Network Coding on Line Networks with Broadcast

Gerhard Kramer
Math. of Networks and Commun. Dept.
Bell Labs, Alcatel-Lucent
Murray Hill, NJ 07974
gkr@bell-labs.com

Sadegh Tabatabaei Yazdi
Dept. Electrical Engineering
Texas A&M University
College Station, TX
sadegh@neo.tamu.edu

Serap A. Savari
Dept. Electrical Engineering
Texas A&M University
College Station, TX
savari@ece.tamu.edu

***Abstract*—An achievable rate region for line networks with edge and node capacity constraints and broadcasting is derived. The region is the capacity region if the broadcast channels are orthogonal or physically degraded.**

***Index Terms*—broadcasting, network coding, relaying**

I. Introduction

Consider a line network with edge and node capacity constraints as shown in Fig. 1. "Supernode" u, $u = 1, 2, 3, 4$, consists of two nodes ui, uo where the "i" represents "input" and "o" represents "output". For N supernodes, let

$$u \to \mathcal{D}(u) = \{v(1), v(2), \ldots, v(L)\} \tag{1}$$

denote a multicast traffic session, where $u, v(1), \ldots, v(L)$ are supernodes. The meaning is that a source message is available at supernode u and is destined for supernodes in the set $\mathcal{D}(u)$. We associate sources with input nodes labeled ui and sinks with output nodes labeled uo. The capacity region of such line networks was recently established in [13]. A binary linear network code achieves capacity and progressive d-separating edge-cut (PdE) bounds [9] provide the converse.

The goal of this work is to extend the results of [13] to wireless networks by using insights from two-way relaying [12], broadcasting with cooperation [11], and broadcasting with side-information [10]. The model is shown in Fig. 2 where the difference to Fig. 1 is that supernode u transmits over a two-receiver broadcast channel (BC) $P(y_{u-1}, y_{u+1}|x_u)$ to supernodes $u-1$ and $u+1$ (see [2]). The channel outputs at supernode u are

$$Y_{u-1,u} = f_{u-1,u}(X_{u-1}, Z_{u-1}) \tag{2}$$
$$Y_{u+1,u} = f_{u+1,u}(X_{u+1}, Z_{u+1}) \tag{3}$$

for some functions $f_{u-1,u}(\cdot)$ and $f_{u+1,u}(\cdot)$, and where the Z_u, $u = 1, 2, \ldots, N$, are statistically independent. We permit the noise random variables Z_u to be common to $Y_{u,u-1}$ and $Y_{u,u+1}$ to include any type of memoryless BC. The X'_u in Fig. 2 represent the bits transmitted through node u, and we require that $H(X'_u) \le C_u$ where $H(X)$ is the entropy of X.

II. Wireline Capacity

We review the main result from [13]. Let $m(u \to \mathcal{D}(u))$ and $R(u \to \mathcal{D}(u))$ denote the message bits and rate, respectively,

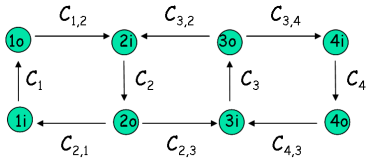


Fig. 1. A line network with edge and node capacity constraints.

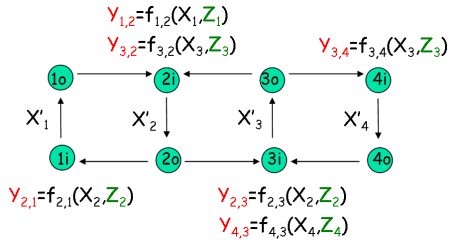


Fig. 2. A line network with broadcasting and node capacity constraints.

of traffic session $u \to \mathcal{D}(u)$. Consider node u. We collect the bits traversing node u's edge into six sets:

$$m_{LR}^{(u)} = \{m(i \to \mathcal{D}(i)) : 1 \le i \le u-1, \mathcal{D}(i) \cap \{u+1, \ldots, N\} \ne \emptyset\} \tag{4}$$
$$m_{RL}^{(u)} = \{m(i \to \mathcal{D}(i)) : u+1 \le i \le N, \mathcal{D}(i) \cap \{1, \ldots, u-1\} \ne \emptyset\} \tag{5}$$
$$m_{u,LR} = \{m(u \to \mathcal{D}(u)) : \mathcal{D}(u) \cap \{1, \ldots, u-1\} \ne \emptyset, \mathcal{D}(u) \cap \{u+1, \ldots, n\} \ne \emptyset\} \tag{6}$$
$$m_{u,R} = \{m(u \to \mathcal{D}(u)) : \mathcal{D}(u) \cap \{1, \ldots, u-1\} = \emptyset, \mathcal{D}(u) \cap \{u+1, \ldots, n\} \ne \emptyset\} \tag{7}$$
$$m_{u,L} = \{m(u \to \mathcal{D}(u)) : \mathcal{D}(u) \cap \{1, \ldots, u-1\} \ne \emptyset, \mathcal{D}(u) \cap \{u+1, \ldots, n\} = \emptyset\} \tag{8}$$
$$m_u = \{m(i \to \mathcal{D}(i)) : \mathcal{D}(i) = \{u\}\} \tag{9}$$

The idea is that $m_{LR}^{(u)}$ and $m_{RL}^{(u)}$ represent traffic flowing from left-to-right and right-to-left, respectively, through node u. Similarly, $m_{u,LR}$, $m_{u,R}$, and $m_{u,L}$ represents traffic originating at node u and destined for nodes on both the left and right, right only, and left only, respectively. Finally, m_u represents traffic destined for node u only. The non-negative message rates are denoted $R_{LR}^{(u)}$, $R_{RL}^{(u)}$, $R_{u,LR}$, $R_{u,R}$, $R_{u,L}$, and R_u. We can now write the capacity result from [13] as follows.

Theorem 1: The capacity region of a line network with N nodes is defined by the bounds, for all u,

$$\max(R_{LR}^{(u)}, R_{RL}^{(u)}) + R_{u,LR} + R_{u,R} + R_{u,L} + R_u \le C_u \quad (10)$$

$$R_{RL}^{(u)} + R_{u,LR} + R_{u,L} \le C_{u,u-1} \quad (11)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} \le C_{u,u+1} \quad (12)$$

Remark 1: The converse in [13] follows by PdE arguments [9] and achievability follows by using rate-splitting, routing, copying, and "butterfly" binary linear network coding.

Remark 2: The bounds (11) and (12) are vertex-partitioning cut bounds [3, Sec. 14.10]. Furthermore, without node constraints ($C_u = \infty$) the bounds (11) and (12) are routing bounds, so routing is optimal (see [1]).

III. Achievable Rates with Broadcast

We separate channel and network coding, which sounds simple enough. However, every BC receiver has side information about some of the messages being transmitted, so we will need the methods of [10]. We further use the theory in [11] to describe our achievable rate region.

We begin by splitting $m_{u,R}$ and $m_{u,L}$ into two parts each, namely the respective $[m'_{u,R}, m''_{u,R}]$ and $[m'_{u,L}, m''_{u,L}]$. The rates of $m'_{u,R}$ and $m''_{u,R}$ are the respective $R'_{u,R}$ and $R''_{u,R}$, and similarly for $R'_{u,L}$ and $R''_{u,L}$. We choose a joint distribution $P_{S_uT_uW_uX_u}(\cdot)$ and generate a codebook of size

$$2^{n\left(R_{LR}^{(u)}+R_{RL}^{(u)}+R_{u,LR}+R'_{u,R}+R'_{u,L}\right)}$$

with codewords

$$\underline{w}_u\left(m_{LR}^{(u)}, m_{RL}^{(u)}, m_{u,LR}, m'_{u,R}, m'_{u,L}\right)$$

by choosing every letter of every codeword independently using $P_{W_u}(\cdot)$. Next, for every such $\underline{w}_u$, we choose $2^{n(R''_{u,R}+R_{T_u})}$ codewords $\underline{t}_u$ by choosing the ith letter $t_{u,i}$ of $\underline{t}_u$ via the distribution $P_{T_u|W_u}(\cdot|w_{u,i})$ where $w_{u,i}$ is the ith letter of $\underline{w}_u$. We label $\underline{t}_u$ with the arguments of $\underline{w}_u$, $m''_{u,R}$, and a "bin" index from $\{1, 2, \ldots, 2^{nR_{T_u}}\}$. Similarly, for every $\underline{w}_u$ we generate $2^{n(R''_{u,L}+R_{S_u})}$ codewords $\underline{s}_u$ generated via $P_{S_u|W_u}(\cdot)$ and label $\underline{s}_u$ with the arguments of $\underline{w}_u$, $m''_{u,L}$, and a "bin" index from $\{1, 2, \ldots, 2^{nR_{S_u}}\}$.

Next, for every six-tuple of messages, the encoder tries to find a pair of bin indices such that $(\underline{w}_u, \underline{t}_u, \underline{s}_u)$ is jointly typical according to one's favorite flavor of typicality. Using standard typicality arguments (see, e.g., [11]) a typical triple exists with high probability if n is large and

$$R_{S_u} + R_{T_u} > I(S_u; T_u|W_u). \quad (13)$$

Once this triple is found, we transmit a signal $\underline{x}_u$ that is generated via $P_{X_u|S_uT_uW_u}(\cdot)$.

The receivers use joint typicality decoders to recover their messages. They further use their knowledge about some of the messages (side-information). The result is that decoding is reliable if n is large and (see [10], [11])

$$R''_{u,L} + R_{S_u} < I(S_u; Y_{u,u-1}|W) \quad (14)$$

$$R_{RL}^{(u)} + R_{u,LR} + R'_{u,R} + R_{u,L} + R_{S_u} < I(S_uW_u; Y_{u,u-1}) \quad (15)$$

$$R''_{u,R} + R_{T_u} < I(T_u; Y_{u,u+1,}|W) \quad (16)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} + R'_{u,L} + R_{T_u} < I(T_uW_u; Y_{u,u+1}). \quad (17)$$

Finally, we use Fourier-Motzkin elimination (see [11]) to remove R_{S_u}, R_{T_u}, $R'_{u,L}$, $R'_{u,R}$, $R''_{u,L}$, and $R''_{u,R}$ from the above expressions and obtain the following result.

Theorem 2: An achievable rate region for a line network with broadcasting and node constraints and with N nodes is given by the bounds, for any choice of $P(s_u, t_u, w_u, x_u)$ and for all u,

$$\max(R_{LR}^{(u)}, R_{RL}^{(u)}) + R_{u,LR} + R_{u,R} + R_{u,L} + R_u \le C_u \quad (18)$$

$$R_{RL}^{(u)} + R_{u,LR} + R_{u,L} \le I(S_uW_u; Y_{u,u-1}) \quad (19)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} \le I(T_uW_u; Y_{u,u+1}) \quad (20)$$

$$R_{RL}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L} \le I(S_uW_u; Y_{u,u-1}) + I(T_u; Y_{u,u+1}|W_u) \quad (21)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L} \le I(T_uW_u; Y_{u,u+1}) + I(S_u; Y_{u,u-1}|W_u) \quad (22)$$

$$R_{RL}^{(u)} + R_{LR}^{(u)} + 2R_{u,LR} + R_{u,R} + R_{u,L} \le I(S_uW_u; Y_{u,u-1}) + I(T_uW_u; Y_{u,u+1}) - I(S_u; T_u|W_u). \quad (23)$$

Remark 3: The bound (18) is the same as (10).

Remark 4: The bounds (19)-(23) are similar to the bounds of [11, Theorem 5]. However, a few rates are "missing" here because nodes $u-1$ and $u+1$ know $m_{LR}^{(u)}$ and $m_{RL}^{(u)}$, respectively, when decoding.

IV. Special Channels

A. Orthogonal Channels

A BC $P_{Y_1Y_2|X}$ is called *orthogonal* if one can write $X = (X_1, X_2)$ and $P_{Y_1Y_2|X} = P_{Y_1|X_1}P_{Y_2|X_2}$ (see [3, p. 419]). In fact, if all N broadcast channels in Fig. 2 are orthogonal, then the model reduces to that of Fig. 1 so hopefully we recover Theorem 1 from Theorem 2.

Let $X_u = (X_{u,u-1}, X_{u,u+1})$. We choose $S_u = X_{u,u-1}$, $T_u = X_{u,u+1}$, $W_u = 0$ and $X_{u,u-1}, X_{u,u+1}$ independent so that (19)-(23) reduce to

$$R_{RL}^{(u)} + R_{u,LR} + R_{u,L} \le I(X_{u,u-1}; Y_{u,u-1}) \le C_{u,u-1} \quad (24)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} \le I(X_{u,u+1}; Y_{u,u+1}) \le C_{u,u+1}. \quad (25)$$

We have equality in the rightmost inequalities in (24) and (25) by optimizing over $X_{u,u-1}$ and $X_{u,u+1}$. The region of Theorem 1 is therefore achievable. The converse follows by using the same steps as in the converse of Theorem 1. Thus, Theorem 2 reduces to Theorem 1, as desired.

B. Physically Degraded Channels

A BC $P_{Y_1Y_2|X}$ is said to be *physically degraded* if either

$$X - Y_1 - Y_2 \quad \text{or} \quad X - Y_2 - Y_1$$

form Markov chains (see [3, p. 422]). Consider Theorem 2 and suppose that $X - Y_{u,u-1} - Y_{u,u+1}$ forms a Markov chain for all u (the direction of degradation can be adjusted either way for any u for the capacity theorem below). We choose $S_u = X_u, T_u = 0$ and the bounds (18)-(23) reduce to

$$\max(R_{LR}^{(u)}, R_{RL}^{(u)}) + R_{u,LR} + R_{u,R} + R_{u,L} + R_u \leq C_u \quad (26)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} \leq I(W_u; Y_{u,u+1}) \quad (27)$$

$$R_{RL}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L} \leq I(X_u; Y_{u,u-1}) \quad (28)$$

$$R_{LR}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L} \leq I(W_u; Y_{u,u+1}) + I(X_u; Y_{u,u-1}|W_u) \quad (29)$$

We prove the following result.

Theorem 3: The capacity region of a line network with physically degraded BCs is achieved by separating channel and network coding. For example, if $X - Y_{u,u-1} - Y_{u,u+1}$ forms a Markov chain for all u, then the capacity region is defined by (26)-(29) for any choice of $P(w_u, x_u)$ for all u.

Proof: The bound (26) is based on an extension of PdE bounds to mixed wireline/wireless networks [8]. The rest of the converse follows by modifying the steps of [4]. Let $\mathcal{S}$ be a subset of the set $\mathcal{V}$ of network supernodes and let $\mathcal{S}^C$ be the complement of $\mathcal{S}$ in $\mathcal{V}$. We define

$$Y_{\mathcal{S},\mathcal{T}} = \{Y_{u,v} : u \in \mathcal{S}, v \in \mathcal{T}\} \quad (30)$$

and $Y_{u,\mathcal{S}} = Y_{\{u\},\mathcal{S}}$ and $Y_{\mathcal{S},u} = Y_{\mathcal{S},\{u\}}$. Let $M_{u,L}$ be the message random variable corresponding to $m_{u,L}$, and similarly for the other messages. The messages are independent and have entropy equal to n times their rate, where n is the number of times we use each BC. Let $M(\mathcal{S})$ be the set of messages originating at nodes in $\mathcal{S}$, and let $M_{u,L}^C$ to be the set of all network messages except for $M_{u,L}$.

For (27), we let $\mathcal{S} = \{1, 2, \ldots, u\}$ and

$$\tilde{W}_{u,i} = \left(M_{u,L}^C, Y_{u,u+1}^{i-1}\right) \quad (31)$$

where $Y_{u,u+1}^{i-1} = Y_{u,u+1,1}, Y_{u,u+1,2}, \ldots, Y_{u,u+1,i-1}$ and the third subscript index denotes time. Fano's inequality [3, p. 38] bounds the rates for reliable communication as

$$\begin{aligned}
&n(R_{LR}^{(u)} + R_{u,LR} + R_{u,R}) \\
&\leq I\left(M_{LR}^{(u)} M_{u,LR} M_{u,R}; Y_{\mathcal{S},\mathcal{S}^C}^n M(\mathcal{S}^C)\right) \\
&= I\left(M_{LR}^{(u)} M_{u,LR} M_{u,R}; Y_{u,u+1}^n | M(\mathcal{S}^C)\right) \\
&= \sum_{i=1}^n I\left(M_{LR}^{(u)} M_{u,LR} M_{u,R}; Y_{u,u+1,i} | M(\mathcal{S}^C) Y_{u,u+1}^{i-1}\right) \\
&\leq \sum_{i=1}^n I\left(M_{u,L}^C Y_{u,u+1}^{i-1}; Y_{u,u+1,i}\right) \\
&= nI\left(\tilde{W}_{u,Q}; Y_{u,u+1,Q} | Q\right) \\
&\leq nI\left(W_u; Y_{u,u+1}\right) \quad (32)
\end{aligned}$$

where the fourth step follows because

$$I(A; B|C) \leq I(AC; B) \leq I(ACD; BE)$$

for any random variables A, B, C, D, E. The fifth step follows by defining Q to be a *time-sharing* random variable that is uniform over $1, 2, \ldots, n$, and the last step follows by defining $W_u = (\tilde{W}_{u,Q}, Q)$ and $Y_{u,u+1} = Y_{u,u+1,Q}$.

Next, consider (28) and let $\mathcal{S} = \{u, u+1, \ldots, N\}$. For reliable communication, we bound

$$\begin{aligned}
&n(R_{RL}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L}) \\
&\leq I\left(M_{RL}^{(u)} M_{u,LR} M_{u,R} M_{u,L}; Y_{\mathcal{S},\mathcal{S}^C}^n M(\mathcal{S}^C)\right) \\
&= I\left(M_{RL}^{(u)} M_{u,LR} M_{u,R} M_{u,L}; Y_{u,u-1}^n | M(\mathcal{S}^C)\right) \\
&\leq I\left(M(\mathcal{S}); Y_{u,u-1}^n Y_{u,u+1}^n | M(\mathcal{S}^C)\right) \\
&= \sum_{i=1}^n H\left(Y_{u,u-1,i} Y_{u,u+1,i} | Y_{u,u-1}^{i-1} Y_{u,u+1}^{i-1} M(\mathcal{S}^C)\right) \\
&\quad - H\left(Y_{u,u-1,i} Y_{u,u+1,i} | Y_{u,u-1}^{i-1} Y_{u,u+1}^{i-1} M(\mathcal{V})\right) \\
&\leq \sum_{i=1}^n H\left(Y_{u,u-1,i} Y_{u,u+1,i}\right) - H\left(Y_{u,u-1,i} Y_{u,u+1,i} | X_{u,i}\right) \\
&= nH\left(Y_{u,u-1,Q} Y_{u,u+1,Q} | Q\right) - nH\left(Y_{u,u-1,Q} Y_{u,u+1,Q} | X_{u,Q}\right) \\
&\leq nI\left(X_{u,Q}; Y_{u,u-1,Q} Y_{u,u+1,Q}\right) \\
&= nI\left(X_u; Y_{u,u-1}\right) \quad (33)
\end{aligned}$$

where the fourth step follows by Markovity, and the final step follows by defining $X_u = X_{u,Q}$, $Y_{u,u-1} = Y_{u,u-1,Q}$, and by using the Markov chain due to physical degradation.

Finally, for (29) we use (32) and $\mathcal{S} = \{u, u+1, \ldots, N\}$ to bound

$$\begin{aligned}
&n(R_{LR}^{(u)} + R_{u,LR} + R_{u,R} + R_{u,L}) \\
&\leq nI\left(W_u; Y_{u,u+1}\right) + I\left(M_{u,L}; Y_{\mathcal{S},\mathcal{S}^C}^n M(\mathcal{S}^C)\right) \quad (34)
\end{aligned}$$

and

$$\begin{aligned}
&I\left(M_{u,L}; Y_{\mathcal{S},\mathcal{S}^C}^n M(\mathcal{S}^C)\right) \\
&\leq I\left(M_{u,L}; Y_{u,u-1}^n Y_{u,u+1}^n M_{u,L}^C\right) \\
&= I\left(M_{u,L}; Y_{u,u-1}^n Y_{u,u+1}^n | M_{u,L}^C\right) \\
&= \sum_{i=1}^n H\left(Y_{u,u-1,i} Y_{u,u+1,i} | Y_{u,u-1}^{i-1} Y_{u,u+1}^{i-1} M_{u,L}^C\right) \\
&\quad - H\left(Y_{u,u-1,i} Y_{u,u+1,i} | Y_{u,u-1}^{i-1} Y_{u,u+1}^{i-1} M(\mathcal{V})\right) \\
&\leq \sum_{i=1}^n H\left(Y_{u,u-1,i} Y_{u,u+1,i} | \tilde{W}_{u,i}\right) \\
&\quad - H\left(Y_{u,u-1,i} Y_{u,u+1,i} | X_{u,i} \tilde{W}_{u,i}\right) \\
&= nI\left(X_{u,Q}; Y_{u,u-1,Q} Y_{u,u+1,Q} | \tilde{W}_{u,Q} Q\right) \\
&\leq nI\left(X_u; Y_{u,u-1} Y_{u,u+1} | W_u\right) \\
&= nI\left(X_u; Y_{u,u-1} | W_u\right) \quad (35)
\end{aligned}$$

where the last step follows by physical degradation. Collecting the bounds (32)-(35) proves Theorem 3. ∎

C. Physically Degraded Gaussian Channels

The additive white Gaussian noise (AWGN) physically degraded BC has (see [5])

$$Y_{u,u-1} = X_u + Z_{u,u-1} \tag{36}$$

$$Y_{u,u+1} = Y_{u,u-1} + Z'_{u,u+1} \tag{37}$$

where X_u is real with power constraint $\sum_{i=1}^{n} X_{u,i}^2 \leq nP_u$ for all u, and $Z_{u,u-1}$ and $Z'_{u,u+1}$ are independent Gaussian random variables with variances $N_{u,u-1}$ and $N'_{u,u+1}$, respectively (again, the direction of degradation can be swapped for any u without changing the results conceptually).

The capacity region is given by Theorem 3 and it remains to optimize $P(w_u, x_u)$. The variances of $Y_{u,u-1}$ and $Y_{u,u+1}$ are at most $P_u + N_{u,u-1}$ and $P_u + N_{u,u-1} + N'_{u,u+1}$, respectively, so the maximum entropy theorem (see [3, p. 234]) gives

$$I(W_u; Y_{u,u+1}) \leq \frac{1}{2}\log(2\pi e(P_u + N_{u,u+1})) - h(Y_{u,u+1}|W_u) \tag{38}$$

$$I(X_u; Y_{u,u-1}) \leq \frac{1}{2}\log(1 + P_u/N_{u,u-1}) \tag{39}$$

$$I(X_u; Y_{u,u-1}|W_u) \leq h(Y_{u,u-1}|W_u) - \frac{1}{2}\log(2\pi e N_{u,u-1}) \tag{40}$$

where $N_{u,u+1} = N_{u,u-1} + N'_{u,u+1}$ and $h(Y|W)$ is the differential entropy of Y conditioned on W. Observe that

$$\begin{aligned} \frac{1}{2}\log(2\pi e N_{u,u+1}) &\leq h(Y_{u,u+1}|W_u) \\ &\leq \frac{1}{2}\log(2\pi e(P_u + N_{u,u+1})) \end{aligned} \tag{41}$$

so there is an α_u, $0 \leq \alpha_u \leq 1$, such that

$$h(Y_{u,u+1}|W_u) = \frac{1}{2}\log(2\pi e(\alpha_u P_u + N_{u,u+1})). \tag{42}$$

Furthermore, a conditional version of the entropy power inequality (see [3, p. 496]) gives

$$\begin{aligned} h(Y_{u,u+1}|W_u) &= h(Y_{u,u-1} + Z'_{u,u+1}|W_u) \\ &\geq \frac{1}{2}\log\left(e^{2h(Y_{u,u-1}|W_u)} + 2\pi e N'_{u,u+1}\right). \end{aligned} \tag{43}$$

Collecting the bounds, and inserting (42) and (43) into (40), we have

$$I(W_u; Y_{u,u+1}) \leq \frac{1}{2}\log\left(1 + \frac{(1-\alpha_u)P_u}{\alpha_u P_u + N_{u,u+1}}\right) \tag{44}$$

$$I(X_u; Y_{u,u-1}) \leq \frac{1}{2}\log(1 + P_u/N_{u,u-1}) \tag{45}$$

$$I(X_u; Y_{u,u-1}|W_u) \leq \frac{1}{2}\log(1 + \alpha_u P_u/N_{u,u-1}). \tag{46}$$

But we achieve equality in (44)-(46) by choosing

$$X_u = V_u + W_u \tag{47}$$

where V_u and W_u are independent Gaussian random variables with zero-mean and variances $\alpha_u P_u$ and $(1-\alpha_u)P_u$, respectively. The optimal $P(w_u, x_u)$ is therefore zero-mean Gaussian, and the capacity region is given by inserting (44)-(46) with equality into (27)-(29), and taking the union over the rates permitted by varying α_u.

V. Discussion

The capacity results in Sec. IV-A and Sec. IV-B imply that one may as well use *decode-and-forward* (DF) relaying, i.e., *amplify-and-forward* (AF) and *compress-and-forward* (CF) do not increase rates (see [7, Ch. 4]). However, this insight applies only to orthogonal and physically degraded BCs for which feedback does not increase capacity. AF and CF strategies are useful for other classes of BCs, as shown in [6].

Finally, we remark that our model applies to wireless problems where every node has a dedicated tone and/or time slot for transmission. If nodes use the same tone at the same time then one should consider the effects of interference. For example, scheduling transmission and reception with half-duplex protocols is an interesting problem for further study.

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