$, such that, for all $x \in H$

$$A\|x\|^2 \le \sum_{i \in I} |\langle \varphi_i, x \rangle|^2 \le B\|x\|^2$$

- A,B — frame bounds

- A=B — tight frame (TF)
- $\|\varphi_i\| = c$, for all i — uniform frame (UF)
- A=B=1 — normalized tight frame (NTF)

- all and norm 1 — uniform normalized tight frame with norm 1 (UNTF1)
  ONB

Basics on frames

Frame operator: $F: H \to l_2(\Im)$

transform coefficient $(Fx)_i = X_i = \langle \varphi_i, x \rangle$

Dual frame: $\{\tilde{\varphi}_i\}_{i \in I}$

$\tilde{\varphi}_i = (F^*F)^{-1}\varphi_i$

Dual frame operator:

$$\tilde{F} = F(F^*F)^{-1}$$

$$B^{-1}\|x\|^2 \le \sum_{i \in I} |\langle \tilde{\varphi}_i, x \rangle|^2 \le A^{-1}\|x\|^2$$

In operator notation

$AI \le F^*F \le BI$ and $B^{-1}I \le (\tilde{F}^*\tilde{F})^{-1} = (F^*F)^{-1} \le A^{-1}I$

Reconstruction

$$x = \sum_{i \in I} \langle \varphi_i, x \rangle \tilde{\varphi}_i = \sum_{i \in I} \langle \tilde{\varphi}_i, x \rangle \varphi_i$$

Tight frames (TF)

$A=B$

$$F^*F = AI \quad \text{or} \quad \|X\|^2 = A\|x\|^2$$

- scaled conservation of energy

Dual frame:

$$\tilde{\varphi}_i = (F^*F)^{-1}\varphi_i = A^{-1}\varphi_i$$

Reconstruction

$$x = \sum_i \langle \varphi_i, x\rangle \tilde{\varphi}_i = A^{-1}\sum_i \langle \varphi_i, x\rangle \varphi_i$$

Uniform frames (UF)

$\|\varphi_i\| = c$, **for all** i

Uniform frames with norm 1 (UF1)

$\|\varphi_i\| = 1$, **for all** i

Uniform tight frames (UTF)

$A=B$ **and** $\|\varphi_i\| = c$, **for all** i

$$F^*F = AI$$

Normalized tight frames (NTF)

$A=B=1$

$$F^*F = I$$

Theorem: [Naimark (NTh)] A family $\{\varphi_i\}_{i\in I}$ in a Hilbert space H is a normalized tight frame for H if and only if there is a larger Hilbert space $H \subset K$ and an orthonormal basis $\{e_i\}_{i\in I}$ for K so that the orthogonal projection P of K onto H satisfies:

$$Pe_i = \varphi_i, \text{ for all } i \in I.$$

Uniform normalized tight frames (UNTF)

$A=B=1$ **and** $\|\varphi_i\| = c$**, for all i**

$$\|\varphi_i\|^2 = \frac{M}{N}$$

Uniform normalized tight frames with norm 1 (UNTF1)

$A=B=1$ **and** $\|\varphi_i\| = 1$**, for all i**

$$F^*F = F F^* = I$$

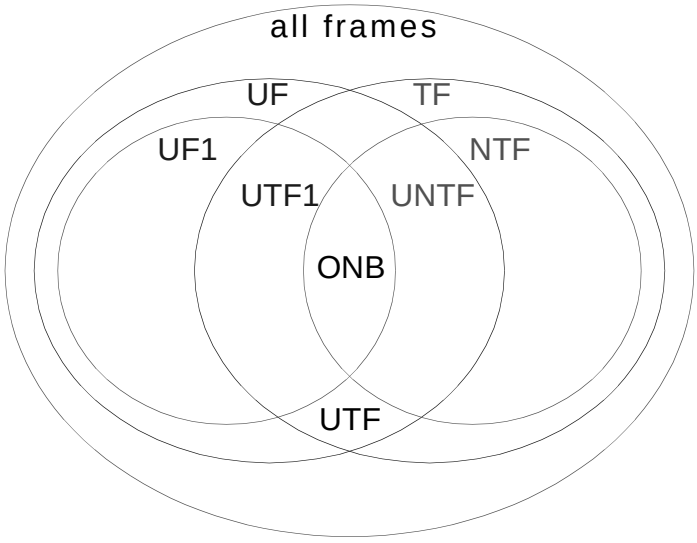
- this is an orthonormal basis

Dual frame:

$$\tilde{\varphi}_i = (F^*F)^{-1}\varphi_i = \varphi_i$$

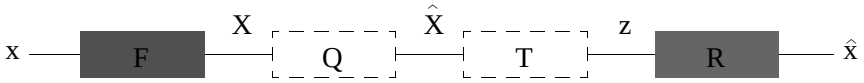
Reconstruction

$$x = A^{-1}\sum_i \langle \varphi_i, x\rangle \varphi_i = \sum_i \langle \varphi_i, x\rangle \varphi_i$$



- UF uniform frames
- UF1 uniform frames with norm 1
- UTF1 uniform tight frames with norm 1
- UTF uniform tight frames
- ONB orthonormal bases
- TF tight frames
- NTF normalized tight frames
- UNTF uniform normalized tight frames

Frames



Finite-dimensional frames in $\Re^N$

A finite set of vectors $\{\varphi_i\}_{i=1}^{M}$ **in** $\Re^N$**, with** $M>N$

$$A\|x\|^2 \le \sum_{i=1}^{M} |\langle \varphi_i, x\rangle|^2 \le B\|x\|^2$$

Frame operator F **adjoint operator** $F^* = F^T$

$$F = \begin{bmatrix} \varphi_1^T \\ \vdots \\ \varphi_M^T \end{bmatrix}_{M\times N} \qquad F^T = \begin{bmatrix} \varphi_1 \,..\, \varphi_M \end{bmatrix}_{N\times M}$$

Dual frame operator, $\tilde{\varphi}_i = (F^*F)^{-1}\varphi_i$, $\tilde{F} = F(F^TF)^{-1}$

$$\tilde{F} = \begin{bmatrix} \varphi_1^T(F^TF)^{-1} \\ \vdots \\ \varphi_M^T(F^TF)^{-1} \end{bmatrix}_{M\times N}$$

Conservation of energy $AI \le F^TF \le BI$

Finite-dimensional frames in $\Re^N$... ...properties

Denote by λ_i the eigenvalues of F^TF, v_i the eigenvectors

Result: For any frame, the eigenvalues of F^TF lie in $[A, B]$.

Since all the eigenvalues are strictly positive $\Rightarrow$ F^TF is invertible $\Rightarrow$ dual frame exists

Result: For any frame, the sum of eigenvalues of F^TF equals the sum of the lengths of the frame vectors

Proof:
$$\sum_{i=1}^{N} \lambda_i = \mathrm{tr}(F^TF) = \mathrm{tr}(FF^T) = \sum_{i=1}^{M} \varphi_i^T \varphi_i = \sum_{i=1}^{M} \|\varphi_i\|^2$$

Finite-dimensional tight frames in $\Re^N$

A=B

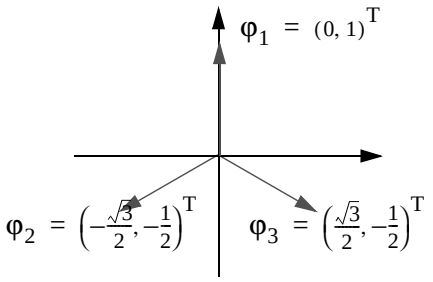
$F^TF = AI_N \Rightarrow F^TF$ has eigenvalue A with multiplicity N

Result: The sum of eigenvalues of F^TF equals the sum of the lengths of the frame vectors:
$$\sum_{i=1}^{N} \lambda_i = NA = \sum_{i=1}^{M} \|\varphi_i\|^2$$

Result: [NTh] F is the first N columns of an $M \times M$ matrix with orthogonal columns.

Example: N=2, M=3, MB frame

$$F = \begin{bmatrix} 0 & 1 \\ -\sqrt{3}/2 & -1/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$



Finite-dimensional uniform tight frames in $\Re^N$

A=B **and** $\|\varphi_i\| = 1$**, for all** i

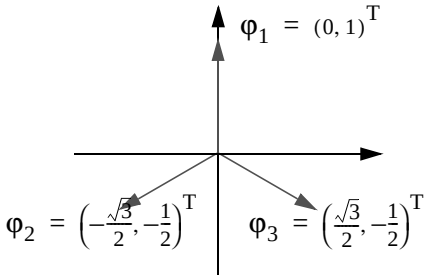
Result: The sum of eigenvalues of $F^T F$ equals the sum of the lengths of the frame vectors:

$$\sum_{i=1}^{N} \lambda_i = NA = \sum_{i=1}^{M} \|\varphi_i\|^2 = M \quad \Rightarrow \quad A = \frac{M}{N} \text{ redundancy}$$

Result: [NTh] F is the first N columns of $\sqrt{\frac{M}{N}}U$, with U some MxM unitary matrix.

Example:

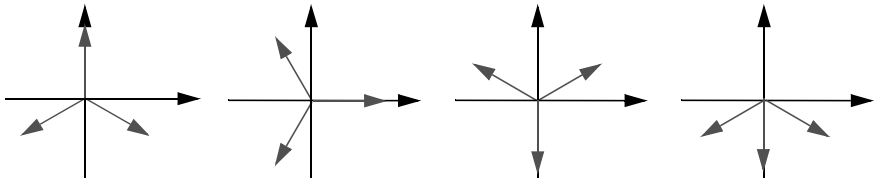
- $\|\varphi_i\| = 1$**, for** i=1,2,3
- 3 vectors in 2-dim space $\Rightarrow$ redundancy 3/2
- $x = \frac{2}{3}\sum_i \langle \varphi_i, x \rangle \varphi_i$
- $\|X\|^2 = \frac{3}{2}\|x\|^2$



Classifying uniform tight frames in $\Re^N$

Def: An equivalence class consists of all frames resulting from a rigid rotations or reflections of the entire frame, as well as negation of individual vectors.

Example: N=2, M=3, MB frame



original frame | rotation by 270 degrees | reflection around horizontal axis | negation of φ_1

Classifying uniform tight frames in $\Re^N$

New frame vectors

$$\gamma_i = \sigma_i U \varphi_i, \quad i = 1, \ldots, M$$

where $\sigma_i = \pm 1$ and U is a unitary matrix

New frame operator

$$G = \Sigma F U^T$$

where $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_M)$

Result: If F is tight, then G is tight as well.

Classifying uniform tight frames in $\Re^N$

Theorem: There is exactly one equivalence class of UTF for $M=N+1$.

Proof: [Goyal, Kovacevic & Kelner, 2001]

Example: $N=2$, $M=3$

- only one class characterized by MB frame
- obtained all from a harmonic tight frame (HTF) with $N=2$, $M=3$

For $M=N+1$, all UTF are obtained from a HTF with N, $M=N+1$.

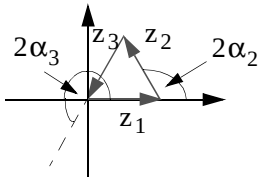
Characterization of uniform tight frames in $\Re^2$

Theorem: The following are equivalent:

$\{\varphi_i = (\cos\alpha_i, \sin\alpha_i)^T\}_{i=1}^M$ is a UTF $\Leftrightarrow$ $\sum_i z_i = 0$, $z_i = e^{j2\alpha_i}$, all i

Example: M=3

$$z_1 + z_2 + z_3 = 0, \ \|z_i\| = 1$$



$$2\alpha_1 = 2k\pi \qquad \alpha_1 = k\pi$$

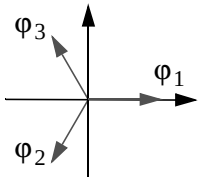
$$2\alpha_2 = 2m\pi + \frac{2\pi}{3} \qquad \alpha_2 = m\pi + \frac{\pi}{3}$$

$$2\alpha_3 = 2n\pi + \frac{4\pi}{3} \qquad \alpha_3 = n\pi + \frac{2\pi}{3}$$

Choose $k = 0, m = 1, n = 0,$

$$\alpha_1 = 0, \ \alpha_2 = \pi + \frac{\pi}{3}, \ \alpha_3 = \frac{2\pi}{3}$$

leading to MB frame!

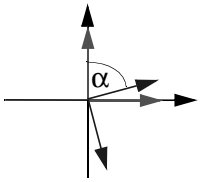


Characterization of uniform tight frames in $\Re^2$

Example: M=4

$$z_1 + z_2 + z_3 + z_4 = 0, \ \|z_i\| = 1$$

Result: All UTF frames with N=2, M=4, are unions of two orthonormal bases, parameterized by the angle between them.



Construction of uniform tight frames in $\Re^N$

[NTh] **Taking the first N columns of an $M \times M$ unitary matrix**

$\Rightarrow$ TF

$\Rightarrow$ not necessarily a UTF

Example: Harmonic tight frames (HTF)

- [NTh] projection of an $M \times M$ DTFT matrix onto an N-dimensional subspace
- complex version

$$(\varphi_{k+1})_{i+1} = \frac{1}{\sqrt{N}} W_M^{ki}, \qquad \begin{matrix} i = 0, \ldots, N\text{-}1 \\ k = 0, \ldots, M\text{-}1 \end{matrix},$$

- real version

  N even

$$\varphi_{k+1} = \sqrt{\frac{2}{N}}\left[\cos\frac{k\pi}{M}, \cos\frac{3k\pi}{M}, \ldots, \cos\frac{(N-1)k\pi}{M}, \sin\frac{k\pi}{M}, \sin\frac{3k\pi}{M}, \ldots, \sin\frac{(N-1)k\pi}{M}\right]$$

  N odd

$$\varphi_{k+1} = \sqrt{\frac{2}{N}}\left[\frac{1}{\sqrt{2}}, \cos\frac{2k\pi}{M}, \cos\frac{4k\pi}{M}, \ldots, \cos\frac{(N-1)k\pi}{M}, \sin\frac{2k\pi}{M}, \sin\frac{4k\pi}{M}, \ldots, \sin\frac{(N-1)k\pi}{M}\right]$$

Harmonic TF

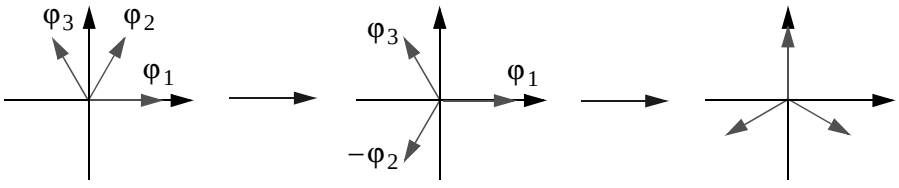
Example: $N=2$, $M=3$

- complex version

$$F = \frac{1}{\sqrt{2}}\begin{bmatrix} W_3^0 & W_3^0 \\ W_3^0 & W_3^1 \\ W_3^0 & W_3^2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & e^{j2\pi/3} \\ 1 & e^{j4\pi/3} \end{bmatrix}$$

- real version

$$F = \begin{bmatrix} \cos 0 & \sin 0 \\ \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$



Reconstruction from frame coefficients in $\Re^N$

We have to solve

$$G_{N\times M}\cdot F_{M\times N} = I$$

$$V\begin{bmatrix}\sigma_1^{-1} & & \\ & \vdots & \\ & & \sigma_N^{-1}\end{bmatrix}\Big|\, W\Big] U^T U \begin{bmatrix}\sigma_1 & & \\ & \vdots & \\ & & \sigma_N \\ -- & -- & -- \\ & 0 & \end{bmatrix} V^T = I$$

- many left inverses
- $W = 0$ gives a special left inverse: "pseudoinverse"

$$G = F^\dagger = (F^T F)^{-1} F^T$$

check: $F^\dagger F = [(F^T F)^{-1} F^T]F = I_N$

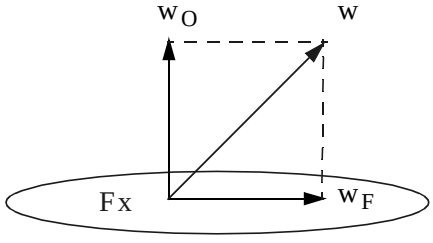
special inverse equal to the transpose of the dual

$$F^\dagger = \tilde{F}^T = (F(F^T F)^{-1})^T$$

Reconstruction from frame coefficients in $\Re^N$

What is so special about $F^\dagger$?

- Fx lies in a particular N-dimensional subspace
- it does not matter what G does to vectors orthogonal to that subspace
- thus, if $X = Fx$ is not known exactly, $F^\dagger$ eliminates the influence of errors orthogonal to the range of F



$$\hat{X} = X + w$$

$$F^\dagger\hat{X} = F^\dagger X + F^\dagger w =$$

$$= F^\dagger Fx + F^\dagger w_F =$$

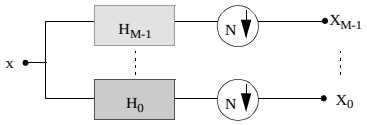
$$= x + F^\dagger w_F$$

Equivalence of finite-dimensional frames in $\Re^N$ and oversampled filter banks

- frame $\Phi = \{\varphi_i\}_{i=1}^{M}$ in $\Re^N$, with $M>N$ with a frame operator F

$\Leftrightarrow$

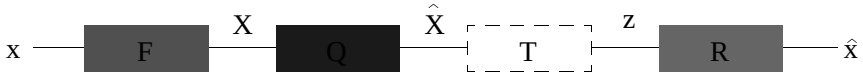
- rectangular block-transform filter bank, that is, a filter bank with:
  M channels,
  sampling by N and
  M filters of length N



$$X = \begin{bmatrix} \vdots & \vdots \\ \cdots & F\ 0 \cdots \\ \cdots & 0\ F \cdots \\ \vdots & \vdots \end{bmatrix} x$$

$$F = \begin{bmatrix} \varphi_1^T \\ \vdots \\ \varphi_M^T \end{bmatrix}_{M \times N} = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \cdots & \cdots & \cdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} = H_p = \begin{bmatrix} H_{0,0} & \cdots & H_{0,N-1} \\ \cdots & \cdots & \cdots \\ H_{M-1,0} & \cdots & H_{M-1,N-1} \end{bmatrix}$$

Quantized frame expansions in $\Re^N$



Quantized frame expansions in $\Re^N$

Quantization models

- quantization noise $w = \hat{X} - X$ is tricky
- use the model $E[w_i] = 0$ and $E[w_i w_j] = \delta_{ij}\sigma^2$
- deterministic quantization:
  model accurate with piecewise constant approximation
  no way to justify independence between x and w
- randomized (subtractive dithered) quantization
  model accurate
  signal x and noise w independent

Linear reconstruction

- we wish to estimate x given $\hat{X} = Fx + w$
  intuition: minimize residual $\|F\hat{x} - \hat{X}\|_2$
  solution: $\hat{x} = F^\dagger \hat{X} = F^\dagger(Fx + w)$
- reconstruction formula

$$\hat{x} = \sum_{k=1}^{M} \tilde{X}_k \tilde{\varphi}_k = \sum_{k=1}^{M} (\langle x, \varphi_k \rangle + w_k)\tilde{\varphi}_k, \qquad x - \hat{x} = -\sum_{k=1}^{M} w_k \tilde{\varphi}_k$$

Quantized frame expansions in $\Re^N$

MSE of linear reconstruction under noise model

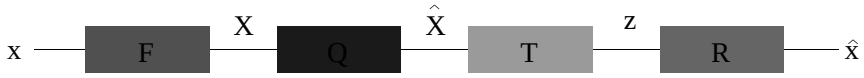
Theorem: Encode with a uniform frame and assume linear reconstruction and the noise model. Then MSE is minimum is and only if the frame is tight and is equal to

$$MSE_0 = \frac{N}{M}\sigma^2$$

Theorem: Bounds on MSE:

- any frame: $B^{-1}\sigma^2 \le MSE \le A^{-1}\sigma^2$
- UF: $(N/M)\sigma^2 \le MSE \le A^{-1}\sigma^2$
- UTF1: $MSE = (N/M)\sigma^2 = \sigma^2/r$

Introducing erasures



Introducing erasures

First question: after deletion of elements, is Φ still a frame?

- deletion of more than $(M\text{-}N)$ vectors: no
- deletion of less than or equal to $(M\text{-}N)$ vectors: not necessarily

How about a TF?

- still not necessarily

$$\begin{bmatrix} 0 & \sqrt{1/3} & \sqrt{2/3} \\ 0 & -\sqrt{1/3} & \sqrt{2/3} \\ 0 & 1 & 0 \\ \sqrt{5/6} & 0 & \sqrt{1/6} \\ -\sqrt{5/6} & 0 & \sqrt{1/6} \end{bmatrix}$$

Theorem: Let Φ be an HTF. Then any subset of N or more vectors from Φ forms a frame.

Theorem: Let Φ be a UTF. Deletion of any element results in a frame. The new frame has $A_1 = M/N\text{-}1$, $B_1 = M/N$, as its bounds.

Introducing erasures

Theorem: Encode with a uniform frame and assume linear reconstruction and the noise model. The MSE averaged over all possible erasures of one is minimum if and only if the starting frame is tight.

- thus, we consider only UTF

MSE with one erasure

- assume erased component is $\langle x, \varphi_1 \rangle$
- F_1 is the new frame operator
- the elements of the dual frame are

$$\tilde{\varphi}_k^{(1)} = (F_1^T F_1)^{-1} \varphi_k$$

- MSE

$$MSE_1 = \left(1 + \frac{1}{M-N}\right) MSE_0, \text{ with } MSE_0 = \frac{N}{M}\sigma^2$$

Introducing erasures

MSE with e erasures

- assume erased component are $\langle x, \varphi_1 \rangle, \ldots, \langle x, \varphi_e \rangle$
- $\varphi = [\varphi_1, \ldots, \varphi_e]$
- F_E is the new frame operator
- the elements of the dual frame are

$$\tilde{\varphi}_k^{(E)} = (F_E^T F_E)^{-1} \varphi_k$$

- eigenvalues of $\varphi^T \varphi$ are μ_i
- MSE

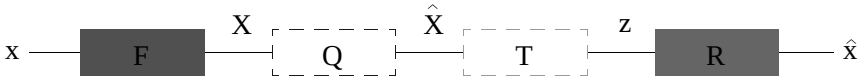
$$MSE_E = \left(1 + \sum_{i=1}^{e} \frac{\mu_i}{M - N\mu_i}\right) MSE_0, \text{ with } MSE_0 = \frac{N}{M}\sigma^2$$

- orthogonal erasures

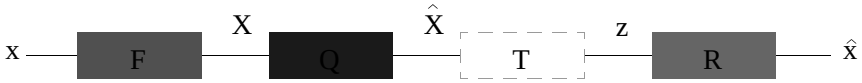
$$MSE_e = \left(1 + \frac{e}{M-N}\right) MSE_0$$

Robust transmission using frames

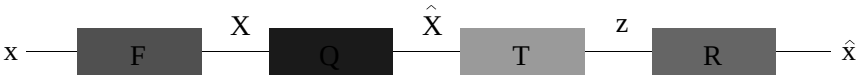
Summary of optimality knowledge



• all frames are optimal



• UF1 is optimal if and only if it is tight



• one erasure

$x \in \Re^N$: UF1 is optimal if and only if it is tight

$\Rightarrow$ **use tight frames!**

Other applications

• oversampled perfect reconstruction filter banks
• resilience to additive noise
• in multiple-antenna wireless systems
• surprise

Oversampled filter banks -- frames in $l_2(\mathfrak{I})$

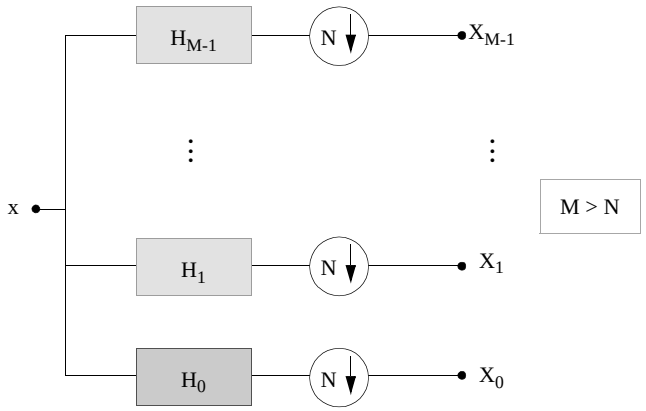
[Cvetkovic & Vetterli, 1998]

What if the input signal is a sequence?

- signal $x \in l_2(\mathfrak{I})$

Can we still implement a frame expansion using a filter bank?

- yes, except that filters (basis functions) are now longer



- we can do the same analysis as for $x \in \mathfrak{R}^N$

Oversampled filter banks -- frames in $l_2(\mathfrak{I})$

Polyphase representation of ith filter

$H_i(\omega) = [H_{i_1}(\omega), \ldots, H_{i_N}(\omega)]^*$

Frame operator is the polyphase matrix $[H_p(\omega)]_{ij} = H_{ij}(\omega)$

Result: [Cvetkovic & Vetterli] A filter bank implements a frame expansion if and only if $H_p(\omega)$ is of full rank on the unit circle

Result: [Cvetkovic & Vetterli] A filter bank implements a tight frame expansion if and only if $[H_p(\omega)]^* H_p(\omega) = A \cdot I_N$

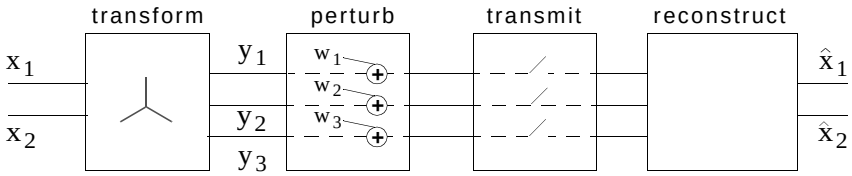
Call $\lambda_i(\omega)$ spectral eigenvalues of $H^T(\omega)H(\omega)$

- UF: integral sum equals M
- TF: $\lambda_i(\omega) = A$
- UTF: $\lambda_i(\omega) = M/N$

Pyramid coding [Burt & Adelson]

- oldest example of frames in image processing

Resilience to additive noise



Perturbation

- additive white noise $E[w_i] = 0$ and $E[w_i w_j] = \delta_{ij}\sigma^2$
- error of the reconstruction

$$MSE_{MB} = \frac{1}{2}E\left\|x - \hat{x}\right\|^2 = \frac{1}{2}E\left\|\frac{2}{3}\sum_{i=1}^{3} w_i\varphi_i\right\|^2 = \frac{2}{3}\sigma^2$$

- with an ONB

$$MSE_{ONB} = \sigma^2 = \frac{3}{2}MSE_{MB}$$

- with a biorthogonal basis (minimum for α=0, ONB)

$$MSE_{BI} = \sigma^2\frac{2}{1+\cos 2\alpha}$$

Multiple-antenna wireless systems

Design of unitary space-time constellations

[Hochwald, Marzetta, Richardson, Sweldens, Urbanke, 2000]

- systems using multiple antennas can have high channel capacities (M transmit antennas)
- unitary space-time signals are well tailored for flat fading channels where neither the transmitter nor the receiver knows the fading coefficients

$S_l = \sqrt{T}\Phi_l$, $l = 1, \ldots, L$ with $\Phi_1^\dagger \Phi_1 = \ldots = \Phi_L^\dagger \Phi_L = I$, Φ are $T \times M$ matrices and $M \le T$

- simple to decode and achieve capacity when used in conjunction with coding
- how to design large constellations?

Example: $M=1$, L vectors in T-dimensional space, $L \gg T$

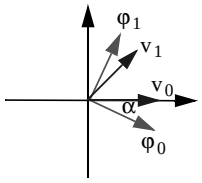
- overcomplete representation, unit-norm vectors $\Rightarrow$ UF
- [NTh] used HTF

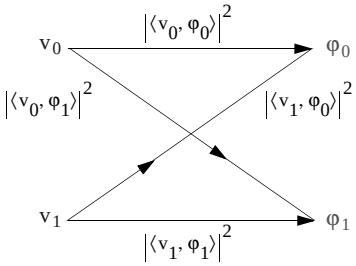
Frames and quantum information theory

[Soljanin, 2000]

Find as much information as possible about source through measurements

- we are given source vectors v_0 and v_1
- measurements are projections -- choice of an ONB





- most information obtained with BSC

Get even more information by obtaining more measurements in a higher-dimensional space

- ONB in a higher-dimensional space
- however, we still look at it in the two-dimensional space
- [NTh] projection is an NTF

Conclusions

New tool: alternative to bases

- provides resilience to noise, quantization, packet losses
- relaxes strong basis constraints
- allows greater freedom in design

Copy of the talk at

- `http://cm.bell-labs.com/cm/ms/events/DIMACSBL`

More details at

- `http://cm.bell-labs.com/who/jelena`
  → Interests → Multiple descriptions