Optimal Control and Hedging of Operations in the Presence of Financial Markets

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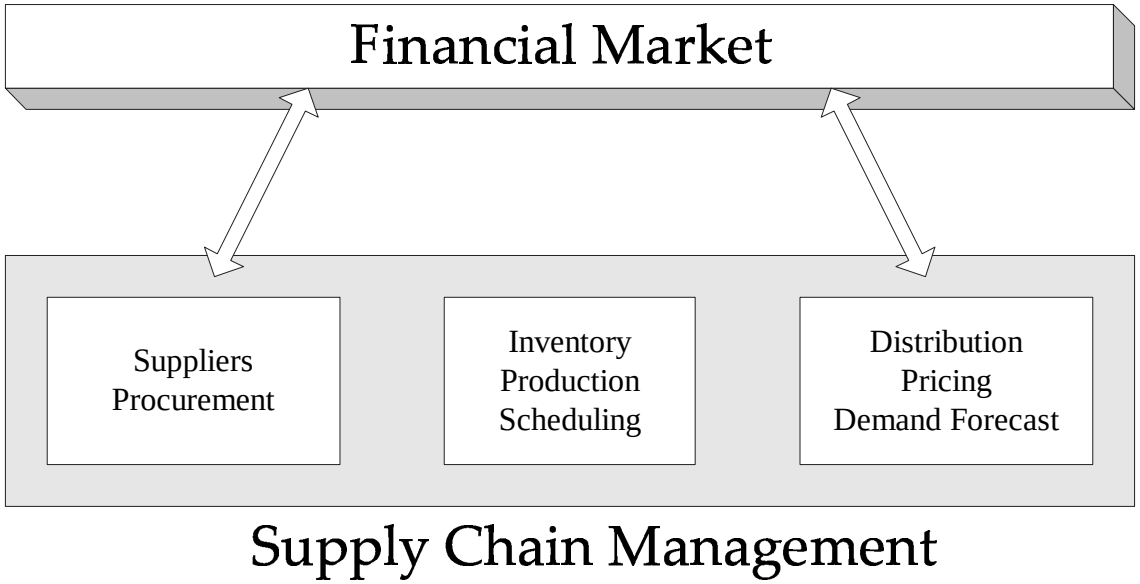
Columbia University and Stern NYU

Value Chain Academic-Industry Consortium
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Motivation

- Operations and services models usually maximize expected profit
  - i.e. risk neutrality assumed
  - so no need for hedging
  - consistent with Modigliani - Miller
- But sometimes there are good reasons for hedging
  - costs associated with financial distress
  - taxation issues
  - other market frictions
- Principal - Agent problems
- And in practice corporations often do hedge

Motivation



Problem: Select a control I and a financial strategy θ to optimize a given performance

$$\max_{\gamma,\theta} \Pi^*(\gamma) = \mathrm{E}_\omega \left[\mathcal{U}(H(\gamma,\omega) + G(\theta,\omega)) \right].$$

Mean-Variance Corporations

Implication

- Corporations should hedge using financial markets
- Leads to a hedging problem in incomplete markets
  - an active area in math finance

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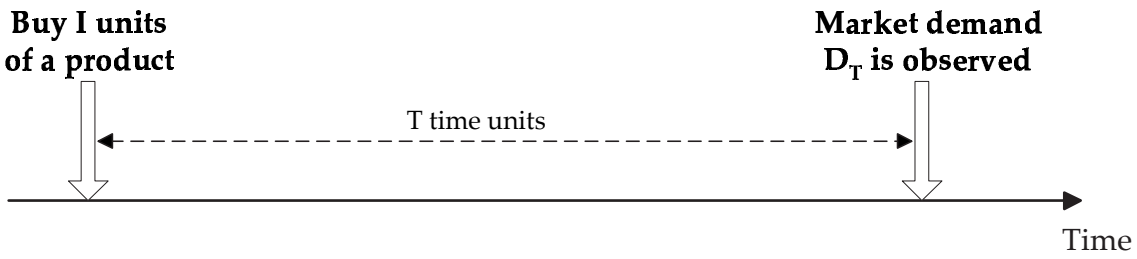
Assumptions

- Will assume that the corporation is a mean-variance optimizer
- It solves

$$\max \frac{L}{2} - \frac{1}{2L} \mathrm{E}_0\left[(H_T - L)^2\right] \quad \Longleftrightarrow \quad \min \mathrm{E}_0\left[(H_T - L)^2\right]$$

- T is the horizon
- $H_T =$ operating profits at time T
- L is target profit
- Mean-variance not perfect; excess profits penalized
- But tractable and often a "good first approximation"

Example: Newsboy



The Newsboy profit is

$$H_T(I) = (R-r)D_T + (r-p)I - (R+b-r)(D_T - I)^+$$

R : Unit selling price of a newspaper

r : Salvage value of unsold units

b : Penalty cost per unit of unsatisfied demand

p : Per unit purchasing cost of a newspaper

and $R > p > r$

Related Literature

- Mean-Variance hedging
  - Follmer and Sonderman (1986)
  - Follmer and Schweizer (1991)
  - Duffie and Richardson (1991)
  - Schweizer (1992, 1995, 1996)
  - Laurent and Pham (1999)
  - Heath, Platen and Schweizer (2001)

- OM Literature (Newsboy)
  - Chen and Federgruen (2000), Lau (1980)
  - Anvari (1987) and Chung (1990)
  - Gaur and Seshadri (2002)

Problem Formulation

- For a fixed operating policy, γ, the problem is

$$\min_{G_T(\theta)} \ \mathrm{E}\left[\left(H_T(\gamma) + G_T(\theta) - L\right)^2\right]$$

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- More generally, we would like to solve

$$\min_{\gamma} \ \min_{G_T(\theta)} \ \mathrm{E}\left[(H_T(\gamma) + G_T(\theta) - L)^2\right]$$

  - $H_T(\gamma) =$ operating profits
  - $\theta =$ trading strategy
  - $G_T(\theta) =$ gains or losses from self-financing trading strategy

Outline of Remainder of Talk

- Notation
- Incomplete Information model
- Complete Information model
  - Use results of Schweizer (1992)
- Examples
  - Newsboy problem
  - Production problem
- Conclusions
- Further research

Notation

- Probability space $(\Omega, \mathcal{F}, \mathcal{P})$
- $(B_{1,t}\ , B_{2,t})$ a standard Brownian motion
- A single risky stock with price dynamics

$$dX_t = \mu X_t\ dt + \sigma X_t\ dB_{1,t}$$

- Cash account available with $r \equiv 0$
- $\mathcal{F}_t$ is the "usual" filtration generated by $(B_{1,t}\ , B_{2,t})$
- $\mathcal{F}_t^F$ is the "usual" filtration generated by X_t
- Operating profits, $H_T(\gamma)$, a function of $(B_{1,t}\ , B_{2,t})$

The Complete Information (CI) Model

- Use Schweizer (1992) to solve

$$S_T := \min_{G_T(\theta)} \mathrm{E}\left[\left(H_T(\gamma) + G_T(\theta) - \frac{1}{2L}\right)^2\right]$$

  where θ_t is $\mathcal{F}_t$ predictable

- Note that strategy, θ_t, can be a function of time t financial and operating information
  - not always appropriate

- Need to assume deterministic market price of risk, μ/σ
  - restrictive, but maybe not so much for a corporation

Solving Schweizer's Problem

1. Find, $\widehat{P}$, the *minimal* equivalent martingale measure (MEMM)
   - this is straightforward
   - X and B_2 are $\widehat{P}$ orthogonal martingales

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3. Find the Galtchouk - Kunita - Watanabe (GKW) decomposition of $\widehat{V}_t$

$$\widehat{V}_t = \widehat{H}_0(\gamma) + \int_0^t \hat{\phi}_s(\gamma)\ dX_s + \int_0^t \hat{\delta}_s(\gamma)\ dB_{2,s}$$

   - not always easy
   - if $\widehat{V}_t = \mathrm{E}_{\widehat{P}}\left[H_T(\gamma)|\mathcal{F}_t\right]$ available, then ok
   - otherwise hope problem is Markovian
   - obtain

$$S_T(\gamma) = e^{-\eta^2 T}\left(\frac{1}{2L} - \widehat{H}_0(\gamma)\right)^2 + \int_0^T e^{-\eta^2(T-t)}\mathrm{E}[\hat{\delta}^2(\gamma)]\ dt$$

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4. Now solve $\min_\gamma S_T(\gamma)$

The Incomplete Information (II) Model

- Not always appropriate that θ be $\mathcal{F}_t$ predictable
  - demand, product quality, customer tastes not always observable
  - so might need θ to be $\mathcal{F}_t^F$ predictable
- No longer need μ/σ deterministic
- Instead assume "financial markets" complete
  - so any random variable that is $\mathcal{F}_t^F$ adapted can be replicated
  - possible justifications
    - corporation happy to hedge "approximately"
    - can trade in derivatives

Incomplete Information Solution

Problem:

$$\min_{G_T} \ \mathrm{E}\left[\left(H_T(\gamma) + G_T - \frac{1}{2L}\right)^2\right] \tag{1}$$

subject to

$$\mathrm{E}\left[\pi_T^F G_T\right] = G_0 \tag{2}$$

$$\text{and } \mathcal{F}_t^F \text{ measurability of } G_T \tag{3}$$

- π_t^F is the unique SPD for the financial market
- Solve by expanding and using iterated expectations in (1)

Incomplete Information Model

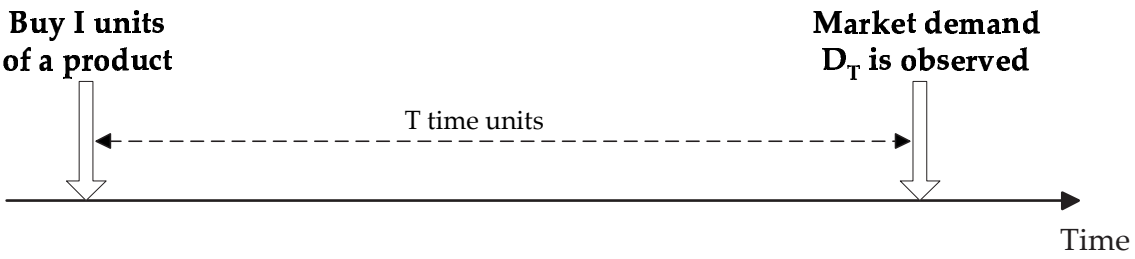
New Problem:

$$\min_{G_T} \ \mathrm{E}\left[\mathrm{E}\left[H_T^2 - \frac{H_T}{L}|\mathcal{F}_T^F\right] \ + \ 2G_T\mathrm{E}[H_T|\mathcal{F}_T^F] \ + \ G_T^2 - \frac{G_T}{2L} \ + \ 1/4L^2\right]$$

subject to (2)

- This problem easily solved using standard static methods
- If CI model not solvable, II model might be a good approximation
  - particularly when financial variables contribute significantly to profit variance

Example: The Newsboy Problem



The Newsboy profit is

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and $R > p > r$

We assume:

$$D_T := \zeta + \beta \ln(X_T) + \alpha B_{2T}$$

Example: The Newsboy Problem

Table 1: Root-Mean-Squared-Error and Optimal Ordering Level, I^*, as a Function of $\eta = \mu/\sigma$.

	Root-Mean-Squared-Error			Optimal Ordering Level I^*		
η	No Hedging	Incomplete	Complete	No Hedging	Incomplete	Complete
0.0	468244	461523	458068	288	266	274
0.2	464468	443593	440376	296	265	273
0.4	460696	394555	391414	305	260	271
0.6	456926	325142	321994	314	252	265
0.8	453160	248909	245818	322	236	253
1.0	449397	177911	175123	331	210	229
1.2	445637	120179	117570	339	179	190
1.4	441881	78519	75327	348	147	138
1.6	438129	53423	47767	356	123	91
1.8	434380	41927	33172	365	114	60
2.0	430635	38205	27651	373	114	45

$$R = \$1000,\ r = \$200,\ b = \$300,\ p = \$500,\ T = 2 \text{ years},\ L = 5^5.$$

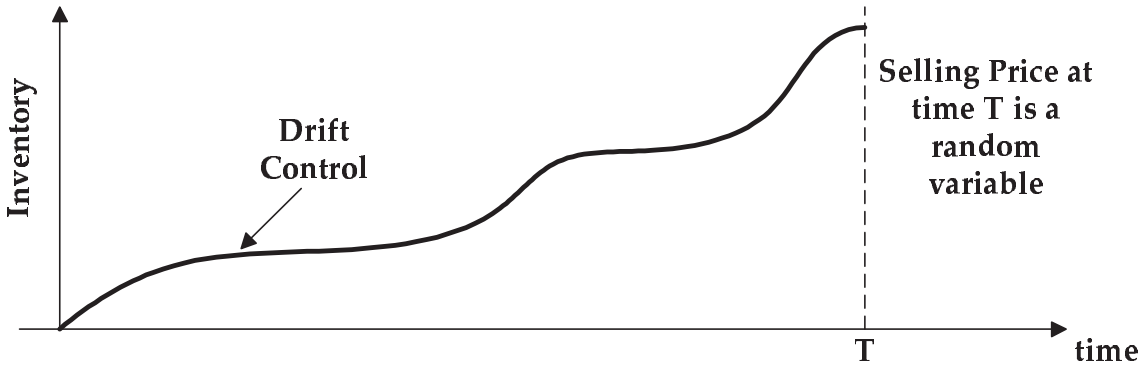
$$\zeta = 200,\ \beta = \alpha = 100,\ X_0 = 1.0,\ \mu = 0.1,\ \sigma = 0.2.$$

Example: The Newsboy Problem

Table 2: Root-Mean-Squared-Error and Optimal Ordering Level as a Function of α.

	Root-Mean-Squared-Error			Optimal Ordering Level I^*		
α	No Hedging	Incomplete	Complete	No Hedging	Incomplete	Complete
50	424719	336563	335819	267	235	238
70	437673	346045	344522	284	245	250
90	451572	356245	353696	301	253	262
110	466220	367000	363221	318	260	274
130	481505	378212	373027	334	266	285
150	497345	389808	383077	350	272	295
170	513677	401736	393342	366	276	304
190	530447	413951	403786	381	279	314
210	547610	426414	414400	396	281	323
230	565159	439101	425166	411	284	332
250	582963	451979	436055	426	285	339

Example: A Production Problem



- Corporation manufactures widgets between 0 and T
- Number of widgets, I_t, satisfies

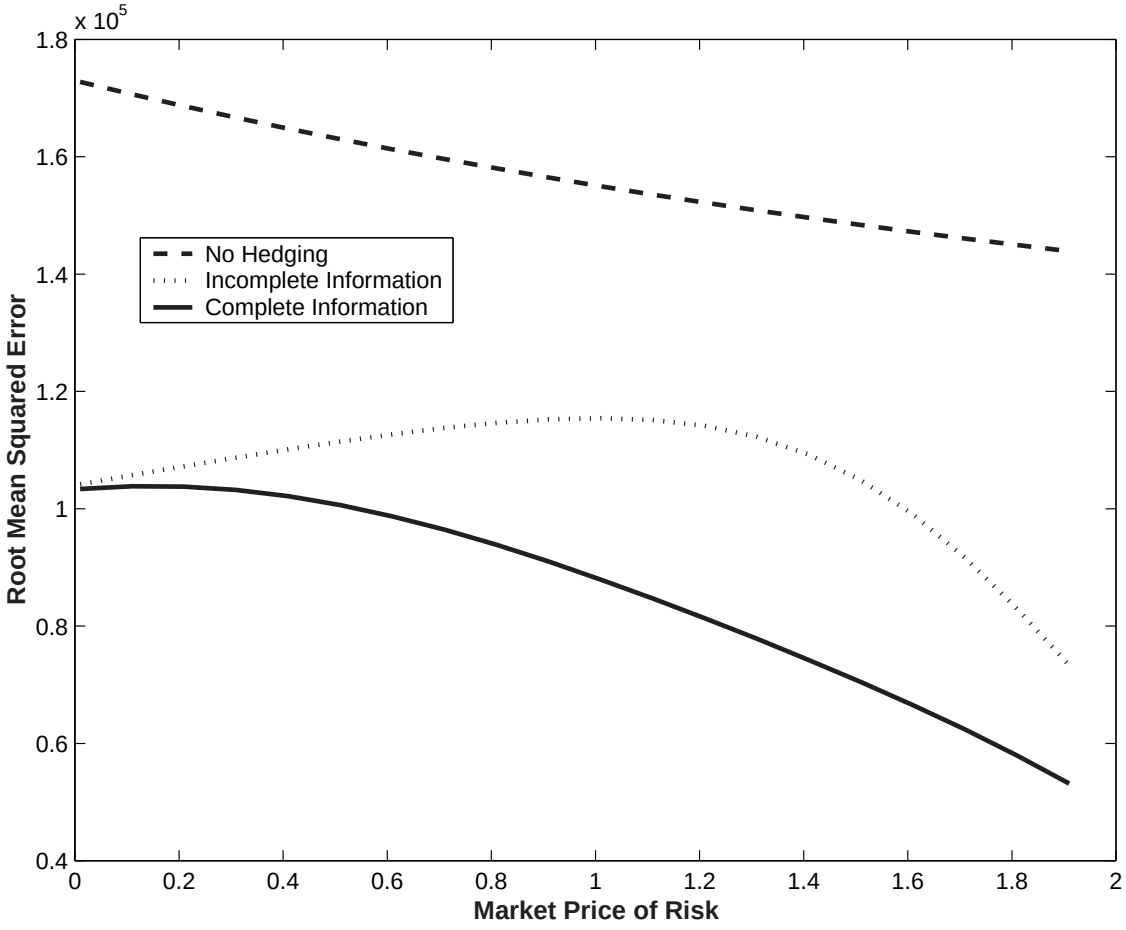
$$dI_t = (aX_t + b)\ dt + c\ dB_{2,t}$$

- $X_t \sim$ GBM(μ, σ), a traded stock or market index
- Operating control is the pair (a, b) to be chosen at time $t = 0$
- All widgets sold at time T with profit

$$H_T(a, b) = I_T(a, b)\ \ (X_T(A + B_{2,T}) - C_0)$$

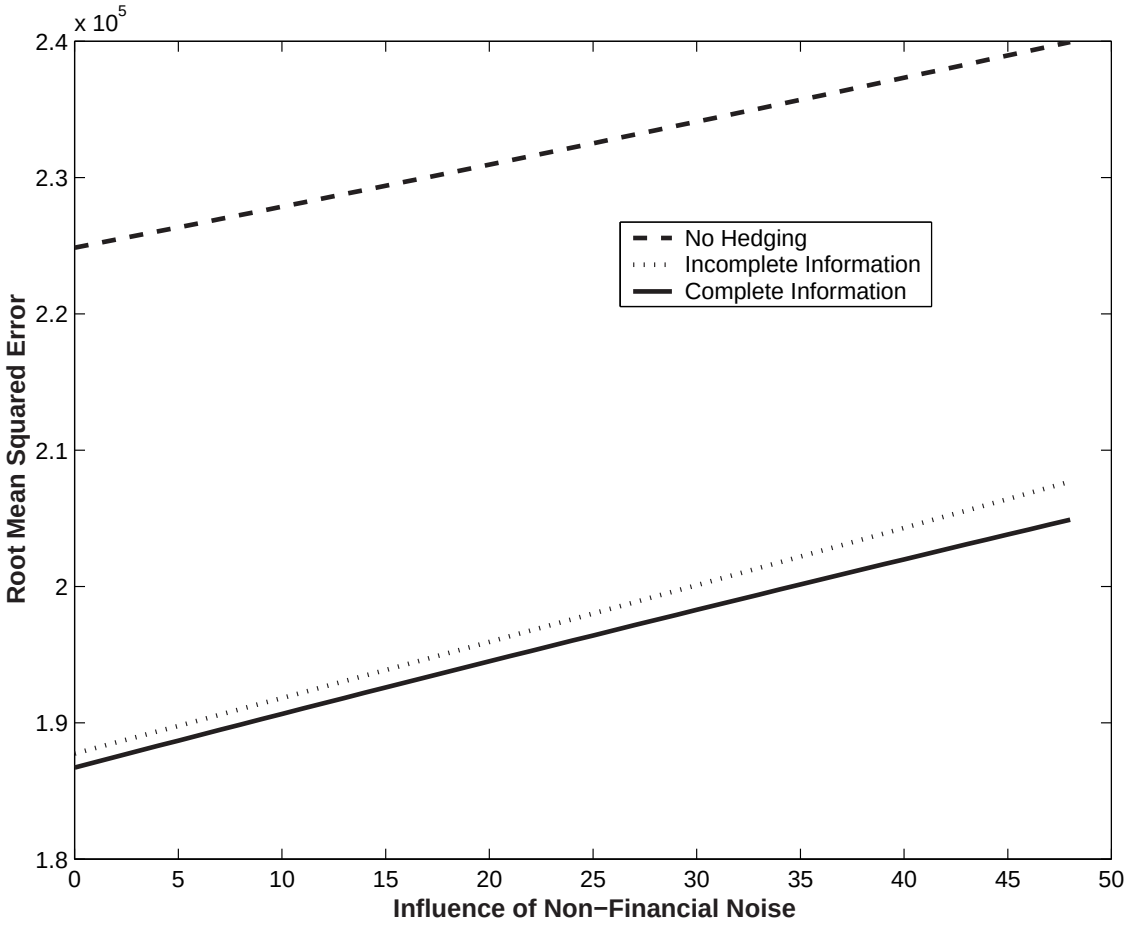
- Financial noise is hedged using operating policy *and* trading in financial markets

Example: A Production Problem



$T = 1\,\text{year},\ C_0 = 500,\ X_0 = 100,\ L = 500,000,\ c = 10.$

Example: A Production Problem



$T = 1\,\text{year},\ C_0 = 500,\ X_0 = 100,\ L = 500,000,\ \eta = 0.01.$

Conclusions and Further Research

Conclusions

- A methodology for mean-variance corporations that simultaneously selects optimal operating policy **and** hedging strategy
- A methodology that is optimal under certain informational restrictions
  - and otherwise often approximately optimal
- Extended applicability of results on portfolio optimization in incomplete markets

Further Research

- Consider other operations and services models
- Extend to other price processes
  - drop deterministic market price of risk assumption for complete information model
- Extend to competitive game theoretic models
- Solve for more general utility functions
- Hedge using *local risk minimization* techniques