The Martingale Model of Forecast Evolution

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Executive Summary

- *The Martingale Model of Forecast Evolution (Graves et al., Heath and Jackson) is a realistic representation of how demand forecasts evolve over time*
- *The parameters of the model can be estimated from historical forecast data*
- *The model is robust: does not require specification of the particular forecasting techniques used*
- *The model allows analysis and simulation of the impact of forecast uncertainty on production/transportation/inventory systems*

Overview

- *Forecast Changes*
- *Forecast Change Variability*
- *Total Forecast Change Variability*
- *Forecast Change Correlations*
- *Simulated Forecast Evolution (Additive)*
- *Applications and Related Research*
- *Multiplicative Model*

Forecasts and Sales: Sample Product, 4 Months

Month	1	2	3	4	5
Sales	120,402	69,116	104,907	57,781	83,028
One-Month-Out Forecast	22,933	45,466	26,894	42,112	232,999
Two-Month-Out Forecast	45,466	26,894	42,112	232,999	145,689

No Forecast Updating

Forecasts and Sales: Different Product

Month	1	2	3	4	5
Sales	15,050	466	2,084	6,036	9,254
One-Month-Out Forecast	1,006	1,512	7,550	6,660	3,942
Two-Month-Out Forecast	2,845	5,890	7,585	3,555	5,931

Forecast Updating

Forecast Changes

Month	1	Change	2	Change	3	Change	4	Change	5	Average
Sales	15,050		466		2,084		6,036		9,254	
		-540		572		-1,514		2,594		278
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942	
		-1,333		1,660		-925		387		-53
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931	

Average change ≅ 0

Forecast Change Variability

Month	1	Change	2	Change	3	Change	4	Change	5	Average	Std. Dev.
Sales	15,050		466		2,084		6,036		9,254		
		-540		572		-1,514		2,594		278	1,764
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942		
		-1,333		1,660		-925		387		-53	1,357
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931		

Variability of forecast change can be large

Forecast Change Notation

Let $D(t)$ denote the actual observed demand in period t.

Let D_{st} denote the forecast of $D(t)$ made in period s, $s \leq t$.

$D_{tt} = D(t)$, the forecast in period t equals the demand in period t.

Let $\varepsilon_{st} = D_{st} - D_{s-1,t}$, the change in forecast for period t made in period s, $s \leq t$.

Month	1	Change	2	Change	3	Change	4	Change	5	Change	6
Sales	15,050		466		2,084		6,036		9,254		D_{66}
		-540		572		-1,514		2,594		ε_{66}	
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942		D_{67}
		-1,333		1,660		-925		387		ε_{67}	
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931		D_{68}

Graves, Meal, Dasu, Qui (1986)

Month	1	Change	2	Change	3	Change	4	Change	5	Change	6
Sales	15,050		466		2,084		6,036		9,254		D_{66}
		-540		572		-1,514		2,594		ε_{66}	
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942		D_{67}
		-1,333		1,660		-925		387		ε_{67}	
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931		D_{68}

$$E[\varepsilon_{st}] = 0$$

$$Var[\varepsilon_{st}] = \sigma^2_{(t-s+1)}$$

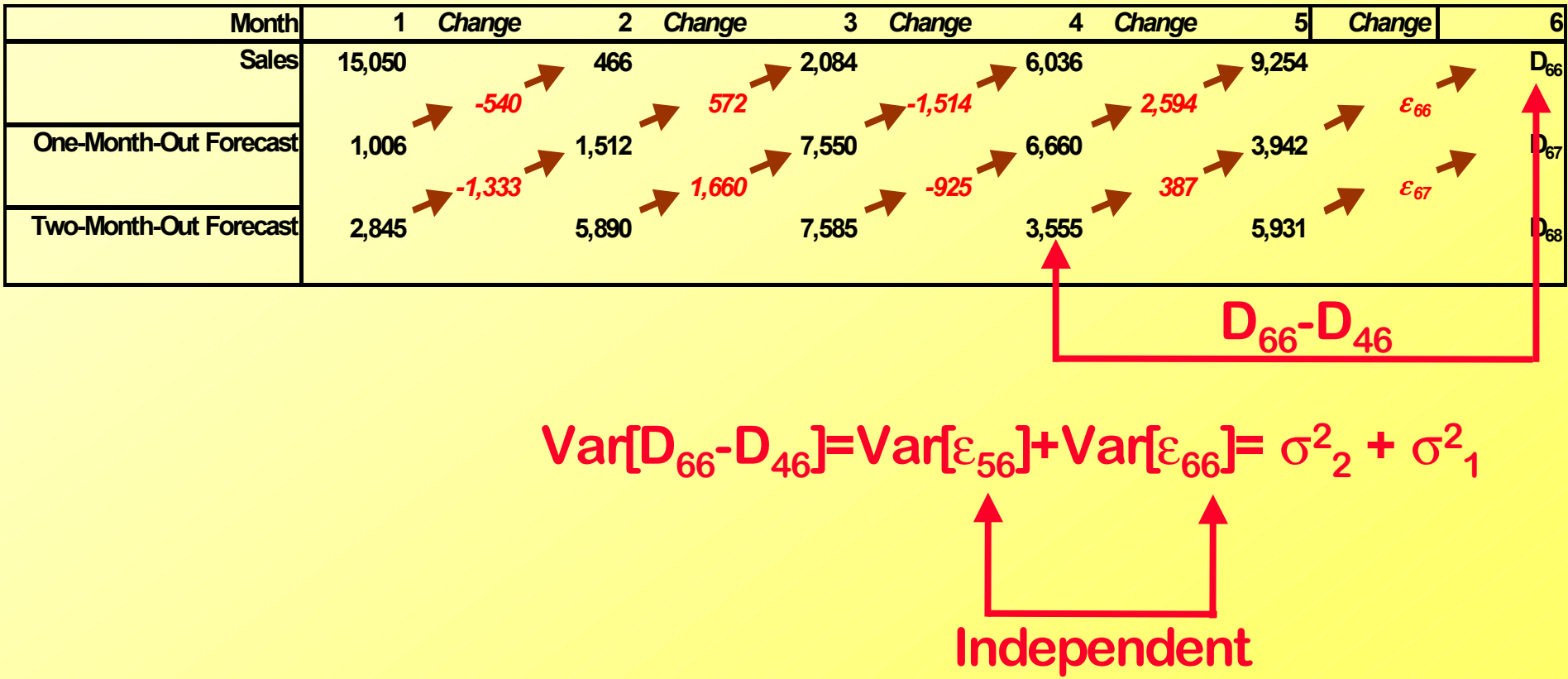
ε_{st} and $\varepsilon_{s't}$ are independent for all $s \neq s'$

Independent

$Var[\varepsilon_{66}]=\sigma^2_1$

$Var[\varepsilon_{67}]=\sigma^2_2$

Total Forecast Change Variability



Estimated Total Forecast Change Variability

Variances of forecast change can be estimated from historical data

Month	1	Change	2	Change	3	Change	4	Change	5	Average	Std. Dev.
Sales	15,050		466		2,084		6,036		9,254		
		-540		572		-1,514		2,594		278	1,764
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942		
		-1,333		1,660		-925		387		-53	1,357
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931		

Std. Deviation of Two-Month Forecast Change

$$\cong \sqrt{1{,}764^2 + 1{,}357^2} = 2{,}225$$

Roughly Speaking

- Safety stock investment is proportional to standard deviation of forecast change over a lead time
- If production lead time is two months, safety stock investment will be proportional to 2,225
- If production lead time can be reduced to one month, safety stock investment will be proportional to 1,764
- A 50% reduction in lead time can lead to a 21% reduction in safety stock (in this example)

Heath, Jackson (1994)

Assume $D_{st} = E[D(t)|\mathcal{I}_s]$ where $\mathcal{I}_s$ denotes all of the information known at time s.

Then, $E[\varepsilon_{st}] = E[D_{s,t} - D_{s-1,t}|\mathcal{I}_{s-1}] = 0$. The expected change in the forecast is zero (i.e. the process $\{D_{1t}, D_{2t}, ..., D_{tt}\}$ is a Martingale).

Also, $E[\varepsilon_{st}\varepsilon_{s't}] = 0$ for all $s \neq s'$. Forecast changes in different periods are uncorrelated.

Also, $E[\varepsilon_{st}\varepsilon_{s't'}] = 0$ for all $s \neq s'$, and for all t and all t'. Forecast changes for any period in different periods are uncorrelated.



Estimating Correlations

Covariances, correlations of forecast change can be estimated from historical data

Month	1	Change	2	Change	3	Change	4	Change	5	Average	Std. Dev.	Correlation
Sales	15,050		466		2,084		6,036		9,254			
		-540		572		-1,514		2,594		278	1,764	
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942			0.575
		-1,333		1,660		-925		387		-53	1,357	
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931			

One-month-out and two-month-out forecast changes are positively correlated (in this example)

Correlation coefficient of one-month-out and two-month-out forecast changes is 0.575

What Could Cause Correlations?

- *Good news in a period would cause forecasts of sales in all future periods to increase (positive correlation)*
- *Bad news would cause all forecasts to decrease (positive correlation)*
- *If total sales are based on fixed quantity contracts, an increase in sales orders against those contracts in current period would cause forecasts of sales in future periods to decrease (negative correlation)*

Why Are Forecast Correlations Important?

- More safety stock may be required:
  - If forecast changes across future time periods are highly positively correlated (inventory is less effective)
  - If forecast changes across locations are highly positively correlated (transshipment is less effective)
  - If forecast changes across products are highly positively correlated (production capacity is less effective)

Representing Correlations

Assume stationarity: $E\left[\varepsilon_{st}\varepsilon_{st'}\right] = \sigma_{(t-s+1),(t'-s+1)}$.

Let $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1H} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2H} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{H1} & \sigma_{H2} & \cdots & \sigma_H^2 \end{pmatrix}$, the variance-covariance matrix (VCV),

where H is the number of forecast changes.

$$\Sigma = \begin{pmatrix} \sigma^2_1 & \sigma_{12} \\ \sigma_{21} & \sigma^2_2 \end{pmatrix} \cong \begin{pmatrix} 1{,}764^2 & 0.575*1{,}764*1{,}357 \\ 0.575*1{,}764*1{,}357 & 1{,}357^2 \end{pmatrix}$$

An Additive MMFE Simulation Model

- Given:
  - An infinite horizon vector of initial forecasts
  - A VCV matrix, Σ, of forecast changes
- Simulate:
  - A new infinite horizon vector of forecasts for every period simulated
- By:
  - Shifting the previous infinite horizon forecast vector up by one
  - Generating a random vector of forecast changes according to a multi-variate normal distribution with mean 0 and VCV matrix Σ
  - Adding the random vector to the shifted vector

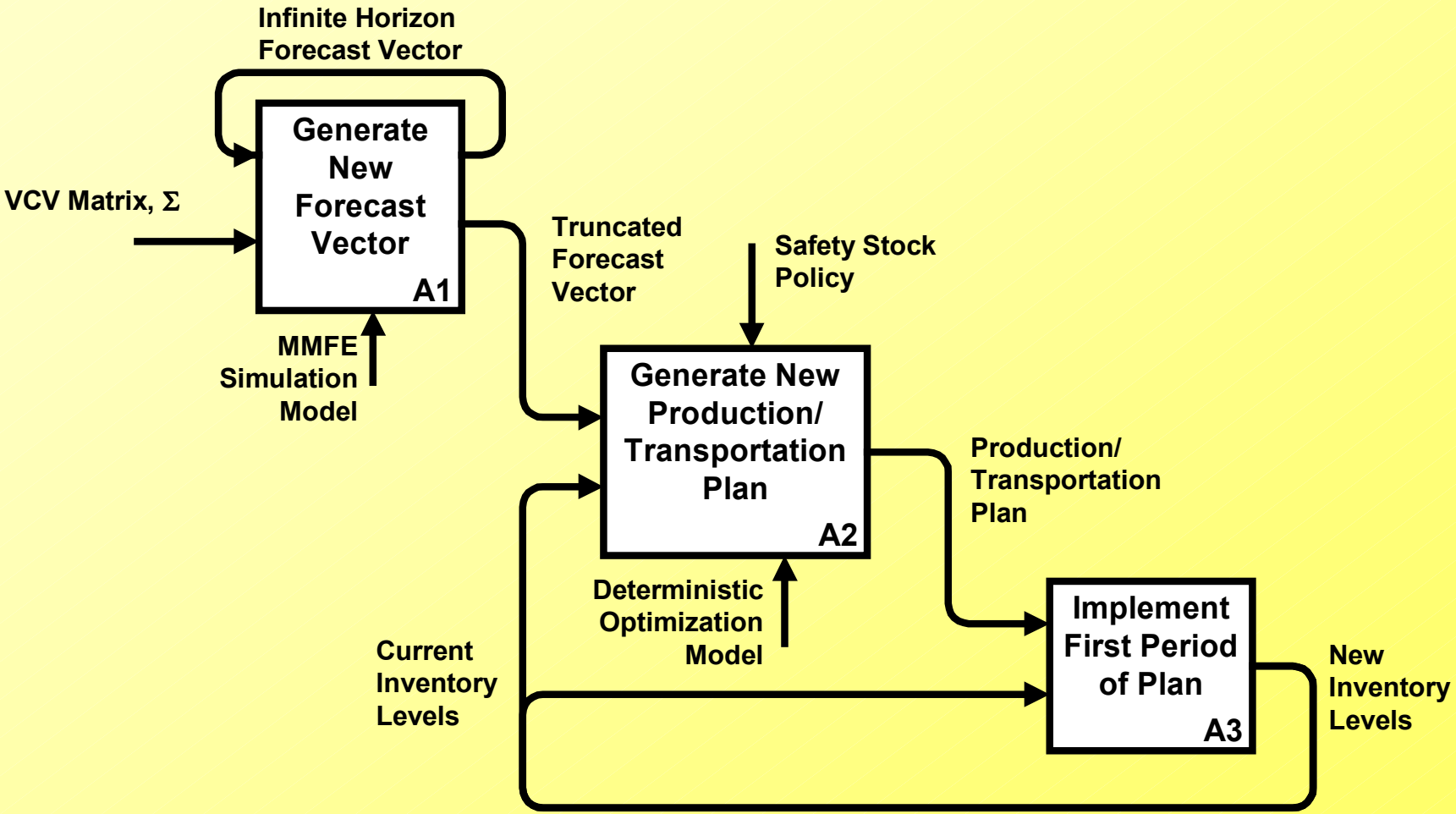
Simulated Forecast Evolution

Randomly generated, using Σ

Month/Month-Out	1	Shift	Change	2	Shift	Change	3	Shift	Change	4
0	15,050	1,006	-293	713	879	603	1,482	9,305	-1946	7,359
1	1,006	2,845	-1966	879	5,890	3415	9,305	7,585	-1502	6,083
2	2,845	5,890		5,890	7,585		7,585	3,555		3,555
3	5,890	7,585		7,585	3,555		3,555	5,931		5,931
4	7,585	3,555		3,555	5,931		5,931	15,050		15,050
5	3,555	5,931		5,931	15,050		15,050	713		713
6	5,931	15,050		15,050	713		713	1,482		1,482
etc.	etc.	etc.		etc.	etc.		etc.	etc.		etc.

A 'realistic' simulation: captures correlations

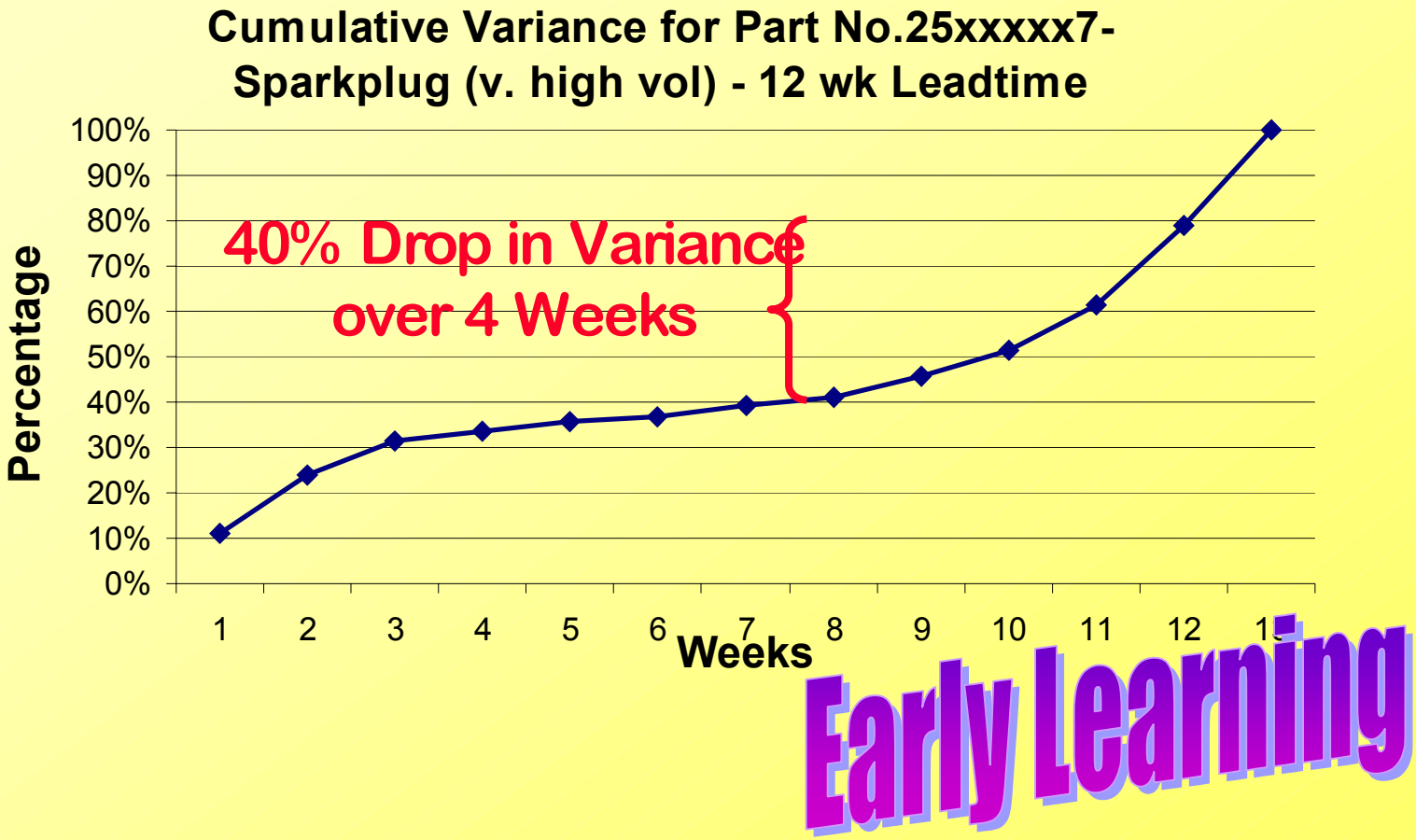
What Can Be Done With Simulated Forecast Evolution?



Applications

- Quaker Oats
  - Captured forecast change correlations across time, location, and product
  - Recommended safety stock levels for seasonal product line
  - Demonstrated cost savings possible with improved forecasting (alternative Σ)
- General Motors Service Parts Operations
  - Captured forecast change correlations across time
  - Related procurement lead time to safety stock requirements

Lead Time Impact



Applications (cont'd)

- General Foods
  - Captured forecast change correlations across time and product
  - Delivered production planning simulation tool for seasonal product
- IBM
  - Captured forecast change correlations across time and product
  - Delivered procurement planning simulation tool

Related Research: Analytic Use of MMFE

- Graves, Meal, Dasu, Qui, 1986
  - Safety stock formulas (uncapacitated system)
- Güllü, 1993
  - Characterization of optimal policies
- Graves, Kletter, Hetzel, 1998
  - Production schedule change optimization
- Toktay, Wein, 2001
  - Safety stock formula (capacitated system)

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- **Multiplicative Model**

Express Change as Ratio

Month	1	Ratio	2	Ratio	3	Ratio	4	Ratio	5	Average
Sales	15,050		466		2,084		6,036		9,254	
		0.463		1.378		0.799		1.389		1.008
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942	
		0.531		1.282		0.878		1.109		0.950
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931	

Average ratio of change (new/old) $\cong$ 1

Taking Logarithms

Assume $D_{st} = E\left[D(t)|\mathcal{I}_s\right]$ where $\mathcal{I}_s$ denotes all of the information known at time s.

Then, $E\left[\varepsilon_{st}\right] = E\left[D_{s,t} - D_{s-1,t}|\mathcal{I}_{s-1}\right] = 0$. The expected change in the forecast is zero (i.e. the process $\left\{D_{1t}, D_{2t}, ..., D_{tt}\right\}$ is a Martingale).

Equivalently, $E\left[\frac{D_{s,t}}{D_{s-1,t}}|\mathcal{I}_{s-1}\right] = 1$. The expected ratio of the new forecast (including actual demand) to the old forecast is 1.

Let $R_{s,t} = \frac{D_{st}}{D_{s-1,t}}$, the ratio of new forecast to old.

Let $\varepsilon_{s,t} = \ln(R_{s,t})$, the natural log of the ratio of new forecast to old.

The Martingale Assumption implies $E\left[e^{\varepsilon_{s,t}}\right] = 1$.

Interpreting Logarithms

Month	1	Ratio-1	2	Ratio-1	3	Ratio-1	4	Ratio-1	5
Sales	15,050		466		2,084		6,036		9,254
		-0.537		0.378		-0.201		0.389	
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942
		-0.469		0.282		-0.122		0.109	
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931

Ratio-1 measures the rate of forecast change

Month	1	Ln(Ratio)	2	Ln(Ratio)	3	Ln(Ratio)	4	Ln(Ratio)	5
Sales	15,050		466		2,084		6,036		9,254
		-0.770		0.321		-0.224		0.329	
One-Month-Out Forecast	1,006		1,512		7,550		6,660		3,942
		-0.632		0.248		-0.130		0.103	
Two-Month-Out Forecast	2,845		5,890		7,585		3,555		5,931

$\text{Ln(Ratio)} \cong \text{Ratio-1}$ since $e^x \cong 1+x$ for small x

Ln(Ratio) is roughly comparable to the rate of forecast change

Distribution of Log Change Ratios

As before, assume stationarity: $Cov\left[\varepsilon_{st}, \varepsilon_{st'}\right] = \sigma_{(t-s+1),(t'-s+1)}$.

Let $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1H} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2H} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{H1} & \sigma_{H2} & \cdots & \sigma_H^2 \end{pmatrix}$, the variance-covariance matrix (VCV), where H is the number of forecast changes.

Assume vector $\varepsilon_s = (\varepsilon_{s,t})_{t=s}^{s+H}$ is multi-variate normal (μ, Σ). i.e. Change ratios are log-normally distributed.

Now, however, to ensure $E\left[e^{\varepsilon_{s,t}}\right] = 1$ we must have $\mu_{(t-s+1)} = -\sigma^2_{(t-s+1)}/2$.

Estimation Problem

Given history of forecast log ratios, ε_s, $s = 1, 2, ..., N$, estimate Σ subject to constraint $\mu_j = -\sigma_j^2/2$.

Approach: Method of Moments

Sample code and data available upon request:
pj16@cornell.edu

Advantages of Multiplicative Model

- *Log-normal distribution ensures that simulated demands are never negative*
- *Yields better fit to data, particularly when demands are seasonal*

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- *The model allows analysis and simulation of the impact of forecast uncertainty on production/transportation/inventory systems*