# ♦ Performance Results for Digital Subscriber Line Precoders

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*Performance results are presented for digital subscriber line (DSL) precoders, also known as dynamic spectrum management (DSM) level 3, in terms of improvement of signal-to-noise ratio (SNR) and achievable bit rates under the very high speed digital subscriber line 2 (VDSL2) standard. Simulations were conducted using channel coefficients obtained from both empirical models and line measurements. The performance impact of various key factors is examined, including the number of lines, line lengths, frequency range, precoder design, channel measurement accuracy, and implementation accuracy (quantization errors). The results provide estimates of the achievable rate gains and offer guidance for the design and implementation of DSL precoders. © 2008 Alcatel-Lucent.*

## Introduction

Crosstalk is one of the major impairments in digital subscriber line (DSL) technology. Its level may be 10 dB to 15 dB or more above the level of the other impairments present in a typical DSL binder. Due to the many different deployments of copper, the crosstalk levels from binder to binder, as well as within a binder, may vary greatly. Furthermore, there is a non-neglectable variation of crosstalk coupling between two lines within the same binder. All this makes the task of crosstalk cancellation challenging, but it can lead to large improvements in rate and reach. As such, it can delay the need to deploy fiber in the access network and can help DSL operators compete with other access technologies such as wireless or cable.

This document provides performance results for DSL precoders, also known as dynamic spectrum management (DSM) level 3, in terms of improvement of signal-to-noise ratio (SNR) and achievable bit rates under the very high speed digital subscriber line 2 (VDSL2) standard [1–7]. We investigate the impact of various key factors, such as the number of lines, line lengths, frequency range, choice of precoder, channel measurement accuracy, and implementation accuracy (quantization errors).

## Preliminaries

The following sections describe the problem and the data used for our performance analysis.

### SNR and Crosstalk

We focus on VDSL2-standard parameter values and systems with VDSL2-12a and VDSL2-17a profiles [5]. We consider wires having diameters 0.5 mm, American wire gauge (AWG) 24. We further focus on the downstream and $L = 4$ or $L = 6$ lines.

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The SNR for a given line $l$, $l = 1, 2. . ., L$, at frequency (or tone) $f$ is calculated as

$$\gamma_f^l = \frac{P_f^l |h_f^{l,l}|^2}{\sum_{m \neq l} P_f^m |\hat{h}_f^{l,m}|^2 + N_f^l}$$

where $P_f^l$ is the transmit power, $N_f^l$ is the noise power, and $\hat{h}_f^{l,m}$ is the channel gain from line $m$ to line $l$. The $\hat{h}_f^{l,m}$ are determined by the concatenation of the precoder and the physical channel: i.e., we have, $\hat{h}_f^l = C_f h_f^l$, with $C_f$ representing the precoder matrix (taken to be the identity matrix in the absence of a precoder), $\hat{h}_f^l = [\hat{h}_f^{l,1}, \ldots, \hat{h}_f^{l,L}]^T$ and $h_f^l = [h_f^{l,1}, \ldots, h_f^{l,L}]^T$. The noise is taken to be Gaussian with uniform power spectral density (PSD) $N_f^l \equiv N = 10^{-14}$ mW/Hz (or $-140$ dBm/Hz). The transmit PSD is taken to be flat with $P_f^l \equiv P = 10^{-6}$ mW/Hz (or $-60$ dBm/Hz). The noise and transmit power per tone is approximately 4000 times these numbers since each tone is used 4000 times per second. However, the SNR depends only on the ratios, $\frac{P_f^l}{N_f^l}$ so we focus on the PSDs.

The total rate $R^l$ for a given line $l$ is determined as

$$R^l = \sum_f R_f^l$$

where

$$R_f^l = s \cdot \min\left\{ \log_2\left(1 + \frac{\gamma_f^l}{\Gamma}\right), 15 \right\}$$

denotes the rate for line $l$ for a given tone $f$, with $s$ representing the symbol rate and $\Gamma$ the so-called SNR gap, which models the effect of DSL error coding. Here, $s = 4000$ symbols per second. Note that $R_f^l$ is rounded down to be no more than 15 bits per tone corresponding to the VDSL2 bit-loading limitation. The SNR gap $\Gamma$ in decibels is computed as

$$\Gamma = \gamma_m - \gamma_c + \gamma_e$$

where $\gamma_e$ is the margin for error rate (taken to be 9.8 dB for a bit-error rate of $10^{-7}$), $\gamma_c$ is the coding gain (3 dB), and $\gamma_m$ is the margin (6 dB). The highest required SNR, corresponding to 15 bits per symbol, is approximately 45 dB, or actually 58 dB true SNR (45 dB + $\Gamma$).

For the VDSL2-12a downstream there are 1604 tones, so the maximum bit rate is

$$1604 \times 4000 \times 15 = 96.24 \text{ Mbit/s}$$

For the VDSL2-17a downstream there are 2917 tones, so the maximum bit rate is

$$2917 \times 4000 \times 15 = 175.02 \text{ Mbit/s}$$

It will be convenient to define the relative crosstalk coefficient for a victim line $l$ and crosstalker $m$ at tone $f$ as

$$\varepsilon_f^{l,m} := \frac{|h_f^{l,m}|}{|h_f^{l,l}|}$$

Also, define

$$\varepsilon_f^l := \max_{m \neq l} \varepsilon_f^{l,m}$$

to be the largest relative crosstalk coefficient for a given line $l$ at a given tone $f$, and let

$$\varepsilon_f := \max_l \varepsilon_f^l$$

be the largest crosstalk coefficient at a given tone $f$. The latter quantity plays a crucial role in determining the scale for making crosstalk corrections. Specifically, the smaller the value of this coefficient, the more difficult it is to cancel the crosstalk because the corrections are implemented using finite-precision arithmetic.

**Panel 1. Abbreviations, Acronyms, and Terms**

A/D—Analog/digital
AWG—American wire gauge
DS—Downstream
DSL—Digital subscriber line
DSM—Dynamic spectrum management
FFT—Fast Fourier transform
IFFT—Inverse fast Fourier transform
PSD—Power spectral density
SNR—Signal-to-noise ratio
VDSL2—Very high speed digital subscriber line 2
XTCC—Crosstalk correction coefficient

### Precoding

Consider a tone $f$ and its $L \times L$ channel matrix $H$, where we suppress the tone subscript for convenience. Let diag($H$) be the diagonal of $H$, and $I$ be the $L \times L$ identity matrix. We suppose that the ideal precoding matrix $C$ is such that $HC = \text{diag}(H)$, i.e.,

we desire $C = H^{-1}\text{diag}(H)$ (see [2]). Let $E$ be such that $H = \text{diag}(H)(I + E)$: i.e., we set the $l,m$ entry of $E$ to be $\varepsilon^{l,m}$. We then have

$$H^{-1} = (I + E)^{-1}\text{diag}(H)^{-1}$$
$$= (I - E + E^2 - E^3 + \dots)\ \text{diag}(H)^{-1}$$

and hence

$$C = (I + E)^{-1} = (I - E + E^2 - E^3 + \dots).$$

Since the values $\varepsilon^{l,m}$ are typically quite small, a natural first-order approximate inverse is

$$C^{(1)} = (I - E)$$

and a second-order approximation is

$$C^{(2)} = (I - E + E^2)$$

We compare the inverse, first-order, and second-order precoders in more detail below.

### Channel Coefficients

The performance of a precoder strongly depends on the channel coefficients. In order to obtain representative results, we have used channel values from various sources:

1. Empirical models,
2. 6-line, 1000 meter (3281 feet), Whippany loop measurements to 3.5 MHz,
3. 8-line, 1663 meter (5456 feet), Whippany loop measurements to 3.5 MHz, and
4. 10-line, 61 meter (200 feet), 122 meter (400 feet), and 244 meter (800 feet), Whippany loop measurements to 40.1 MHz.

In cases 2 and 3 above, the measured values were supplied as 801 real and imaginary components for each of the victim-crosstalker pairs, spaced 4312.5 Hz apart starting from 10 kHz. The measured direct channel gains agreed very well with the results obtained empirically using the model in case 1.

In case 4, the measurements supplied 401 values for each of the victim-crosstalker pairs, but no direct channel gains. The measurements were spaced at 0.1 MHz and covered the band up to 40.1 MHz, starting at 0.1 MHz. The direct gains were again determined by the empirical model.

The performance results in the next sections will be provided for a frequency range up to 12 MHz. In case 2 and case 3, the available data was extrapolated beyond the 3.5 MHz range for which the measurements were conducted. Additionally, in cases 2, 3, and 4, interpolation was applied. These extensions of the measurements enabled the calculation of bit rates.

## Precoder Performance

This section presents the precoder performance for the channel coefficients described above. In addition, the performance of partial crosstalk cancellation is evaluated.

### Performance Using Empirical Models

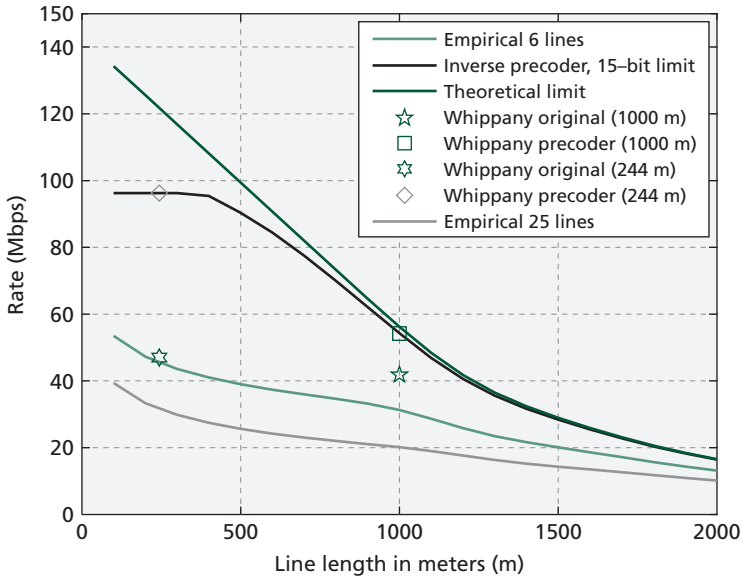
The 99 percent empirical model is conservative in the sense that it is based on the 99% percentile values of the crosstalk gains: i.e., empirically 99 percent of all lines have a smaller crosstalk value than the values given by the model.

**Figure 1** show several curves discussed further below. One curve shows rates with and without precoding as a function of the line length for a VDSL2-12a system with L = 6 lines. The figure indicates that shorter lines benefit more from the precoder than longer ones. Also, taking 30 Mbps as a benchmark, there is roughly a 40 percent increase in reach from about 1000 meters to 1400 meters. Alternatively, taking 1000 meters as a benchmark, there is roughly an 80 percent increase in rate from about 30 Mbps to 54 Mbps.

### Performance for Whippany Measurements

There are five sets of Whippany measurements for line lengths of 61 meters (200 feet), 122 meters (400 feet), 244 meters (800 feet), 1000 meters (3281 feet), and 1663 meters (5456 feet). **Figure 2** plots the relative crosstalk coefficients for the 244 meter lines. As the graphs show, the relative crosstalk values can be significant even for short lines.

The 1000 meter (1663 meter) measurements were taken for six (eight) lines in a bundle for frequencies up to roughly 3.5 MHz, yet modeling was required for frequencies up to 12 MHz. To compensate for the missing measurement data, the log crosstalk gain was extrapolated linearly, with the coefficients being estimated using linear regression. It is stressed that these estimates are inherently crude. As far as the measurement data is concerned, there are rapid fluctuations in the relative crosstalk coefficient at the frequencies 2 MHz and above.




VDSL2—Very high speed digital subscriber line 2

***Figure 1.***
***VDSL2-12a downstream rates with Whippany rates shown as averages over 6 lines.***

The phase values were extrapolated by using the average phase change with respect to frequency. The measurements suggest linear changes in phase over broad frequency ranges.

Taken as a whole, the results suggest that a considerable reduction in estimation effort might be made by first accurately determining phase and amplitude at only a fraction of the tones evenly spaced.

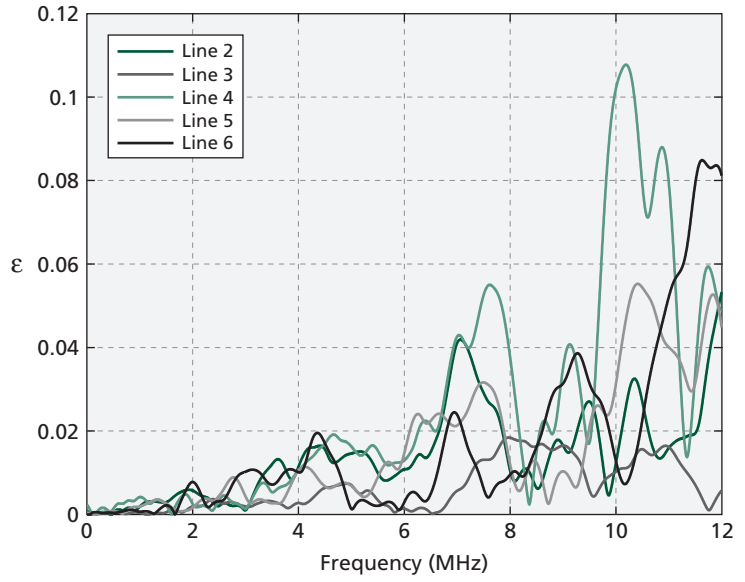


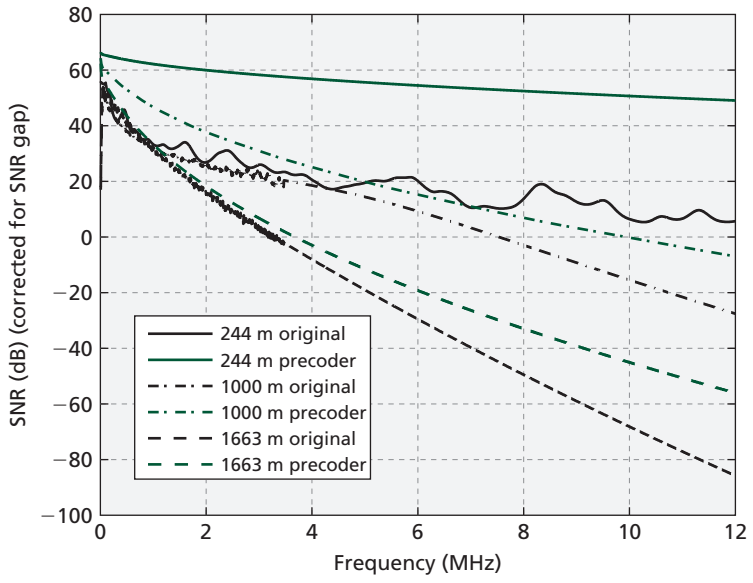

***Figure 2.***
***Relative crosstalk coefficients for Whippany loops of 244 meters, lines 2, 3, 4, 5, 6 into line 1.***

At the second stage, interpolated estimates can be used as initial estimates for the remaining tones. These by themselves are not likely to be all that good. However, an additional small number of guesses on either side of these starting points could be made, the guesses would be supplied to the precoder, and SNR measurements then performed. Good guesses would result in significant increases in SNR for the given tone as the SNR rapidly increases near the correct crosstalk values.

Variations on this strategy can be made by using more or less widely spaced tones in combination with more or fewer guesses, to gain higher efficiency. In some cases, a sequence of guesses may fail to improve SNR, but performance-wise this affects one tone only and can, in any case, be addressed by then obtaining a direct estimate. Elementary optimization techniques coupled with statistical knowledge of channel propagation should allow such approaches to be evaluated in more detail.

**Figure 3** plots SNR (corrected for the $\Gamma = 12.8$ dB gap) as a function of frequency for line 1 (of the 6 lines) and various lengths. Note that the SNR with inverse precoding has the same form for all lines, since it corresponds to a single line in isolation and thus only depends on the direct gain which is determined using the empirical model. The results suggest that the 244 meter lines examined in the Whippany measurements offer substantial scope for rate gains over nearly the entire frequency range, with the exception of very low frequencies. Indeed, at 244 meters, the maximum VDSL2-12a rate is achievable as the SNR is above 45 dB over the entire range.

The rates without and with precoding are given in **Table I**, **Table II,** and **Table III**. The unparenthesized numbers are for VDSL2-12a using both the downstream 1 (DS1) and downstream 2 (DS2) bands. The DS1 band includes tones 32 to 869, and the DS2 band tones 1206 to 1971. The numbers in parentheses are computed by assuming the entire 12 MHz is devoted to downstream communications alone. The results show that gains of around 100 percent are achievable for 244 meter lines. At longer lengths, the gains are limited to around 20 percent to 30 percent for 1000 meters, and around 10 percent for 1663 meters.




SNR—Signal-to-noise ratio

***Figure 3.***
***SNR of line 1 for Whippany loops of 244 meters, 1000 meters, and 1663 meters, and 6 lines.***

**Table I. Rates for Whippany loops of 1000 meters and 6 lines.**

| Line number | Original rate (Mbps) | Precoder rate (Mbps) |
|---|---|---|
| 1 | 37.1 (46.46) | 54.2 (69.1) |
| 2 | 39.5 (49.4) | 54.2 (69.1) |
| 3 | 42.5 (53.2) | 54.2 (69.1) |
| 4 | 42.3 (53.0) | 54.2 (69.1) |
| 5 | 48.5 (62.4) | 54.2 (69.1) |
| 6 | 41.0 (53.1) | 54.2 (69.1) |

**Table II. Rates for Whippany loops of 1663 meters and 6 lines.**

| Line number | Original rate (Mbps) | Precoder rate (Mbps) |
|---|---|---|
| 1 | 21.3 (23.3) | 23.7 (26.0) |
| 2 | 21.3 (23.1) | 23.7 (26.0) |
| 3 | 21.0 (22.8) | 23.7 (26.0) |
| 4 | 21.4 (23.2) | 23.7 (26.0) |
| 5 | 21.2 (23.1) | 23.7 (26.0) |
| 6 | 22.2 (24.1) | 23.7 (26.0) |

**Table III. Rates for Whippany loops of 244 meters and 6 lines.**

| Line number | Original rate (Mbps) | Precoder rate (Mbps) |
|---|---|---|
| 1 | 50.4 (71.3) | 96.2 (166.9) |
| 2 | 45.1 (65.5) | 96.2 (166.9) |
| 3 | 50.4 (75.0) | 96.2 (166.9) |
| 4 | 43.9 (61.6) | 96.2 (166.9) |
| 5 | 46.1 (67.8) | 96.2 (166.9) |
| 6 | 46.0 (69.4) | 96.2 (166.9) |

Comparison results for six 1000 meter lines are shown in Table I. Table II gives the rates for six 1663 meter lines. Due to the increased line length, there is a significant decrease in the rates without precoding to about 21 Mbps. There is a corresponding drop in the rates with precoding to about 24 Mbps. As before, the inverse precoding rates across lines are constant as these correspond to a line in isolation and thus depend only on the direct gain. Table III shows the total rates for six 244 meter lines. Here the precoding gains are much more significant than in Table II. The precoding rates are 75 percent greater than those for 1000 meter lines and coincide with the maximum bit loading at this line length. Figure 1 further suggests that lines as long as 500 meters may exhibit higher relative gains than those shown in Table III. It also follows that the bit loadings are saturated for line lengths shorter than 244 meters, and so the precoding rates are omitted for 122 meter and 61 meter lines.

### Partial Cancellation

We next examine the performance if we only partially remove the crosstalk. Suppose there are six lines and the three lines producing the worst crosstalk at each tone have their crosstalk removed: i.e., only two crosstalkers remain. Observe from Figure 2 that the three dominant crosstalkers are not always the same for every tone. **Figure 4** graphs the SNR for a 244 meter line where most of the SNR gains are limited to about 10 dB to 15 dB. Assuming the precoder bit loadings are not saturated, which on the whole they are not, this translates into a 3 bit gain per tone or about 20 Mbps. The actual rate gains are given at **Table IV**. Further gains would be obtained in VDSL2-17a.

## Impact of Channel Measurement Errors

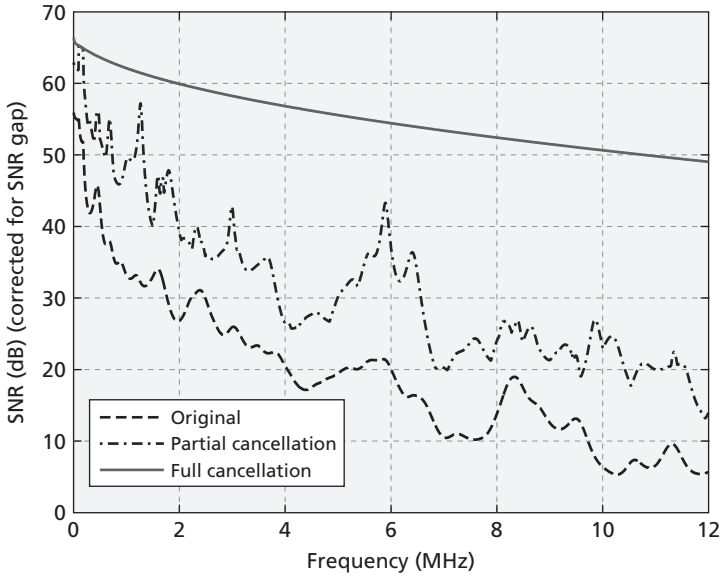
We now examine the performance degradation due to measurement errors in the channel coefficients. A standard method for measuring the amplitude and phase of the channel gains is using a network analyzer. Such devices have an approximate amplitude error of 0.5 dB ($\approx 6\%$) and phase error of around 5°. Similar measurement errors would also occur if SNR feedback or some other scheme is employed. For all such schemes, it is important to know what degree of error can be tolerated.

### Precoder Comparison

Recall that we write $H = \mathrm{diag}(H)(I - E)$ and suppose the channel measurement gives the estimates

$$\mathrm{diag}(\hat{H}) = \mathrm{diag}(\hat{H}) + \Delta_H,\ \hat{E} = E + \Delta_E,$$

where $\Delta_H$ and $\Delta_E$ are measurement errors. The inverse precoder is $\hat{C} = (I + \hat{E})^{-1}$ and gives the overall channel




SNR—Signal-to-noise ratio

***Figure 4.***
***SNR for Whippany loops of 244 meters and 6 lines, where the worst 3 crosstalkers per tone are canceled.***

$$\begin{aligned} H\hat{C} &= \mathrm{diag}(H)\,(I+E)\,(I+E+\Delta_E)^{-1} \\ &= \mathrm{diag}(H)\,[I+(I+E)^{-1}\Delta_E]^{-1} \\ &\approx \mathrm{diag}(H)\,[I+(I-E)\,\Delta_E]^{-1} \\ &\approx \mathrm{diag}(H)\,[I-(I-E)\,\Delta_E+(I-E)^2\Delta_E^2] \\ &\approx \mathrm{diag}(H)\,[I-\Delta_E+E\Delta_E+\Delta_E^2] \end{aligned}$$

where we assume that the entries of $E$ and $\Delta_E$ have magnitudes much less than 1. We further neglected third- and higher-order terms such as $E^2\Delta_E$. The error in the ideal diag($H$) channel is therefore approximately

$$\mathrm{diag}(H)\,(-\Delta_E+E\Delta_E+\Delta_E^2).$$

**Table IV. Rates for Whippany loops of 244 meters and 6 lines with partial cancellation.**

| Line number | Original rate (Mbps) | Partial cancellation (Mbps) | Full cancellation (Mbps) |
|---|---|---|---|
| 1 | 50.4 (71.3) | 74.2 | 96.2 (166.9) |
| 2 | 45.1 (65.5) | 68.1 | 96.2 (166.9) |
| 3 | 50.4 (75.0) | 77.2 | 96.2 (166.9) |
| 4 | 43.9 (61.6) | 62.8 | 96.2 (166.9) |
| 5 | 46.1 (67.8) | 68.4 | 96.2 (166.9) |
| 6 | 46.0 (69.4) | 75.6 | 96.2 (166.9) |

Obviously, if $\Delta_E$ is small then this error is small. Note also that the error $\Delta_H$ does not affect the overall channel but it might lead to incorrect bit loading.

The first-order precoder is $\hat{C}=I-\hat{E}$ and gives the overall channel

$$\begin{aligned} H\hat{C}^{(1)} &= \mathrm{diag}(H)\,(I+E)\,(I-E-\Delta_E) \\ &= \mathrm{diag}(H)\,(I-\Delta_E-E^2-E\Delta_E) \end{aligned}$$

We observe that the error in the ideal channel diag($H$) is small only if both $\Delta_E$ and $E$ are small. Furthermore, the inverse precoder should perform better than the first-order precoder if $\Delta_E$ has magnitudes that are smaller than those of $E$ (so that $\Delta_E^2$ causes less error than $E^2$). In other words, as long as we can estimate the entries of $E$ with error $\Delta_E$ less than these entries, then we expect the inverse precoder to perform better than the first-order precoder.

Similarly, the second-order precoder is $\hat{C}=I-\hat{E}+\hat{E}^2$ and gives

$$\begin{aligned} H\hat{C}^{(2)} &= \mathrm{diag}(H)\,(I+E)\,(I-E-\Delta_E+E^2 \\ &\quad +E\Delta_E+\Delta_E E+\Delta_E^2) \\ &\approx \mathrm{diag}(H)\,(I-\Delta_E+\Delta_E E+\Delta_E^2), \end{aligned}$$

where we have neglected third-order terms such as $E^3$ and $E\Delta_E^2$. Comparing the above expressions with the second-order error terms for the inverse precoder, we expect the performance of the second-order precoder to be comparable to that of the inverse precoder.

### Modeling of Channel Measurement Errors

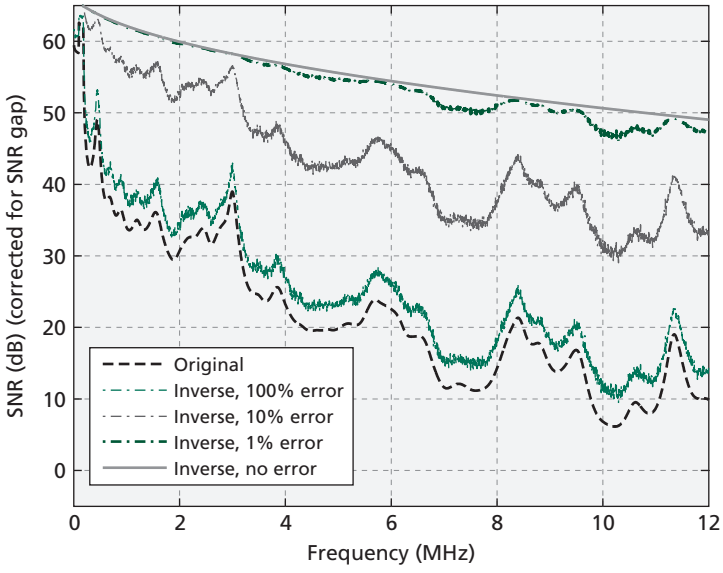
The statistical nature of channel measurement errors depends on the measurement methodology. Since we do not consider any specific method here, the following simple stochastic model has been adopted. Phase and amplitude errors are presumed independent. In the case of phase error, we assume there is a maximum error of $x_\phi/100$ radians, so the actual error is uniform on $\lfloor -x_\phi/100, x_\phi/100 \rfloor$. We further assume that the error in amplitude is multiplicative: i.e., larger amplitudes are obtained with larger absolute errors. (The motivation for this assumption is that the SNR feedback provides measurements accurate to within 0.1 dB, apparently independent of the absolute amplitude.) We model this by assuming that the error in log amplitude is uniformly distributed on the interval $\lfloor -x_\psi/100, x_\psi/100 \rfloor$.

**Table V** gives rates for the inverse precoder and a range of errors. The results are averaged over the four lines considered, and 100 trials were used for each table entry. At these modest error levels there are still considerable rate gains. Nevertheless, these results may not reflect the performance of an actual estimation algorithm, which may impact the errors differently than our model statistics described earlier.

**Table V. Rates for Whippany loops of 244 meters and 122 meters, and 4 lines.**

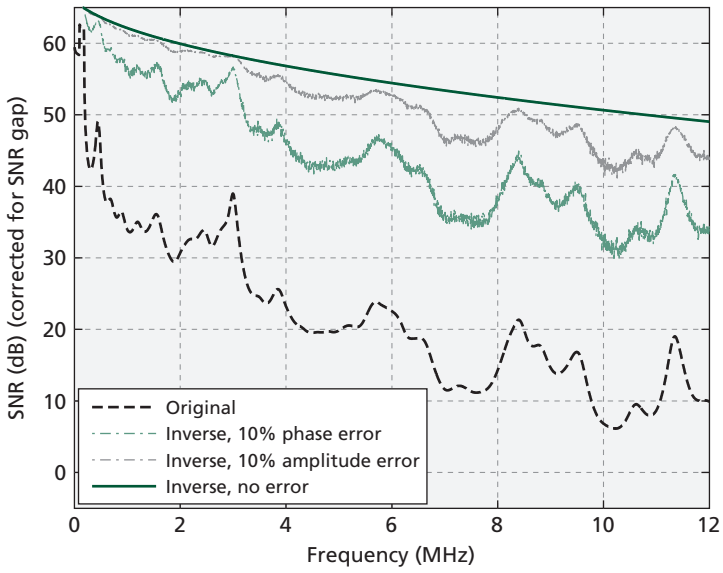
| Error (%) | Rate (Mbps) – 244 m | Rate (Mbps) – 122 m |
|---|---|---|
| 0 | 96.2 | 96.2 |
| 10 | 88.3 | 93.6 |
| 15 | 84.4 | 90.9 |
| 20 | 81.2 | 88.4 |
| 25 | 78.4 | 86.1 |
| 30 | 75.9 | 83.9 |

Finally, we present graphs showing results for a system with 244 meter length lines and various error levels, as shown in **Figure 5**. Consider the results for 100 percent error, i.e., a uniform error of 1 dB either way, and a phase error of up to a radian either way. This degree of error almost completely negates the gains from precoding.




SNR—Signal-to-noise ratio

***Figure 5.***
***Average SNR for Whippany loops of 244 meters at various error levels.***




SNR—Signal-to-noise ratio

***Figure 6.***
***Average SNR for Whippany loops of 244 meters with various errors.***

In a further experiment at 244 m, alternately the amplitude and the phase error were set to zero while leaving the other at 10 percent. The results for this experiment are plotted in **Figure 6**. The results show that phase error is the predominant contributor to performance as the results for phase error alone are only marginally better than the error from phase and amplitude combined.

## Impact of Fixed-Point Arithmetic

This section develops theory for precoders implemented with finite-precision arithmetic. The goal is to determine the accuracy of computation needed to cancel crosstalk.

### Precision in the Computation of the Crosstalk Correction Term

The crosstalk correction terms are added to the original signals; the original signals can thus be retained at (nearly) full precision in what follows. It is further worth noting that power matching of the launched signals can be supported by performing additional operations.

Consider a system with $L$ lines. The precoder computation may be written as

$$Y = X + AX$$

with $X$ the original signal vector and the matrix $A$ depending on the form of precoding (e.g., inverse, first-order, second-order), but approximately equal to the matrix $-E$ when the relative crosstalk coefficients are small.

Suppose the entries, $x_l$, $1 \leq l \leq L$, of $X$ coming out of the fast Fourier transform (FFT) are $2 \times 26$ bit complex numbers: i.e., the $x_l$ have 26-bit real and 26-bit imaginary parts. Consider the calculation of the output signal for a given user, say user $l$:

$$y_l = x_l + \sum_{\substack{m=1 \\ m \neq 1}}^{L} a_{l,m} x_m$$

We assume that hardware constraints dictate the output, $y_l$, $1 \leq l \leq L$, to be represented as a $2 \times 20$ bit complex number going into the inverse fast Fourier transform (IFFT).

Two questions are

1. How many bits should we use to represent the coefficients $a_{l,m}$, and
2. How many bits should we use for the above calculation?

In what follows, we will make some preliminary comments as to how these questions can be answered.

The answer to question 1 is driven by the accuracy with which we wish to represent the precoder coefficients and the dynamic range (smallest to largest) of the coefficients. Specifically, let

$$a_{max,l} = \max_{1 \le m \le L,\, m \ne l} \max(|\Re(a_{l,m})|, |\Im(a_{l,m})|)$$

be the "largest" crosstalk correction coefficient (XTCC) for row $l$, of matrix $A$. Then suppose we want to be able to represent XTCC values that are 100 times smaller than $a_{max,l}$ with 1 percent accuracy (both the 100 and the 1 percent are design parameters); smaller XTCC values have a minimal contribution to the overall crosstalk correction term and therefore do not require the same degree of accuracy.

Clearly in representing real numbers by integers, the rounding error made depends on where in the integer scale the number falls. An integer rounded near 10 produces an error of around 10 percent, while one rounded near 100 produces an error of around 1 percent. Thus, in the example above, we require that all the coefficients to be corrected fall in a range for the fixed point of, $\pm$[100, 10000], or let's say, $[2^{-14}, 2^{14}]$ i.e., 15 bits. In addition, we need a separate scaling coefficient $S$, which we assume to be of the form, $2^{-n-14}$, with $n$ a number in the range [0, 15], i.e., 4 bits.

### Numerical Example

Suppose the XTCCs are 0.0001, 0.002, 0.06, which we take as purely real numbers. Then, $n = 4$ since $2^4 \cdot 0.06 \approx 1$, so that the 0.06 will map to 15729 (rounded off) in the fixed-point representation, the 0.002 will map to 524 (rounded off), and the 0.0001 will map to 26, with a significant rounding error of four percent.

The answer to question 2 relates to the issue of how much we are prepared to truncate $x_1, \ldots, x_L$ before multiplication. The less we truncate, the more bits we need to prevent the multiplication from overflowing.

For instance, if we allocate $b$ bits for the calculation of the term $a_{l,m} \cdot x_m$ and $a_{l,m}$ is a 15-bit quantity, then we need to truncate $x_m$ to $b - 15$ bits (e.g., if $b = 32$ bits then $b - 15 = 17$ bits, of which one bit is a sign bit) to prevent overflow. For $L = 8$ there are seven further complex additions (ignoring the $x_l$ term for now) so overall we require 14 real and 14 imaginary additions. We could thus truncate by a further 4 bits to prevent overflow of the summation of these terms. However, a better approach is as follows: group the 14 terms to be added into four groups of at most four terms each, truncate each term by 2 bits, perform the sum within each group, and then scale down by another 2 bits before adding to $x_l$ (which we do not wish to truncate).

### Analysis

We consider the computation of a single correction term for a crosstalker $m$ into a line $l$, and for now suppress the indices $l$ and $m$. The error model for imprecision may be described as follows. First, as a result of rounding, we obtain $a + u = a_B$ in place of $a$ in the crosstalk correction. The error $u$ is taken to be deterministic and depends on all the $a$ values at the frequency considered; $a_B$ is the rounded value. Second, we replace the complex signal $x$ by $x + v = x_B$, where $v$ is due to the truncation which we described earlier and $x_B$ is the truncated value. In actual fact, $v$ depends on $x$ as it is obtained by rounding. However, we will suppose that $v$ is a random variable independent of $x$, which nevertheless has a probability distribution similar to that for rounding. It follows that the error resulting from the multiplication stage is given by

$$a_B x_B - ax = av + uv + xu = a_B v + xu.$$

However, since only $x$ and $v$ are random, we obtain the mean square error resulting from multiplying the two rounded terms as

$$\mathrm{E}\left[\,|a_B v + xu|^2\,\right] = |a_B|^2 \mathrm{E}\left[\,|v|^2\,\right] + P|u|^2$$

which follows on substituting $a + u$ for $a_B$ and using the assumed independence of $x$ and $v$. The term $|a_B|^2 \mathrm{E}[|v|^2]$ corresponds to the random rounding of $x$ (we emphasize this rounding depends on $x$ itself). The term $P|u|^2$ can be interpreted as noise power due to the rounding of $a$, where $P$ is the power of the random variable $x$. Ignoring additional errors from scaling and subsequent analog/digital (A/D) conversion, we focus on the above expression.

The distribution of $v = v_R + jv_I$, $j = \sqrt{-1}$, is taken to be: $v_R$ and $v_I$ are independent and uniform

and jointly independent of $x$. We assume that the covered range for the signal $x$ is a square with sides length $2\sqrt{P_{\max}}$ centered on the origin, with a grid width of $2\sqrt{P_{\max}}10^{-d(x)}$, where $d(x)\log_2(10) = b(x)$, with $b(x)$ bits allocated to represent $x_R$ and a further $b(x)$ bits to represent $x_I$. Here $b(x)$ is determined by the number of bits in $x$ after truncation. The distribution of $v_R$, $v_I$ is

$$v_R, v_I \sim U\left(\left[-\frac{2\sqrt{P_{\max}}}{10^{d(x)}}, \frac{2\sqrt{P_{\max}}}{10^{d(x)}}\right]\right)$$

independently, where $U([a, b])$ is the uniform distribution on $[a, b]$. Since $E[W^2] = \frac{1}{3}$ if $w^2 \sim U([-1,1])$ we obtain

$$|a_B|^2 \mathrm{E}\left[|v|^2\right] = \frac{8}{3} \cdot P_{\max} \cdot 10^{-2d(x)} \cdot |a_B|^2.$$

We suppose that $b(a)$ bits are allocated to the real and imaginary components of $a$ and $d(a)$ is defined as $d(x)$ above. We further define

$$a_{\max} := \max_{1 \le l \le L} a_{\max,l}.$$

The errors $a_R$, $a_I$ are then given by $0.05 \cdot 10^{-d(a)} \cdot a_{\max}$ because of the discrete representation which places $a_{\max}$ at roughly $10^{d(a)}$ and because we assume rounding to the nearest value as described in the previous section. Thus unless $a_R$, $a_I \le 0.05 \cdot 10^{-d(a)} \cdot a_{\max}$, it is better to use the discrete representation than neglecting the crosstalk. We thus have

$$P|u|^2 \le 0.5 \cdot P \cdot 10^{-2d(a)} \cdot a_{\max}^2$$

which is also an upper bound in the case where $a$ under-flows.

Since we wish to avoid overflow after multiplication, we must have $b(a) + b(x) \le 31$, the number of bits allocated to store the multiplication. Comparison of the error terms suggests we take $b(x) = b(a)$, or equivalently $d(x) = d(a) = d$.

Based on the above analysis, we find that the SNR taking into account implementation error is given by

$$\gamma = \frac{P|h^{l,l}|^2}{(L-1)|h^{l,l}|^2(P|u|^2 + |a_B|^2 E[|v|^2]) + N}$$

$$\ge \frac{P|h|^2}{3.167(L-1)P10^{-2d}a_{\max}^2|h|^2 + N}$$

where for the second step we have taken $h = h^{l,l}$ and $P_{\max} = P$, which is approximately true.

The above formula indicates that implementation error is not likely to cause significant performance degradation, as shown by the following example.

### Numerical Example

We take $L = 8$, $P = 10^{-6}$, $a_{\max} = 0.1$, $b(x) = b(a) = 13$ so that $d(x) = d(a) \approx 4$ The quantization noise contribution from each line would then be

$$|a_B|^2 E\left[|v|^2\right] \le \left(\frac{8}{3}\right) \cdot (10^{-6}) \cdot (2^{-26}) \cdot (0.1^2) \approx 4 \cdot 10^{-16} \text{ mW/Hz}$$

$$P|u|^2 \le 0.5 \cdot (10^{-6}) \cdot (2^{-26}) \cdot (0.1^2) \approx 7.5 \cdot 10^{-17} \text{ mW/Hz}.$$

Both of these quantities are much smaller than the noise PSD, and so there is a negligible reduction in precoding performance from the ideal. Note that there is an additional loss of $10\log_{10}(L-1)$ dB when there are $L$ lines. More seriously, if we reduce the total number of bits available from 32 to 24, there is an additional loss of 24 dB, and if we reduce the total number of bits available from 32 to 16, an additional loss of 48 dB.

Note that if the maximum relative crosstalk coefficient goes up by a factor of 10, then there is an increase of 20 dB in implementation loss.

## Additional Results

**Table VI** shows the rates with precoding for four 1000 meter lines in either the DS1 band or the DS2 band, but not both. The values are therefore the sum of the precoded rates and the original rates for the band which does not use precoding. The numbers in parentheses represent the percentage gains over the original rates which are shown in the first column. "DS1 precoder" in a column means the precoder is being used in the DS1 band only, and similarly "DS2 precoder" means the precoder is being used in the

**Table VI. Rates for Whippany loops of 1000 meters and 4 lines with precoding in one band.**

| Line number | Original rate (Mbps) | DS1 precoder rate (Mbps) | DS2 precoder rate (Mbps) |
|---|---|---|---|
| 1 | 45.7 | 53.5 (17.1%) | 46.4 (1.7%) |
| 2 | 46.0 | 53.8 (16.9%) | 46.5 (1.0%) |
| 3 | 46.1 | 53.4 (15.8%) | 47.0 (1.8%) |
| 4 | 44.3 | 50.7 (14.4%) | 47.8 (8.0%) |

DS—Downstream

DS2 band only. We see that precoding in the DS1 band has greater relative impact than in the DS2 band for the 1000 meter lines.

**Table VII** gives the corresponding results for 244 meter lines. Unlike the previous case, the highest relative improvement is obtained by precoding in the DS2 band rather than in the DS1 band. These results further indicate that greater relative gains could be demonstrated for shorter lines, i.e., those less than 1000 meters.

## Conclusions

We conclude by summarizing some of the insights gained from our experiments and calculations.

### Line Lengths and Bandwidths

The results in the previous sections indicate that lines of 1000 meters and below offer the greatest scope for performance gains, as seen in Figure 1. Figure 1 further shows that below 500 meter line lengths, the actual rates are more and more limited by the VDSL2 bit-loading constraints. Because the performance without precoding improves as the line lengths get shorter, the scope for improvement reduces. The results confirm that there are significant rate gains for line lengths in the range of 100 meters to 250 meters, as evidenced by the star and diamond symbols in Figure 1 that show the average rates for the 244 meter Whippany measurements. The margins at 244 meters leave considerable room for implementation error, while still providing scope for significant performance improvement. Furthermore, Figure 1 shows that large numbers of crosstalkers, e.g., 24, do not significantly change the performance gap. This may be explained from the fact that the performance is determined by a small number of large crosstalkers at any frequency rather than the cumulative effects of equal crosstalkers.

**Table VII. Rates for Whippany loops of 244 meters and 4 lines with precoding in one band.**

| Line number | Original rate (Mbps) | DS1 precoder rate (Mbps) | DS2 precoder rate (Mbps) |
|---|---|---|---|
| 1 | 55.2 | 68.0 (23.2%) | 83.4 (51.3%) |
| 2 | 47.3 | 63.9 (35.2%) | 79.6 (68.4%) |
| 3 | 51.3 | 66.0 (28.6%) | 81.6 (59.0%) |
| 4 | 48.5 | 64.3 (32.5%) | 80.5 (65.8%) |

DS—Downstream

**Figure 7** shows corresponding results for VDSL2-17a and relatively short lines, up to 1000 meters. The 99 percent empirical crosstalk curve tends to underestimate the rates without precoding, especially at the two shortest line lengths. Also note that the precoding curve saturates at 300 m, so that a larger symbol constellation or more bandwidth is needed to increase the rate. However, even with the VDSL2 bit-loading constraints, rate gains on the order of a factor between 2 and 3 can be obtained for most line lengths below about 750 meters for the empirical models. Observe that the measurements suggest this factor may turn out to be somewhat smaller than 2 or 3.

All the results show that the relative crosstalk coefficient tends to increase with frequency; see Figure 2. By itself, this suggests that there is more to be gained by canceling crosstalk at high frequencies. However, at high frequencies there is greater overall signal attenuation, so that noise plays a more significant role. It turns out that low frequencies offer greater scope for performance gains with long lines, while high frequencies are better for short lines, as shown in Tables VI and VII. Thus, if one were to conduct an experiment in a restricted band for short loops, then it seems that the high frequencies offer the best choice. In any event, there does not appear to be any pressing reason to extend to yet higher bandwidths (beyond 12 MHz) in order to demonstrate performance improvement.

### Correctability

As we have already noted, larger values of $\varepsilon_f^{l,m}$ can be corrected with greater precision as they tend to

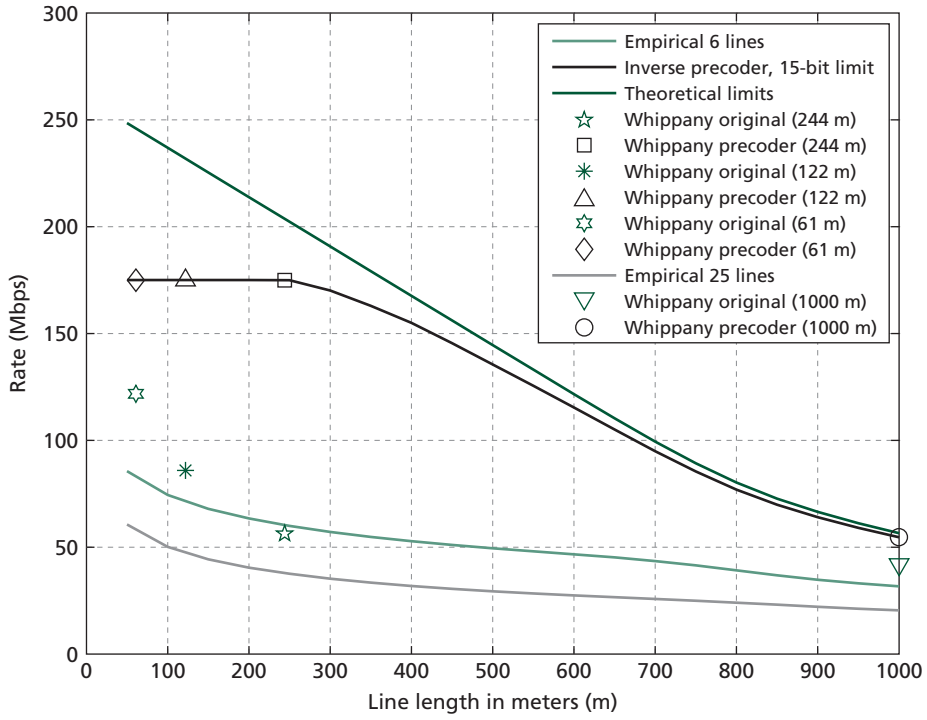



***Figure 7.***
***VDSL2-17a downstream rates with Whippany rates shown as averages over 6 lines.***

fall near the higher end of the quantization scale. Note, however, that this scale will be adjusted with frequency and with the line identity itself. Suppose the precoder, neglecting other sources of error, can correct $\varepsilon_f^{l,m}$ values 100 times smaller than the maximum corrected with an error of around 1 percent. It follows that in the case of the Whippany results, the crosstalk is correctable across the band with performance impact more or less independently of the size of the $\varepsilon_f^{l,m}$ coefficient.

### Precoder Design and Channel Measurement Errors

The inverse precoder has superior performance as it eliminates all crosstalk. Its performance is consistently better than that of the other precoders even though small channel measurement errors cause significant deterioration as compared to no measurement error. The results suggest that with measurement errors as large as 10 percent in amplitude and in phase, significant performance gains can be had. The overriding conclusion however is that accurate estimation of the crosstalk coefficients is central to realizing the performance of the precoder.

### Further Considerations

There are several additional factors affecting performance. These include:

- *Quantization noise.* The level of quantization noise of the original (direct) signal introduced by the precoding is thought not to be significant based on the analysis.
- *Statistical nature of channel measurement errors.* As of yet, little is known about the statistical nature of errors in determining the crosstalk coefficients using a network analyzer or by other means.
- *Heterogeneous launch powers.* High-power signals make small crosstalk coefficients relevant. However, such coefficients are difficult to determine accurately. Moreover, the crosstalk correction is geared to representing crosstalk values above a

certain threshold, and smaller values deteriorate in their accuracy of representation until they underflow, as mentioned earlier. We remark that DSM level-2 power adaptations might assist in overcoming such difficulties.

- *Channel estimation using interpolation.* We would like to infer information on the crosstalk coefficients at tone $f_2$ from knowledge of the values at $f_1$ and $f_3$, where $f_1 \leq f_2 \leq f_3$. If the coefficients vary slowly with frequency, then interpolation might suffice as an estimation method. The Whippany measurements show that this may be possible across a narrow frequency range only.


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