On Relations Between Covering Radius and Dual Distance

Alexei E. Ashikhmin[*], **Iiro S. Honkala**[†], *Member,* IEEE,
Tero K. Laihonen[‡] and **Simon N. Litsyn**[§], *Member*, IEEE

Abstract

The covering radius of a code tells us how far in the sense of Hamming distance an arbitrary word of the ambient space can be from the code. For a few decades this parameter has been widely studied.

In this paper we estimate the covering radius of a code when the dual distance is known. We derive a new bound on covering radii of linear codes. It improves essentially on the previously known estimates in a certain wide range. We also study asymptotic bounds on the cardinality of constant weight codes.

Index Terms: Covering radius, dual distance, Krawtchouk polynomial, constant weight codes.

1 Introduction

Covering radius is a basic geometric parameter of a code. If it is smaller than the minimum distance, then the code is maximal. It also measures the maximal error-correcting capability of a code in the case of minimum distance decoding. The covering radius has applications to problems of data compression and write-once memories [3], for instance.

[*]Los Alamos National Laboratory, Group CIC-3, Mail Stop P990, Los Alamos, NM 87545, e-mail: alexei@c3serve.c3.lanl.gov.

[†]Department of Mathematics, University of Turku, FIN-20014 Turku, Finland, e-mail: honkala@utu.fi.

[‡]Department of Mathematics, University of Turku, FIN-20014 Turku, Finland, e-mail: terolai@utu.fi.

[§]Department of Electrical Engineering-Systems, Tel Aviv University 69978, Israel, e-mail: litsyn@eng.tau.ac.il.

The problem of finding bounds on the covering radius has been an object of growing interest for a couple of decades. Knowledge of another parameter of a code called dual distance enables us to estimate the covering radius reasonably well. The dual distance is often known or it can at least be estimated from the construction of the code. For a linear code it is simply the minimum distance of the dual code.

After the seminal articles due to Delsarte [5] and Helleseth, Kløve and Mykkeltveit [8] there has been several papers concerning bounds on covering radius when the dual distance is known (see, e.g., [3, 7, 9, 23, 28, 29, 30, 16]).

In particular, in 1990 Tietäväinen proved the following remarkable results [30, 31]:

1) Let $\mathcal{C} = (C_n)_{n=1}^{\infty}$ be a sequence of binary codes C_n of length n, dual distance $d' = d'(n)$ and covering radius $R = R(n)$ where $d'/n \to \delta'$ and $R/n \to \rho$ when $n \to \infty$. Then

$$\rho \leq \frac{1}{2}(1 - \sqrt{\delta'(2 - \delta')}). \tag{1}$$

2) Let $0 < \delta' < 1/2$. There are sequences $\mathcal{C}$ such that

$$\rho \geq H_2^{-1}(1 - H_2(\delta'))$$

where $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function.

The upper bound (1) is the best known bound for relatively small values of (normalized) dual distance and although the lower bound is obtained in an obvious way it is still the only estimate in this direction for binary codes. The preceding upper bound was generalized by Fazekas and Levenshtein [7] to other metric spaces.

A new powerful method for binary linear codes was introduced by Litsyn and Tietäväinen [23] in 1996. Using this method and (regular) Chebyshev polynomials they were able to improve on the previous bounds for relatively large values of dual distance. In [10], Honkala, Litsyn and Tietäväinen combined this approach with the so-called Elias argument for further improvements.

In 1997 it was shown by Honkala, Laihonen and Litsyn [9] that using discrete Chebyshev polynomials instead of regular ones yields even better bounds. A refinement of that bound was obtained by Laihonen and Litsyn [17]. At the moment, the best estimate for large values of (normalized) dual distance can be found from [14, Theorem 2], which utilizes the approach of [17] and regular Chebyshev polynomials. The result is as follows:

Theorem 1 ([14, 15]) *Let $(C_n)_{n=1}^{\infty}$ be a family of binary linear codes of length n, dual distance $d' = d'(n)$ and covering radius $R = R(n)$ where $\lim_{n\to\infty} d'/n = \delta'$ and $\lim_{n\to\infty} R/n = \rho$. Let $0 < \delta' < 1/2$ and denote $\zeta = 1/2(1 - \sqrt{1-2\delta'})$. Let $\epsilon > 0$ and $\alpha = \zeta + \epsilon$. Assume $\alpha < \delta'$. Take β $(0 < \beta < \zeta)$ such that*

$$\frac{\alpha(1-\alpha) - \beta(1-\beta)}{1 + 2\sqrt{\beta(1-\beta)}} < \frac{\delta'}{2}.$$

Then we have an upper bound

$$\rho \leq \gamma := \frac{H_2(1/2 - \sqrt{\delta'(1-\delta')})}{\log_2\left(\frac{1-\zeta-\epsilon+\delta'+2\sqrt{\delta'(1-\zeta-\epsilon)}}{1-\zeta-\epsilon-\delta'}\right)} \tag{2}$$

if

$$\left(\frac{1-\zeta+\epsilon-\delta'+2\sqrt{\epsilon(1-\delta'-\zeta)}}{1-\zeta-\epsilon+\delta'+2\sqrt{\delta'(1-\epsilon-\zeta)}}\right)^{\gamma} 2^{H_2(\beta)} < 1.$$

In this paper we shall use a similar method as Tietäväinen [30, 31], which is in a sense a dual approach to the one applied in [23, 10, 9, 14]. In fact, our method is based on an extension of Tietäväinen's approach [30]. This extension allows us to choose our polynomial with less restrictions, that is, we may use polynomials of higher degree than in [30].

We show (in Theorem 5) that if we use the polynomial originally designed for the linear programming method for error-correcting codes [25], we get a relatively strong bound on covering radius as a function of dual distance (see Figure 1).

2 Preliminary results

Let $C \subset \mathbb{F}_2^n$ be a linear code with covering radius R and dual distance d'. Denote the Hamming weight of $\mathbf{x} \in \mathbb{F}_2^n$ by $w(\mathbf{x})$. We define the weight distribution of the translate $\mathbf{x}+C$ $(\mathbf{x} \in \mathbb{F}_2^n)$ to be the $(n+1)$-tuple $\mathbf{A}(\mathbf{x}) = (A_0(\mathbf{x}), \ldots, A_n(\mathbf{x}))$ where $A_i(\mathbf{x}) = |\{\mathbf{c} \in C|\ w(\mathbf{x}+\mathbf{c}) = i\}|$. Let us define the Krawtchouk polynomial $K_k(x)$ via

$$K_k(x) = K_k^n(x) = \sum_{j=0}^{k} (-1)^j \binom{x}{j}\binom{n-x}{k-j}.$$

The MacWilliams transform [3, p. 227] of $\mathbf{A}(\mathbf{x})$ is the $(n+1)$-tuple $\mathbf{A}'(\mathbf{x}) = (A_0'(\mathbf{x}), ..., A_n'(\mathbf{x}))$ where

$$A_j'(\mathbf{x}) = \frac{1}{|C|} \sum_{i=0}^{n} A_i(\mathbf{x}) K_j(i).$$

The covering radius of a code can be estimated using the following result from [3, p. 230]. For completeness, the proof is included. Denote by $M(n, d, w)$ the maximum cardinality of a binary constant weight code of length n, minimum distance d and fixed weight w.

Theorem 2 *Let r be an integer and*

$$f(x) = \sum_{i=0}^{n} f_i K_i(x)$$

a polynomial such that

$$f_0 > \sum_{j=d'}^{n} |f_j| M(n, d', j) \tag{3}$$

and

$$f(i) \leq 0 \quad \text{for } i = r+1, ..., n. \tag{4}$$

Then the covering radius of a linear code C with dual distance d' is at most r.

PROOF. Clearly, $A_0'(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbb{F}_2^n$. Let A_i' denote the number of words of weight i in the dual code $C^{\perp}$. It is known [3, p. 228] that $|A_i'(\mathbf{x})| \leq A_i'$ for all $i = 0, ..., n$ and for any $\mathbf{x} \in \mathbb{F}_2^n$. This implies that $A_i'(\mathbf{x}) = 0$ if $A_i' = 0$. Evidently $A_i' = 0$ at least for $i = 1, ..., d'-1$. Therefore, by the MacWilliams transform

$$\begin{aligned} \frac{1}{|C|} \sum_{i=0}^{n} f(i) A_i(\mathbf{x}) &= \frac{1}{|C|} \sum_{i=0}^{n} \sum_{j=0}^{n} f_j K_j(i) A_i(\mathbf{x}) \\ &= \sum_{j=0}^{n} f_j \frac{1}{|C|} \sum_{i=0}^{n} A_i(\mathbf{x}) K_j(i) \\ &= \sum_{j=0}^{n} f_j A_j'(\mathbf{x}) \end{aligned}$$

$$= f_0 + \sum_{j=d'}^{n} f_j A_j'(\mathbf{x})$$
$$\geq f_0 - \sum_{j=d'}^{n} |f_j| A_j'.$$

Because $A_j' \leq M(n, d', j)$, we get

$$\frac{1}{|C|} \sum_{i=0}^{n} f(i) A_i(\mathbf{x}) \geq f_0 - \sum_{j=d'}^{n} |f_j| M(n, d', j).$$

So, if

$$f_0 - \sum_{j=d'}^{n} |f_j| M(n, d', j) > 0$$

then

$$\sum_{i=0}^{n} f(i) A_i(\mathbf{x}) > 0$$

for all $\mathbf{x} \in \mathbb{F}_2^n$. If furthermore $f(i) \leq 0$ when $i = r+1, ..., n$ then (for every $\mathbf{x}$) $A_j(\mathbf{x}) \neq 0$ for some $j = 0, ..., r$ and hence the covering radius is at most r. □

3 About constant weight codes

Although the results of this section help us in the sequel to decide which bound on constant weight codes to use, this section is also of independent interest. It is known (see, e.g., [2], [16]) that applying recurrence techniques to known bounds on information rate of codes we get better bounds for minimum distance. In this section we investigate, what sort of results we may obtain employing recurrence techniques to the bounds on the cardinality of *constant* weight codes. Let us recall the conventional definition.

Definition 1 *We define the asymptotic rate function $R(\delta, \omega)$ by*

$$R(\delta, \omega) = \sup_{(d_n), (w_n)} \{\limsup_{n \to \infty} \frac{1}{n} \log_2 M(n, d_n, w_n)\}$$

where the supremum is over all sequences (d_n) and (w_n) with $\lim_{n\to\infty} d_n/n = \delta$ and $\lim_{n\to\infty} w_n/n = \omega$.

Various bounds are known for the function $R(\delta, \omega)$. Of the classical ones we may mention the Hamming bound [24, Chapter 17]

$$R(\delta, \omega) \leq H_2(\omega) - \omega H_2(\delta/(4\omega)) - (1-\omega) H_2(\delta/(4(1-\omega)))$$

and the following estimates due to Johnson [24, Chapter 17]:

$$R(\delta, \omega) \leq \begin{cases} H_2(\omega - \delta/2) - \omega H_2((\omega - \delta/2)/\omega) & \text{if } 0 \leq \omega \leq (\omega - \delta/2)/\omega \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For other earlier estimates the interested reader is referred to Levenshtein [18, 19] and Sidelnikov [27].

A very powerful result was obtained in 1977 by McEliece, Rodemich, Rumsey and Welch [25]. They used the linear programming approach to achieve the following bound.

Theorem 3 *We have*

$$R(\delta, \omega) \leq \begin{cases} 0 & \text{if } 0 \leq \omega \leq \zeta^* \\ H_2((1 - \sqrt{1-u^2})/2) & \text{if } \zeta^* \leq \omega \leq 1/2 \end{cases} \tag{5}$$

where $\zeta^* = (1 - \sqrt{1-2\delta})/2$ *and* $u = -\delta + (\delta^2 - 2\delta + 4\omega(1-\omega))^{1/2}$.

This bound is still the best known for a wide range of distances. Indeed, according to the numerical calculations, it is the best for all values of ω ($\zeta^* \leq \omega \leq 1/2$) when $\delta \geq 0.28$. For other values of distances see Table I. There we have abbreviated this bound to MRRWC and we consider only the values of ω which are larger than the Elias range ζ^*.

The previous bounds can be improved (in a certain range) using a recurrence technique explained below. There are many recurrences in the literature, see for example [24, p. 527] and [11, 22]. However, the following one due to Levenshtein [18] turns out to be a good choice.

Theorem 4 *Let* r *and* j *be integers such that* $r \equiv j \pmod 2$. *Then if* $0 \leq r - j \leq 2(n-w)$ *and* $0 \leq r + j \leq 2w$, *it follows that*

$$M(n, d, w) \leq \frac{\binom{n}{w-j}}{\binom{w}{(r+j)/2}\binom{n-w}{(r-j)/2}} M(n, d, r).$$

Table I: The points where MRRWC is the best

δ	ω	δ	ω
0.10	0.06	0.20	0.12-0.18
0.12	0.08	0.22	0.14-0.22
0.14	0.08-0.10	0.24	0.14-0.24
0.16	0.10-0.12	0.26	0.16-0.36
0.18	0.10-0.14		

We may deduce [21, p. 21] from this estimate (denote $j/n \to \xi$ and $r/n \to \eta$ as $n \to \infty$) that

$$R(\delta,\omega) \le H_2(\omega-\xi) - \omega H_2\left(\frac{\eta+\xi}{2\omega}\right) - (1-\omega)H_2\left(\frac{\eta-\xi}{2(1-\omega)}\right) + R(\delta,\eta) \quad (6)$$

under the natural constraints. This recurrence combined with (5) yields relatively good results. Indeed, when $\delta \le 0.26$ and we are not within the areas (of ω) in Table I, then according to numerical calculations (6) gives the best bound. Some values are listed in Table II.

The discussion above was only about upper estimates. For corresponding lower bounds consult [6].

4 A new bound on covering radius for linear binary codes

In this section we present a relatively strong estimate on covering radius when the dual distance is given.

The result is based on Theorem 2 and we apply to it the polynomial

$$f(x) = \frac{1}{a-x}\left\{K_{t+1}(x)K_t(a) - K_t(x)K_{t+1}(a)\right\}^2.$$

Let $x_1^{(t)} (= x_1^{(t,n)})$ denote the least zero of the Krawtchouk polynomial $K_t(x)$. Here we choose (as in [25]) the parameter a $(= a_n = a_{n,t})$ in such a way that $x_1^{(t+1)} < a < x_1^{(t)}$ and $K_t(a) = -K_{t+1}(a)$. From [3, p. 32] one can see that $K_i(a) > 0$ for $0 \le i \le t$.

The goal of this section is to prove Theorem 5. We achieve it by first examining in Section 4.1 lower estimates on the left-hand side of (3), i.e.,

Table II: Bounds on constant weight codes

δ	ω	(6)	δ	ω	(6)	δ	ω	(6)
0.02	0.06	0.2454	0.08	0.24	0.5386	0.16	0.24	0.3456
0.02	0.12	0.4474	0.08	0.30	0.6249	0.16	0.30	0.4319
0.02	0.18	0.5981	0.08	0.36	0.6863	0.16	0.36	0.4933
0.02	0.24	0.7130	0.08	0.42	0.7250	0.16	0.42	0.5320
0.02	0.30	0.7993	0.08	0.48	0.7424	0.16	0.48	0.5494
0.02	0.36	0.8607	0.10	0.12	0.2221	0.18	0.18	0.1856
0.02	0.42	0.8994	0.10	0.18	0.3728	0.18	0.24	0.3006
0.02	0.48	0.9168	0.10	0.24	0.4878	0.18	0.30	0.3868
0.04	0.06	0.1818	0.10	0.30	0.5740	0.18	0.36	0.4482
0.04	0.12	0.3837	0.10	0.36	0.6354	0.18	0.42	0.4870
0.04	0.18	0.5344	0.10	0.42	0.6742	0.18	0.48	0.5044
0.04	0.24	0.6494	0.10	0.48	0.6916	0.20	0.24	0.2564
0.04	0.30	0.7357	0.12	0.12	0.1733	0.20	0.30	0.3427
0.04	0.36	0.7971	0.12	0.18	0.3240	0.20	0.36	0.4040
0.04	0.42	0.8358	0.12	0.24	0.4389	0.20	0.42	0.4428
0.04	0.48	0.8532	0.12	0.30	0.5252	0.20	0.48	0.4602
0.06	0.06	0.1245	0.12	0.36	0.5866	0.22	0.24	0.2129
0.06	0.12	0.3264	0.12	0.42	0.6254	0.22	0.30	0.2992
0.06	0.18	0.4771	0.12	0.48	0.6427	0.22	0.36	0.3606
0.06	0.24	0.5921	0.14	0.12	0.1260	0.22	0.42	0.3993
0.06	0.30	0.6784	0.14	0.18	0.2767	0.22	0.48	0.4167
0.06	0.36	0.7397	0.14	0.24	0.3916	0.24	0.30	0.2563
0.06	0.42	0.7785	0.14	0.30	0.4779	0.24	0.36	0.3177
0.06	0.48	0.7959	0.14	0.36	0.5393	0.24	0.42	0.3564
0.08	0.06	0.0710	0.14	0.42	0.5781	0.24	0.48	0.3738
0.08	0.12	0.2729	0.14	0.48	0.5955	0.26	0.42	0.3140
0.08	0.18	0.4236	0.16	0.18	0.2306	0.26	0.48	0.3314

on f_0 and then treating in Sections 4.2 and 4.3 upper bounds on $|f_j|$ and $M(n,d',j)$ on the right-hand side of (3).

Let us recall two results concerning Krawtchouk polynomials. For the first one see [3, p. 30].

Lemma 1 (Christoffel-Darboux formula)

$$\sum_{i=0}^{k} \frac{K_i(x)K_i(y)}{\binom{n}{i}} = -\frac{k+1}{2\binom{n}{k}} \frac{K_{k+1}(x)K_k(y) - K_k(x)K_{k+1}(y)}{x-y}.$$

Lemma 2 ([25]) *We have*

$$K_r(x)K_s(x) = \sum_{j=0}^{n} \binom{n-j}{\frac{r+s-j}{2}} \binom{j}{\frac{r-s+j}{2}} K_j(x)$$

where the binomial coefficient with fractional or negative lower index is to be interpreted as zero.

Proposition 1 *We may write the polynomial $f(x)$ defined above as $f(x) = \sum_{j=0}^{n} f_j K_j(x)$ where*

$$f_j = \frac{2K_t(a)}{t+1}\binom{n}{t} \sum_{i=0}^{t} \frac{K_i(a)}{\binom{n}{i}} \left\{ \binom{n-j}{\frac{t+1+i-j}{2}} \binom{j}{\frac{t+1-i+j}{2}} + \binom{n-j}{\frac{t+i-j}{2}} \binom{j}{\frac{t-i+j}{2}} \right\}. \tag{7}$$

Proof. Using Lemma 1 and the fact $K_t(a) = -K_{t+1}(a)$ we may write

$$\begin{aligned} f(x) &= \frac{2}{t+1}\binom{n}{t} \{K_{t+1}(x)K_t(a) - K_t(x)K_{t+1}(a)\} \sum_{i=0}^{t} \frac{K_i(x)K_i(a)}{\binom{n}{i}} \\ &= \frac{2K_t(a)}{t+1}\binom{n}{t} \sum_{i=0}^{t} \frac{K_i(a)}{\binom{n}{i}} \{K_{t+1}(x)K_i(x) + K_t(x)K_i(x)\}. \end{aligned}$$

Lemma 2 yields the assertion. □

4.1 Lower bound on f_0

Evidently, we can deduce from (7) that

$$f_0 \geq \frac{2K_t(a)^2}{t+1}\binom{n}{t}. \tag{8}$$

The value of $K_t(a)$ can be lower estimated according to the result below. For the proof see Appendix.

Lemma 3 *Assume that* $1 \leq t \leq \frac{n}{2}$, $x_1^{(t+1)} < a < x_1^{(t)}$ *and* $K_t(a) = -K_{t+1}(a)$. *Then*

$$K_t(a) \geq \frac{1}{3t^3} \max_{i=0,\ldots,t-1} K_i(a).$$

Observe that clearly $\max_{i=0,\ldots,t-1} K_i(a)$ in Lemma 3 can be lower estimated by any $K_i(a)$, $i = 0, ..., t-1$. So, if we choose $i = t - e$, $(e = \lfloor \varepsilon n \rfloor$, $\varepsilon > 0)$, we are able to use Lemma 4 below, which is originally due to Kalai and Linial [12], because we are not at the critical point of this lemma. Let $\tau < 1/2$. Notice that if $\lim_{n\to\infty} t/n = \tau$, then [3, p. 32]

$$\psi = \lim_{n\to\infty} \frac{x_1^{(t)}}{n} = \frac{1}{2} - \sqrt{\tau(1-\tau)}.$$

Lemma 4 ([13]) *Let* $\eta = \lim_{n\to\infty} b/n$. *For* $\eta < \psi$

$$\begin{aligned} \frac{1}{n}\log_2 K_t(b) &= \frac{1}{n}\log_2 \binom{n}{t} \\ &\quad + \int_0^{\eta} \log_2 \left(\frac{1 - 2\tau + \sqrt{(1-2\tau)^2 - 4z(1-z)}}{2-2z} \right) dz + o(1). \end{aligned} \tag{9}$$

This enables us to estimate $K_t(a)$, and hence also f_0, precisely enough as seen in the following corollary. We denote by $\mathcal{I}(\tau, \eta)$ the limit of the right-hand side of (9) as $n \to \infty$.

Let us first introduce a technical lemma. For the proof consult [26, p. 150].

Lemma 5 *Suppose* K *is compact and* $\{\nu_h\}$ $(h = 1, 2, ...)$ *is a sequence of continuous functions on* K. *Assume also that* $\{\nu_h\}$ *converges pointwise to a continuous function* ν *on* K *and* $\nu_h(x) \geq \nu_{h+1}(x)$ *for all* $x \in K$ $(h = 1, 2, 3, ...)$. *Then* $\{\nu_h\}$ *converges uniformly to* ν *on* K.

Corollary 1 *For the sequence* $(a_n)_{n=1}^{\infty}$ *defined in the beginning of this section (below we omit the subscript)*

$$\liminf_{n\to\infty} \frac{1}{n}\log_2 K_t(a) \geq \mathcal{I}(\tau, \psi)$$

where $\psi = \lim_{n\to\infty} a/n = \frac{1}{2} - \sqrt{\tau(1-\tau)}$.

PROOF. By Lemmas 3 and 4

$$\liminf_{n\to\infty} \frac{1}{n} \log_2 K_t(a) \geq \lim_{n\to\infty} \frac{1}{n} \log_2 K_{t-e}(a) = \mathcal{I}(\tau - \varepsilon, \psi).$$

Let $(\varepsilon_h)_{h=1}^{\infty}$ be a fixed sequence of positive real numbers such that $\varepsilon_h \geq \varepsilon_{h+1}$ $(h = 1, 2, ...)$ and $\varepsilon_h \to 0$ as $h \to \infty$. Then

$$\liminf_{n\to\infty} \frac{1}{n} \log_2 K_t(a) \geq \lim_{h\to\infty} \mathcal{I}(\tau - \varepsilon_h, \psi) = H_2(\tau) \tag{10}$$
$$+ \lim_{h\to\infty} \int_0^{\psi} \log_2 \left(\frac{1 - 2(\tau - \varepsilon_h) + \sqrt{(1 - 2(\tau - \varepsilon_h))^2 - 4z(1-z)}}{2 - 2z} \right) dz.$$

Denote

$$\nu_h(z) = \log_2 \left(\frac{1 - 2(\tau - \varepsilon_h) + \sqrt{(1 - 2(\tau - \varepsilon_h))^2 - 4z(1-z)}}{2 - 2z} \right)$$

and

$$\nu(z) = \log_2 \left(\frac{1 - 2\tau + \sqrt{(1 - 2\tau)^2 - 4z(1-z)}}{2 - 2z} \right).$$

Notice that these functions $\nu_h(x)$ and $\nu(x)$ are all continuous on $[0, \psi]$ and $(\nu_h(z))_{h=1}^{\infty}$ converges pointwise to $\nu(z)$ on the same interval. Moreover, $\nu_{h+1}(z) \leq \nu_h(z)$ for all $z \in [0, \psi]$ and h. Then by the preceding lemma we know that $\{\nu_h\}$ converges to ν uniformly on the interval $[0, \psi]$.

Thus,

$$\lim_{h\to\infty} \int_0^{\psi} \nu_h(z) dz = \int_0^{\psi} \nu(z) dz$$

and hence the claim of the lemma follows from (10). □

4.2 Upper bounds on f_j

Besides the lower bound on f_0 we need also upper bounds on f_j. Observe that the natural requirement $\deg f(x) = 2t+1 \leq n$ implies that $t \leq (n-1)/2$ (in fact we assume the stronger restriction $t < (n-1)/2$ to hold). Recalling that [24, p. 309]

$$\binom{n}{s} \leq 2^{nH_2(s/n)}$$

we get by Proposition 1

$$\begin{aligned} f_j &\leq \frac{2K_t(a)}{t+1}\binom{n}{t}\sum_{i=0}^{t}\frac{K_i(a)}{\binom{n}{i}}2^j\left\{\binom{n-j}{\frac{t+1+i-j}{2}}+\binom{n-j}{\frac{t+i-j}{2}}\right\} \\ &\leq \frac{2K_t(a)}{t+1}\binom{n}{t}\sum_{i=\max\{0,j-t-1\}}^{t}\frac{K_i(a)}{\binom{n}{i}}2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}. \end{aligned} \tag{11}$$

For asymptotic estimations we shall also need another upper bound on f_j.

Evidently,

$$2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}$$

grows as i increases ($i \leq t$). Next we show that the other term in the sum of (11), namely $K_i(a)/\binom{n}{i}$, decreases with respect to i.

Lemma 6 *Let a be fixed and $0 < a < x_1^{(t)}$. Then*

$$\frac{K_i(a)}{\binom{n}{i}} \geq \frac{K_{i+1}(a)}{\binom{n}{i+1}}$$

for all $i = 0, ..., t-1$.

PROOF. The claim is equivalent to $(n-i)K_i(a) \geq (i+1)K_{i+1}(a)$. By the well known recurrence relation [3, p. 28]

$$(k+1)K_{k+1}(x) = (n-2x)K_k(x) - (n-k+1)K_{k-1}(x) \tag{12}$$

we get

$$(i+1)K_{i+1}(a) - (n-i)K_i(a) = iK_i(a) - (n-i+1)K_{i-1}(a) - 2aK_i(a).$$

The sought result follows by induction. □

Thus we know that $K_i(a)/\binom{n}{i}$ decreases and

$$2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}$$

increases as i grows. We proceed now using this fact. In the sequel we shall assume that the parameters t_n, j_n and a_n grow with n. To simplify notations we omit the subscript n of the parameters. Let $\lim_{n\to\infty} j/n = \xi$ and $\lim_{n\to\infty} t/n = \tau$. Assume that $\xi < 2\tau$.

Let us divide the interval $[\max\{0, j-t-1\}, t]$ into $p \in \mathbb{N}$ equal-length pieces. We get an upper bound on f_j by evaluating $K_i(a)/\binom{n}{i}$ at the lower end and

$$2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}$$

at the upper end of each sub-interval. More precisely,

$$\begin{aligned} f_j &\leq \frac{2K_t(a)}{t+1}\binom{n}{t}\left\lceil \frac{t-\max\{0,j-t-1\}}{p}\right\rceil \\ &\quad \cdot \sum_{l=0}^{p-1} \frac{K_{i(t,j,l)}(a)}{\binom{n}{i(t,j,l)}} 2^{j+(n-j)H_2\left(\frac{t+1+i(t,j,l+1)-j}{2n-2j}\right)} \end{aligned} \tag{13}$$

where we define $(l = 0, 1, ..., p)$

$$i(t,j,l) = \left\lfloor \max\{0, j-t-1\} + \frac{t-\max\{0,j-t-1\}}{p} l \right\rfloor .$$

Denote

$$M(\xi,\tau) = \max_{l=0,\ldots,p-1} \left\{ \limsup_{n\to\infty} \frac{1}{n} \log_2 \left(\frac{K_{i(t,j,l)}(a)}{\binom{n}{i(t,j,l)}} 2^{j+(n-j)H_2\left(\frac{t+1+i(t,j,l+1)-j}{2n-2j}\right)} \right) \right\}.$$

Observe that we can treat this with the aid of Lemma 4. Let $\epsilon > 0$. Thus, by (13) for large enough n

$$\begin{aligned} f_j &\leq \frac{2K_t(a)}{t+1}\binom{n}{t}\left\lceil \frac{t-\max\{0,j-t-1\}}{p}\right\rceil \sum_{l=0}^{p-1} 2^{n(M(\xi,\tau)+\epsilon)} \\ &= \frac{2K_t(a)}{t+1}\binom{n}{t} p \left\lceil \frac{t-\max\{0,j-t-1\}}{p}\right\rceil 2^{n(M(\xi,\tau)+\epsilon)}. \end{aligned} \tag{14}$$

Note that we deal only with finite number $(= p)$ of limit processes and hence there actually exists n_ϵ such that this holds for every ϵ when $n \geq n_\epsilon$.

4.3 The verification of the main result

Let us now return to the condition (3):

$$f_0 > \sum_{j=d'}^{n} |f_j| M(n, d', j).$$

By virtue of (11) and (8) this is fulfilled if

$$\frac{2K_t(a)^2}{t+1}\binom{n}{t} > \sum_{j=d'}^{2t+1} M(n,d',j)\frac{2K_t(a)}{t+1}\binom{n}{t} \cdot \sum_{i=\max\{0,j-t-1\}}^{t} \frac{K_i(a)}{\binom{n}{i}} 2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}. \quad (15)$$

Notice that we may divide by $K_t(a)\binom{n}{t}\frac{2}{t+1}$. Denote

$$\hat{f}(j) = \sum_{i=\max\{0,j-t-1\}}^{t} \frac{K_i(a)}{\binom{n}{i}} 2^{j+(n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right)}.$$

In order to approximate the right-hand side of (15) we employ the same method as in the calculation of (14). In other words, we divide the interval $[d', 2t+1]$ (in fact sometimes a smaller interval turns out to be sufficient) into several pieces and evaluate decreasing terms like $\hat{f}(j)$ (see below) of the sum $\sum_{j=d'}^{2t+1}$ at the lower end of the sub-interval and possible growing terms at the upper end of it.

Let us denote

$$g(j) = j + (n-j)H_2\left(\frac{t+1+i-j}{2n-2j}\right).$$

Lemma 7

$$g'(j) = \frac{1}{2}\log_2\left(\frac{(t+1+i-j)(2n-t-i-1-j)}{(n-j)^2}\right).$$

PROOF. By routine calculations the derivative

$$\begin{aligned} g'(j) &= 1 + \frac{1}{2}\log_2\left(\frac{t+1+i-j}{2(n-j)}\right) + \frac{1}{2}\log_2\left(1 - \frac{t+1+i-j}{2(n-j)}\right) \\ &= \frac{1}{2}\log_2\left(\frac{(t+1+i-j)(2n-t-i-1-j)}{(n-j)^2}\right). \end{aligned}$$

□

According to this derivative the function $g(j)$ decreases as j grows. Since also the number of terms in the sum $\sum_{i=\max\{0,j-t-1\}}^{t}$ does not increase with respect to j, the term $\hat{f}(j)$ decreases as j grows.

4.3.1 Upper bounds on $M(n,d',j)$

At this point, we need an estimate to the $M(n,d',w)$ for (15). Note that we cannot use the conjecture [24, p. 691] ($w_1 < w_2 \leq n/2$)

$$M(n,d',w_1) \leq M(n,d',w_2)$$

to simplify our argument.

By [25],

$$M(n,d',w) \leq \frac{1}{h_1^{(s+1)}} \binom{n}{s} n^6 \tag{16}$$

provided that $s < w \leq n/2$, $s < (n-1)/2$ and $h_1^{(s)} \leq d'/2$ where $h_1^{(s)}$ is the least zero of the Hahn polynomial [20, p. 513]

$$J_s(x) = J_s^{(n,w)}(x) = \sum_{j=0}^{s} (-1)^j \frac{\binom{s}{j}\binom{n+1-s}{j}}{\binom{w}{j}\binom{n-w}{j}} \binom{x}{j}.$$

By Levenshtein [20, Section 5.2],

$$\begin{aligned} h_1^{(s)} \quad \leq \quad & \frac{w}{l'}\left(1 + \frac{(n+2)^2}{(n-2s)^2}\right) \\ & + \left(1 - \frac{2}{l'}\right)\left(\frac{w(n-w) - (s - \lfloor\sqrt{s}\rfloor)(n - s + \lfloor\sqrt{s}\rfloor)}{n + 2\sqrt{(s - \lfloor\sqrt{s}\rfloor)(n - s + \lfloor\sqrt{s}\rfloor)}}\right) \end{aligned} \tag{17}$$

provided that $1 < s < \frac{1}{2}(n+1-\sqrt{(n+1)^2 - 4w(n-w)})$. Here $l' = \lfloor\sqrt{s}\rfloor + 1$. Denote the right-hand side of (17) by $U(n,w,s)$. Next we estimate $M(n,d',w)$ in an interval $w_{i-1} \leq w \leq w_i$. As an interval we take a piece, which we obtain by dividing $[d', \min\{\lfloor\frac{1}{2}n\rfloor, 2t+1\}]$ into $q \in \mathbb{N}$ sub-intervals. As can be seen later, we have to choose $\tau > \delta'/2$ in order to improve on the Tietäväinen bound (1). Notice that since $\hat{f}(j)$ is decreasing and $M(n,d',w) = M(n,d',n-w)$, there are no difficulties to treat the values $\lfloor n/2\rfloor < j \leq n - d'$ (the tail $n - d' < j \leq 2t+1$ can be handled easily as will be seen in the sequel). More precisely, we get the sub-intervals $[w_{i-1}, w_i]$ ($i = 1, ..., q$) where for $i = 0, ..., q$

$$w_i = \left\lfloor d' + \frac{\min\{\lfloor\frac{1}{2}n\rfloor, 2t+1\} - d'}{q} i \right\rfloor.$$

Evidently, (denoting $\lim_{n\to\infty} w/n = \alpha$ and $\lim_{n\to\infty} s/n = \beta$)

$$\frac{U(n,w,s)}{n} \to \frac{\alpha(1-\alpha)-\beta(1-\beta)}{1+2\sqrt{\beta(1-\beta)}}, \quad (n\to\infty). \tag{18}$$

In order to use the upper bound on $M(n,d',w)$ given above in (16), we need to choose the degrees of the Hahn polynomials, say $s_i = \lceil \beta_i n \rceil$. Let $\lim_{n\to\infty} d'/n \to \delta'$. We do this choice so that the following holds, ($i = 1,...,q$),

$$\frac{\alpha_i(1-\alpha_i)-\beta_i(1-\beta_i)}{1+2\sqrt{\beta_i(1-\beta_i)}} < \frac{\delta'}{2} \tag{19}$$

where

$$\alpha_i = \lim_{n\to\infty} \frac{w_i}{n} = \delta' + \frac{\min\{1/2, 2\tau\} - \delta'}{q} i.$$

Then by virtue of (18) and (19) one may conclude that $U(n,w_i,s_i) < d'/2$ for sufficiently large n. Furthermore, since $U(n,w,s_i)$ is increasing with respect to w (n and s_i are fixed), the same $s_i = \lceil \beta_i n \rceil$ satisfies $U(n,w,s_i) < d'/2$ also for all the smaller values of w in the interval $w_{i-1} \le w \le w_i$. So, we are allowed to use the result (16) for *all* the weights $w_{i-1} \le w \le w_i$ (for large enough n) provided that the condition of (17) is fulfilled. It is guaranteed by requiring

$$0 < \beta_i < \alpha_{i-1} \tag{20}$$

for all $i = 1,...,q$. Observe that for sufficiently large n this implies that

$$s_i + 1 < \frac{1}{2}(n+1-\sqrt{(n+1)^2 - 4w_{i-1}(n-w_{i-1})})$$

as well and hence by [1, p. 123]

$$h_1^{(s_i+1)} \ge 1.$$

Summing up, we know that for large n

$$M(n,d',w) \le \binom{n}{s_i} n^6 \tag{21}$$

for *all* $w_{i-1} \le w \le w_i$.

4.3.2 The main theorem

Let us now return to upper estimating the sum $\sum_{j=d'}^{2t+1}$ on the right-hand side of (15). From the above one can see that we are in a situation where the upper bound $\hat{f}(j)$ on $f_j/(2\binom{n}{t}K_t(a)(t+1)^{-1})$ is decreasing and the bound on $M(n,d',j)$ is like a "step function", that is, the bound on $M(n,d',j)$ is constant in each sub-interval $w_{i-1} \le j \le w_i$ of $[d', 2t+1]$. Hence, evaluating (with respect to j) the upper bound $\hat{f}(j)$ at the beginning of an interval and the bound for $M(n,d',j)$ at the end of the same interval, we get the upper estimate for every interval $[w_{i-1}, w_i]$:

$$\begin{aligned} &\sum_{j=w_{i-1},\ldots,w_i} M(n,d',j) \sum_{\nu=\max\{0,j-t-1\}}^{t} \frac{K_\nu(a)}{\binom{n}{\nu}} 2^{g(j)} \\ &\le \quad (w_i - w_{i-1} + 1)\binom{n}{s_i} n^6 \hat{f}(w_{i-1}) \end{aligned}$$

when n sufficiently large. Moreover, to $\hat{f}(j)$ we can apply the easily computed result (14) (notice Lemma 4).

In Section 4.3.1 we left open the treatment of $M(n,d',j)$ in the case that $n - d' < j \le 2t+1$. If $1 - \delta' \le 2\tau$, then we have to deal with this tail. It is treated as follows.

Let λ be a positive number less than Elias range, i.e., $\lambda < 1/2(1 - \sqrt{1-2\delta'})$. Denote $l = \lfloor \lambda n \rfloor$. Let us first consider the range $n - d' < j \le n - l$ (if $2\tau < 1 - \lambda$ then this is all we have to check). Since $M(n,d',w) = M(n,d',n-w)$, we have as above that (for sufficiently large n)

$$M(n,d',w) \le \binom{n}{s_1} n^6$$

for all $n - d' < w \le n - l$ if $\beta_1 < \lambda$. Hence for large n

$$\begin{aligned} &\sum_{j=n-d'+1,\ldots,n-l} M(n,d',j) \sum_{\nu=\max\{0,j-t-1\}}^{t} \frac{K_\nu(a)}{\binom{n}{\nu}} 2^{g(j)} \\ &\le \quad (d' - l)\binom{n}{s_1} n^6 \hat{f}(n - d' + 1). \end{aligned}$$

Evidently, this can be combined with the corresponding sum over the first interval $[w_0, w_1]$, because $\hat{f}(j)$ is decreasing in j.

The treatment of $n - l < j \leq 2t + 1$ (this should be taken into account if $1 - \lambda \leq 2\tau$) goes similarly and is simpler, since we are allowed to use the result (subexponential in n) of the following well-known lemma instead of (21). By an (n, M, d) code we mean a code of length n, cardinality M and minimum distance d. For the proof of the lemma see [21, p. 61].

Lemma 8 *Assume that all the words of an (n,M,d) code have weight at most r, where $r \leq n/2$ and $r^2 - nr + nd/2 > 0$. Then*

$$M \leq \frac{\frac{1}{2}nd}{r^2 - nr + \frac{1}{2}nd}.$$

Now we are able to present the main result.

Theorem 5 *Let $(C_n)_{n=1}^{\infty}$ be a sequence of binary linear codes of length n, dual distance $d' = d'(n)$ and covering radius $R = R(n)$ where $R/n \to \rho$ and $d'/n \to \delta' \in (0, 1/2)$ when $n \to \infty$. Suppose that the requirements (20) and (19) hold for $i = 1, ..., q$ and $\beta_1 < \zeta$. If $\delta'/2 < \tau < 1/2$ and*

$$\mathcal{I}(\tau, 1/2 - \sqrt{\tau(1-\tau)}) > \max_{v=0,\ldots,q-1} \{H_2(\beta_{v+1}) + M(\alpha_v, \tau)\},$$

then

$$\rho \leq \frac{1}{2} - \sqrt{\tau(1-\tau)}. \tag{22}$$

PROOF. Let us choose $r = \lceil a \rceil$ in the condition (4). Then, by the discussion above, we get using (15) and Corollary 1 the statement by Theorem 2. Notice also that $\lambda < \zeta$. □

In Figure 1 the bounds on binary linear codes are illustrated (there we have used $p = 150$ and $q = 250$).

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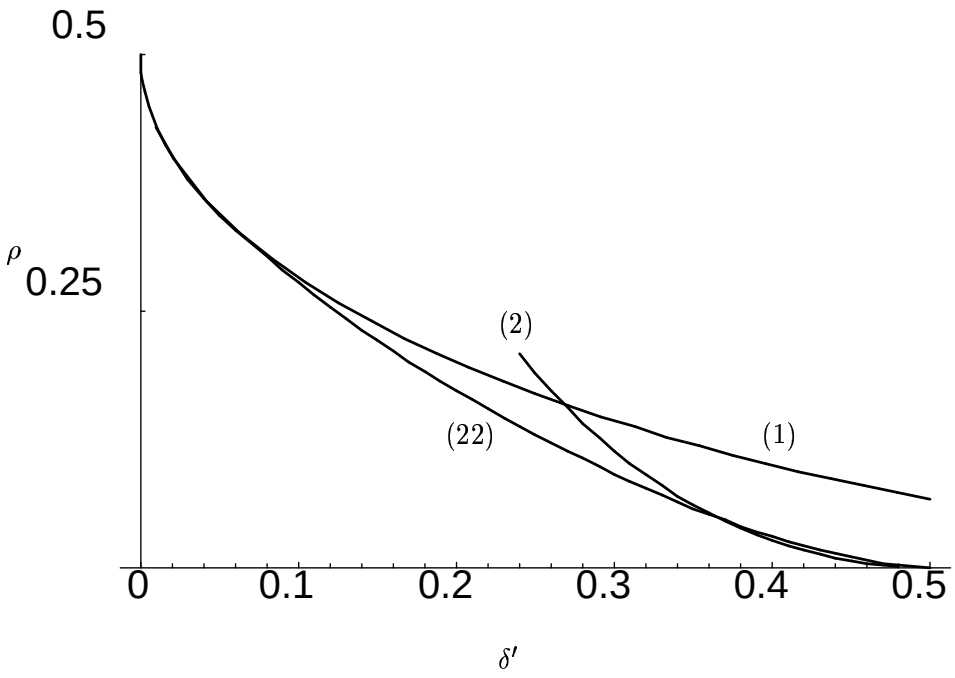


Figure 1: A comparison for binary linear codes

A Appendix

Before going to the proof of Lemma 3 we show the next auxiliary lemma.

Lemma 9 *Let $K_t'(x)$ denote the derivative of the Krawtchouk polynomial of degree t. Then*

$$K_t'(x) = -2 \sum_{i=0}^{\lfloor \frac{t-1}{2} \rfloor} \frac{K_{t-2i-1}(x)}{2i+1}.$$

PROOF. Differentiating the equation (12) and using induction we obtain

$$\begin{aligned}
(t+1)K_{t+1}'(x) &= (n-2x)K_t'(x) - (n-t+1)K_{t-1}'(x) - 2K_t(x) \\
&= -2K_1(x) \sum_{i=0}^{\lfloor \frac{t-1}{2} \rfloor} \frac{K_{t-2i-1}(x)}{2i+1} + 2(n-t+1) \sum_{i=0}^{\lfloor \frac{t-2}{2} \rfloor} \frac{K_{t-2i-2}(x)}{2i+1} - 2K_t(x) \\
&= \sum_{i=0}^{\lfloor \frac{t-1}{2} \rfloor} \frac{-2(t-2i)}{2i+1} K_{t-2i}(x) + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \frac{-2(n-t+2i)}{2i-1} K_{t-2i}(x) \\
&\quad + \sum_{i=1}^{\lfloor \frac{t}{2} \rfloor} \frac{2(n-t+1)}{2i-1} K_{t-2i}(x) - 2K_t(x) \\
&= \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \frac{-2(t-2i)}{2i+1} K_{t-2i}(x) - 2 \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} K_{t-2i}(x) \\
&= -2(t+1) \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \frac{K_{t-2i}(x)}{2i+1}
\end{aligned}$$

where the third equation follows from the formula (see (12))

$$K_1(x)K_{t-2i-1}(x) = (t-2i)K_{t-2i}(x) + (n-t+2i+2)K_{t-2i-2}(x).$$

□

Now we are in the position to prove the needed lemma.

Proof of Lemma 3. Let $y \to x$ in the Christoffel-Darboux formula. Then the left-hand side of it approaches to

$$\sum_{i=0}^{t} \frac{K_i(x)^2}{\binom{n}{i}}.$$

The right-hand side can be written as

$$-\frac{t+1}{2\binom{n}{t}}\frac{(K_{t+1}(x)-K_{t+1}(y))K_t(y)-(K_t(x)-K_t(y))K_{t+1}(y)}{x-y}$$

and this tends to

$$-\frac{t+1}{2\binom{n}{t}}(K'_{t+1}(x)K_t(x)-K'_t(x)K_{t+1}(x)).$$

Therefore we have for a real x

$$\sum_{i=0}^{t}\frac{K_i(x)^2}{\binom{n}{i}}=-\frac{t+1}{2\binom{n}{t}}(K'_{t+1}(x)K_t(x)-K'_t(x)K_{t+1}(x)).$$

Consequently, recalling that $K_t(a)=-K_{t+1}(a)$,

$$\begin{aligned}\sum_{i=0}^{t}\frac{K_i(a)^2}{\binom{n}{i}} &= -\frac{t+1}{2\binom{n}{t}}(K'_{t+1}(a)K_t(a)-K'_t(a)K_{t+1}(a))\\ &= -\frac{t+1}{2\binom{n}{t}}K_t(a)(K'_{t+1}(a)+K'_t(a)).\end{aligned}$$

By the previous lemma we can conclude that

$$\begin{aligned}\frac{K_t(a)^2}{\binom{n}{t}}+\sum_{i=0}^{t-1}\frac{K_i(a)^2}{\binom{n}{i}} &= \frac{t+1}{\binom{n}{t}}K_t(a)\left(\sum_{i=0}^{\lfloor\frac{t}{2}\rfloor}\frac{K_{t-2i}(a)}{2i+1}+\sum_{i=0}^{\lfloor\frac{t-1}{2}\rfloor}\frac{K_{t-2i-1}(a)}{2i+1}\right)\\ &= \frac{t+1}{\binom{n}{t}}K_t(a)\left(K_t(a)+\sum_{i=1}^{\lfloor\frac{t}{2}\rfloor}\frac{K_{t-2i}(a)}{2i+1}+\sum_{i=0}^{\lfloor\frac{t-1}{2}\rfloor}\frac{K_{t-2i-1}(a)}{2i+1}\right).\end{aligned}$$

Thus we may estimate ($1\le t\le\frac{n}{2}$)

$$\begin{aligned}0 &= tK_t(a)^2+(t+1)K_t(a)\left(\sum_{i=1}^{\lfloor\frac{t}{2}\rfloor}\frac{K_{t-2i}(a)}{2i+1}+\sum_{i=0}^{\lfloor\frac{t-1}{2}\rfloor}\frac{K_{t-2i-1}(a)}{2i+1}\right)\\ &\quad -\sum_{i=0}^{t-1}\frac{K_i(a)^2\binom{n}{t}}{\binom{n}{i}}\\ &\le tK_t(a)^2+(t+1)K_t(a)\sum_{i=0}^{t-1}K_i(a)-\sum_{i=0}^{t-1}K_i(a)^2\end{aligned}$$

$$\leq \quad tK_t(a)^2 + \left(3t^2 - \frac{1}{3t^2}\right) K_t(a) \sum_{i=0}^{t-1} K_i(a) - \frac{1}{t} \left(\sum_{i=0}^{t-1} K_i(a)\right)^2$$

$$= \quad \left(K_t(a) + 3t \sum_{i=0}^{t-1} K_i(a)\right) \left(tK_t(a) - \frac{1}{3t^2} \sum_{i=0}^{t-1} K_i(a)\right)$$

and therefore the claim follows from

$$K_t(a) \geq \frac{1}{3t^3} \sum_{i=0}^{t-1} K_i(a).$$

□

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