Network Coding in Star Networks

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***Abstract*—We investigate network coding in star networks with multiple unicast sessions. We use entropy arguments to upper bound the simultaneous rates of communication among the different nodes in the network and prove that in many cases, the optimal network code is related to the combinatorial optimization problem of finding the maximum number of edge disjoint cycles in the demand graph of the network. Finally, we propose a polynomial time algorithm with linear binary operations that achieves the capacity in many cases.**

I. Introduction

Star networks are among the most common computer network topologies. In its simplest form, a star network consists of a central switch or hub processor and several peripheral processors. The central processor (or node) acts as the conduit for transmitting messages and the peripheral nodes communicate by transmitting and receiving messages from the central node only. One reason that star topologies are popular is that they isolate the impact of the failure of a transmission line to a single node while the rest of the system remains unaffected. Star networks also have desirable scalability properties; it is easy to expand one by adding new peripheral devices or by enlarging the capacity of the hub node. Furthermore, since all messages pass through the hub node the task of inspecting or controlling messages can be left to the hub node. Because of these attractive properties the star topology is commonly used in high-speed optical networks and in other applications with high reliability requirements such as sensor networks (see, e.g., [6], [17], and [1]).

Star networks have received considerable attention in the networking literature. A number of routing and packet forwarding algorithms have been proposed for star networks, and these have been optimized in terms of query time [9], echo delay, error probability, scalability, failure tolerance and reliability (see, e.g., [12], [4]).

Fig. 1 depicts a star network with M peripheral nodes $n_1, ..., n_M$ and an interconnection (or central) node n_0. The edges in the figure represent bidirected channels with certain capacities in each direction. We assume that every processor or node in the network also has a limited capacity for processing data. One way to model restricted processing capability is via node capacities or constraints (see, e.g., [8, §11]). The idea here is to represent a network with both node and edge constraints by a second network with edge constraints only. This is done by decomposing each node in the original network into an input and an output node in the modified network; we use the notation from [15] and let I_i and O_i respectively denote the input and output nodes of processor n_i. For each node n_i we add a directed edge from I_i to O_i with a capacity equal to the node capacity for n_i. If there is an edge from n_i to n_j in the original network, then the modified network contains an edge from O_i to I_j. Fig. 2 illustrates the modified network for a node-constrained star network. As in [15], we further assume that the messages transmitted from node n_i originate at I_i and the messages received by node n_i are received at O_i.

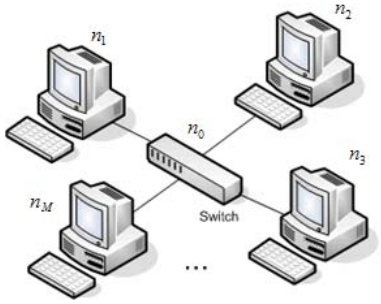


Fig. 1. Star network

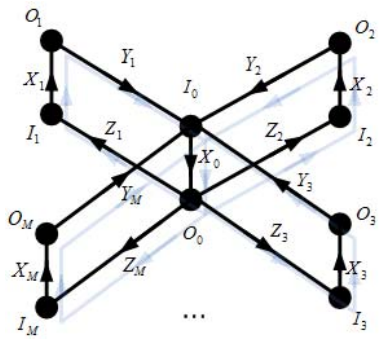


Fig. 2. Communication model for star network

In this paper we initiate the study of node-constrained star topologies from a network coding perspective. Network coding [2] permits the intermediate nodes of a network to encode a combination of their own messages to other nodes and their received messages and to transmit these new encoded messages to their neigboring nodes. This transmission strategy potentially increases the throughput of a network compared

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with routing strategies which only permit messages to be stored and forwarded.

The remainder of the paper is organized as follows. In Section II we formally define the problem and use information theoretic tools such as entropy arguments and cut set bounds [10], [5] to provide an outer bound for the capacity region of node-constrained star networks with multiple unicast sessions. In Section III we describe a coding and decoding scheme for each node in the network; the basis for the coding scheme is solving the optimization problem of finding the maximum number of edge disjoint cycles in the demand graph corresponding to the communication problem. In Section IV we demonstrate that our coding scheme is optimal for a broad class of demand graphs.

II. Network Coding Bounds

For the communication model of a star network depicted by Fig. 2, we introduce the following notation for random variables and capacities.

- X_i^t is the bit stream from I_i to O_i at time instant t for $i \in \{0, \cdots, M\}$.
- Y_i^t and Z_i^t respectively denote the bitstreams at time t from O_i to I_0 and from O_0 to I_i.
- $W_{i\to j}^t$ is the message from node n_i to n_j at time t for $i, j \in \{0, \cdots, M\}$.
- C_i^X, C_i^Y, and C_i^Z respectively denote the capacities of the edges with bit streams X_i, Y_i, and Z_i.
- $R_{i\to j}$ represents the rate of transmission from n_i to n_j.
- T is the final time instant of transmission.

Therefore, by the definitions of capacities and rates it follows that $H(X_i^t) \le C_i^X, H(Y_i^t) \le C_i^Y, H(Z_i^t) \le C_i^Z$, and $H(W_{i\to j}^t) \le R_{i\to j}$. We follow the convention that the omission of a superscript t from a random variable means that we consider the set of that kind of random variable over t ranging from 1 to T. Likewise, we let W denote the set of all messages until time T, i.e., $W = \{W_{i\to j} : i, j \in \{0, \cdots, M\}\}$. We assume throughout that all communication channels are noiseless.

Theorem 1: Let $\Pi(\cdot)$ denote an arbitrary permutation of $(1, \cdots, M)$. The following bound holds for the node-constrained star network of Fig. 2.

$$\sum_{i,j} R_{i\to j} - \sum_{i=1}^{M} \sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)} \le C_0^X. \tag{1}$$

Proof: We first define the following sets of messages:

$$W_{\Pi(i)} = \{W_{\Pi(i)\to\Pi(i+1)}, W_{\Pi(i)\to\Pi(i+2)}, \cdots, W_{\Pi(i)\to\Pi(M)}\}$$
$$W'_{\Pi(i)} = \{W_{\Pi(i)\to\Pi(i-1)}, W_{\Pi(i)\to\Pi(i-2)}, \cdots, W_{\Pi(i)\to\Pi(1)}\}.$$

We split set W into $4M$ disjoint sets of messages according to permutation Π as follows:

$$W = \bigcup_{i=1}^{M} W_{\Pi(i)} \bigcup_{i=1}^{M} W'_{\Pi(i)} \bigcup_{i=1}^{M} W_{i\to 0} \bigcup_{i=1}^{M} W_{0\to i}. \tag{2}$$

Since X_0^t is a function of W, so is X_0. In the following expansion of the entropy term $H(X_0, W)$ we use the fact that $H(X_0|W) = 0$, the independence of the different messages $W_{i\to j}$, and the chain rule for entropy.

$$\begin{aligned}
H\big(W\big) &= H\big(X_0, W\big) \\
&= H\big(X_0\big) + H\Big(\bigcup_{i=1}^{M} W_{i\to 0}\Big|X_0\Big) \\
&\quad + \Big[\sum_{i=1}^{M} H\Big(W_{\Pi(i)}\Big|X_0, \bigcup_{i=1}^{M} W_{i\to 0}, \bigcup_{j=1}^{i-1} W_{0\to\Pi(j)}, \\
&\quad \bigcup_{j=1}^{i-1} W_{\Pi(j)}, \bigcup_{j=1}^{i-1} W'_{\Pi(j)}\Big)\Big] \\
&\quad + \Big[\sum_{i=1}^{M} H\Big(W'_{\Pi(i)}\Big|X_0, \bigcup_{i=1}^{M} W_{i\to 0}, \bigcup_{j=1}^{i-1} W_{0\to\Pi(j)}, \\
&\quad \bigcup_{j=1}^{i} W_{\Pi(j)}, \bigcup_{j=1}^{i-1} W'_{\Pi(j)}\Big)\Big] \\
&\quad + \Big[\sum_{i=1}^{M} H\Big(W_{0\to\Pi(i)}\Big|X_0, \bigcup_{i=1}^{M} W_{i\to 0}, \bigcup_{j=1}^{i-1} W_{0\to\Pi(j)}, \\
&\quad \bigcup_{j=1}^{i} W_{\Pi(j)}, \bigcup_{j=1}^{i} W'_{\Pi(j)}\Big)\Big].
\end{aligned} \tag{3}$$

To simplify (3), we first observe that messages decoded at the hub's output node O_0 are a function of X_0, and so $H\Big(\bigcup_{i=1}^{M} W_{i\to 0}\Big|X_0\Big) = 0$.

Next consider any term in the second summation of (3). For each i, $W'_{\Pi(i)}$ is a union over $j < i$ of the messages $W_{\Pi(i)\to\Pi(j)}$. $W_{\Pi(i)\to\Pi(j)}$ is decoded at $O_{\Pi(j)}$ as a function of $X_{\Pi(j)}$. We observe that $X_{\Pi(j)}$ is a function of $X_0, W_{\Pi(j)}, W'_{\Pi(j)}$, and $W_{\Pi(j)\to 0}$. Since our entropy term is conditional on all such random variables for $j < i$, it is therefore equal to zero.

Finally consider any term in the third summation of (3). Since $W_{0\to\Pi(i)}$ is decoded as a function of $X_0, W_{\Pi(i)}, W'_{\Pi(i)}$, and $W_{\Pi(i)\to 0}$ and since our entropy term is conditional on all these random variables, our entropy term is equal to zero. Finally, to bound the terms in the first summation of (3), observe that

$$H\Big(W_{\Pi(i)}\Big|X_0, \bigcup_{i=1}^{M} W_{i\to 0}, \bigcup_{j=1}^{i-1} W_{0\to\Pi(j)}, \bigcup_{j=1}^{i-1} W_{\Pi(j)}, \bigcup_{j=1}^{i-1} W'_{\Pi(j)}\Big) \le H\Big(W_{\Pi(i)}\Big).$$

As a consequence of these arguments, we obtain the following bound on $H(X_0)$:

$$H\big(X_0\big) \ge H\big(W\big) - \sum_{i=1}^{M} H\big(W_{\Pi(i)}\big). \tag{4}$$

To establish (1), we replace the entropy terms in (4) with their corresponding rates and divide both sides of the resulting inequality by T. ∎

Since (1) is valid for every choice of permutation Π, the most restrictive bound on the rates corresponds to that Π which

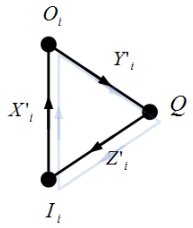


Fig. 3. Modified star network.

minimizes $\sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)}$. In the next section we will discuss a coding scheme which sometimes solves this optimization problem, and we will discuss cases in which our outer bound on the capacity region is tight.

Remark 1: Notice that $\min_{\Pi}\sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)} \leq 0.5\sum_{i=1}^{M}\sum_{j=1}^{M} R_{i\to j}$. This shows that the total capacity consumption of the central link in any coding scheme is never less than half of the capacity consumption by routing schemes.

We next discuss the outer bounds on the capacity region which involve the terms C_i^X, C_i^Y, and C_i^Z for $i \geq 1$. We proceed by constructing another network from the one depicted in Fig. 2; the new network will have a capacity region containing the achievable rate region of Fig. 2. To create the new network, fix the index i and increase to infinity the capacity of all links except the ones used for the transmission of random variables X_i, Y_i, and Z_i. For this new network, we will focus on the rate bounds only involving terms corresponding to a source at I_i or a destination at O_i; i.e., we implicitly set other rate terms to 0. Since for $j,k \neq i$ the path from I_j to O_k has infinite capacity, we will merge all nodes except for I_i and O_i into a single supernode which we label Q; Fig. 3 depicts the modified network. We point out that the network coding bounds for this network are discussed in [10] and the results are obtained using progressive d-separating edge set (*PdE*) bounds. Observe that in the modified network we have a unicast session from I_i to Q with rate $\sum_{j\neq i} R_{i\to j}$, and we have a second unicast session from Q to O_i with rate $\sum_{j\neq i} R_{j\to i}$. Let us respectively denote the corresponding messages by $W_{i\to Q}$ and $W_{Q\to i}$. As illustrated in Fig. 3, the counterpart bitstreams in the modified network for X_i, Y_i, and Z_i are respectively called X_i', Y_i', and Z_i'. Since X_i' is a function of $W_{i\to Q}$ and $W_{Q\to i}$, the chain rule for entropy implies

$$\begin{aligned} H\Big(W_{i\to Q}, W_{Q\to i}\Big) &= H\Big(X_i', W_{i\to Q}, W_{Q\to i}\Big) \\ &= H\Big(X_i'\Big) + H\Big(W_{Q\to i}\Big|X_i'\Big) \\ &\quad + H\Big(W_{i\to Q}\Big|X_i', W_{Q\to i}\Big). \end{aligned} \tag{5}$$

Since the messages destined for O_i are decoded using the bitstream X_i', it follows that $H\Big(W_{Q\to i}\Big|X_i'\Big) = 0$. Furthermore, the messages destined for Q are decoded from Y_i' and $W_{Q\to i}$. Observe that the structure of the network implies that Y_i' is a function of X_i'. Therefore, we have $H\Big(W_{i\to Q}\Big|X_i', W_{Q\to i}\Big) = 0$. These entropy terms and (5) imply

$$H\Big(X_i'\Big) = H\Big(W_{i\to Q}, W_{Q\to i}\Big) \tag{6}$$

By dividing both sides of (6) by T, we obtain

$$\sum_{j\neq i}\Big(R_{i\to j} + R_{j\to i}\Big) \leq C_i^X. \tag{7}$$

Note that (7) clearly holds for the original network as well. We next apply an edge cut argument from [10] to the triangular network in Fig. 3 in order to obtain bounds on achievable rates involving capacities C_i^Y and C_i^Z. Since a similar example is discussed in detail in [10], we only mention the results here. First consider the edge cut which removes the edge from O_i to Q. It is possible to show that

$$\sum_{j\neq i} R_{i\to j} \leq C_i^Y. \tag{8}$$

Next consider the edge cut which removes the edge from Q to I_i. The corresponding bound is

$$\sum_{j\neq i} R_{j\to i} \leq C_i^Z. \tag{9}$$

The network coding scheme that we will discuss in the next section establishes that (7), (8), (9) are tight.

III. Coding Scheme

In this section we describe the encoding and decoding processes used at every node. We assume throughout that there is a unit delay associated with the channel from an output node to the input node of another processor.

Encoding and decoding at I_0: At time t, I_0 has access to Y_i^τ for $\tau \in \{1,\cdots,t-1\}$ and $i \in \{1,\cdots,M\}$. Therefore its output X_0^t is a function of these bit streams as well as its messages $W_{0\to j}, j \neq 0$, to other nodes. We only use the most recent Y_i^{t-1}, which is a function of $W_{i\to j}^{\tau}, j \neq i, \tau \in \{1,\cdots,t-2\}$. We claim that all messages $W_{i\to j}^{t-2}$ are decodable from Y_i^{t-1}; this assertion will become clear when we describe the encoding rules for O_i. Node I_0 transmits $\Big(W_{1\to 0}^{t-2},\cdots,W_{M\to 0}^{t-2}, W_{0\to 1}^{t},\cdots,W_{0\to M}^{t}\Big)$ within X_0^t to O_0. It also performs a series of operations on the messages $W_{i\to j}^{t-2}, i,j \neq 0$.

1) I_0 forms a directed multigraph called a demand graph, which is the multigraph $G(V,E)$ with $V = \{1,\cdots,M\}$ and with $R_{i\to j}$ parallel edges from node i to j if the rate of $W_{i\to j}^{t-2}$ is $R_{i\to j}$ bits/sec for $i,j \neq 0$. Notice that there is a one to one correspondence between the edges of G and the messages $W_{i\to j}^{t-2}, i,j \neq 0$, in binary form.
2) I_0 extracts a set of edge disjoint cycles $\{\mathcal{C}_1,\cdots,\mathcal{C}_P\}$ from G. A cycle is a directed path of edges which starts and ends at the same node and in between visits other nodes, each one at most once.
3) Let $\oplus$ denote the XOR operation and $|\mathcal{C}_k|$ be the number of edges in cycle $\mathcal{C}_k$. For $k \in \{1,\cdots,P\}$, I_0 encodes

the bits corresponding to the edge of cycle $\mathcal{C}_k$ into the following binary vector and sends this to O_0:

$$\left(b_1^k \oplus b_2^k, b_1^k \oplus b_3^k, \cdots, b_1^k \oplus b_{|\mathcal{C}_k|}^k\right). \quad (10)$$

Here the ordering of bits $b_1^k, \cdots, b_{|\mathcal{C}_k|}^k$ corresponds to their ordering along cycle $\mathcal{C}_k$ starting at an arbitrary edge.

4) The remaining bits on G which correspond to the edges in $E \setminus \bigcup_{k=1}^{P} \mathcal{C}_k$ are sent to O_0 without encoding.

The length of X_0^t is therefore $\sum_{i,j} R_{i\to j} - P$ bits.

Encoding and decoding at O_0: By receiving X_0^t, O_0 has to decode the messages $\left(W_{1\to 0}^{t-2}, \cdots, W_{M\to 0}^{t-2}\right)$ which are destined for O_0. This can be done directly, since $\left(W_{i\to 0}^{t-2}\right)$ is a vector included in X_0^t. Then O_0 encodes the bit stream Z_i^{t+1} as a function of X_0^t for $i \in \{1, \cdots, M\}$. The encoding of Z_i^{t+1} is done in the following steps:

1) The vector $\left(W_{0\to i}^t\right)$ is part of Z_i^{t+1} which is directly available from X_0^t.
2) For each cycle $\mathcal{C}_k, k \in \{1, \cdots, P\}$, O_0 determines if node i is a vertex along the cycle. If so, then it lets b_j^k and b_{j+1}^k denote the bits entering and exiting node i along cycle $\mathcal{C}_k$ (using the convention $b_{|\mathcal{C}_k+1|}^k = b_1^k$) and it forms and transmits the bit $b_j^k \oplus b_{j+1}^k = (b_1^k \oplus b_j^k) \oplus (b_1^k \oplus b_{j+1}^k)$.
3) The bits of the messages with destination at O_i that belong to $E \setminus \bigcup_{k=1}^{P} \mathcal{C}_k$ are sent without encoding as part of Z_i^{t+1}.

Since there is one bit corresponding to every bit of messages $W_{j\to i}^{t-2}, j \neq 0$, and $W_{0\to i}^t$, the total length of Z_i^{t+1} is $\sum_{j\neq i} R_{j\to i}$, and so the capacity C_i^Z needed to transmit this bit stream satisfies (9).

Encoding and decoding at I_i: By receiving Z_i^{t+1}, I_i first decodes all messages $W_{j\to i}^{t-2}, j \neq 0, W_{0\to i}^t$ that have destination O_i:

1) The vector $\left(W_{0\to i}^t\right)$ is directly decodable as a part of Z_i^{t+1}.
2) If a bit of a message $W_{j\to i}^{t-2}, j \neq 0$ is part of a cycle $\mathcal{C}_k$ and is represented by b_l^k, then the bit $b_l^k \oplus b_{l+1}^k$ is part of Z_i^{t+1}, and since b_{l+1}^k is part of a message with source I_i then Z_i^{t+1} extracts b_l^k by using $(b_l^k \oplus b_{l+1}^k) \oplus b_{l+1}^k$.
3) If a bit of a message $W_{j\to i}^{t-2}, j \neq 0$ is part of $E \setminus \bigcup_{k=1}^{P} \mathcal{C}_k$, then it is directly available from Z_i^{t+1}.

After the decoding process, I_i chooses the following vector as X_i^{t+2} and sends it to node O_i:

$$\left(W_{1\to i}^{t-2}, \cdots, W_{i-1\to i}^{t-2}, W_{i+1\to i}^{t-2}, \cdots, W_{M\to i}^{t-2}, W_{0\to i}^t, W_{i\to 0}^{t+2}, \cdots, W_{i\to M}^{t+2}\right). \quad (11)$$

The total length of X_i^{t+2} is $\sum_{j\neq i}\left(R_{i\to j} + R_{j\to i}\right)$, and the capacity C_i^X needed to transmit this bit stream is given by (7).

Encoding and decoding at O_i: Node O_i directly decodes the messages $\left(W_{1\to i}^{t-2}, \cdots, W_{i-1\to i}^{t-2}, W_{i+1\to i}^{t-2}, \cdots, W_{M\to i}^{t-2}, W_{0\to i}^t\right)$ with destination O_i and encodes Y_i^{t+3} as the vector $\left(W_{i\to 0}^{t+2}, \cdots, W_{i\to M}^{t+2}\right)$. The capacity C_i^Y needed to transmit Y_i^{t+3} satisfies (8), and this bit stream also includes all of the information that is needed for the encoding process at I_0. We conclude our description of the encoding and decoding processes in the node-constrained star network with the following simple example:

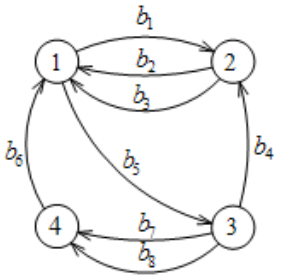


Fig. 4. Demand graph of Example 1 .

Example 1: Assume that the following rates are the demands in a star network with 4 branches: $R_{1\to 2} = 1, R_{2\to 1} = 2, R_{1\to 3} = 1, R_{3\to 4} = 2, R_{4\to 1} = 1$, and $R_{3\to 2} = 1$. Fig. 4 shows the demand graph corresponding to the rates. Here, $b_k, k \in \{1, \cdots, 8\}$, denotes the bits to be transmitted. We describe here part of the operations of nodes I_0, O_0, I_1, and O_1. Node I_0 after receiving the demand graph information from $O_1, \cdots, O_4$, uses the cycles $\mathcal{C}_1 = \{b_1, b_2\}$ and $\mathcal{C}_2 = \{b_6, b_5, b_7\}$ for the coding scheme, and forms $X_0^t = \left(b_3, b_4, b_8, b_1 \oplus b_2, b_7 \oplus b_5, b_7 \oplus b_6\right)$, and sends it to O_0. Now we concentrate on node n_1 and its corresponding bitstreams. n_1 is the source of b_1 and b_5, and is the destination of b_2, b_3, and b_6. O_0 after receiving X_0^t, forms $Z_1^{t+1} = \left(b_3, b_1 \oplus b_2, b_5 \oplus b_6\right)$ and sends it to I_1. Since I_1 knows b_1 and b_5, it forms $X_1^{t+2} = \left(b_3, b_2, b_6\right)$ and sends it to O_1 which is the destination of these bits.

Notice that in order to transmit X_0^t, C_0^X must be at least 6 bits/sec. To show that this requirement on C_0^X is tight for all encoding schemes, we compute the tightest bound of the form (1). In this case the permutation $\Pi(1) = 1, \Pi(2) = 2, \Pi(3) = 4$, and $\Pi(4) = 3$ results in the tightest bound, which is $6 \leq C_0^X$. Since our scheme is always optimal for the other edges, it is optimal for this example.

In the next section we will prove that our coding scheme is optimal for a broad class of demand graphs.

IV. Optimality of Network Coding

We have already established the optimality of our network coding scheme for all edges except for the one from I_0 to O_0; i.e., our scheme matches the bounds (7), (8), and (9) exactly. In this section we focus on the performance of the last edge.

Recall that we have chosen X_0^t to be a function of the bits along P edge disjoint cycles of the demand graph and that X_0^t has $\sum_{i,j} R_{i\to j} - P$ bits. Observe that by (1) the rates are constrained by $\sum_{i,j} R_{i\to j} - \sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)} \leq$

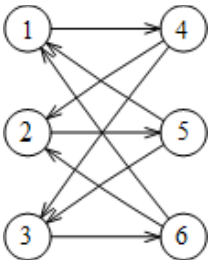


Fig. 5. Directed $\mathcal{K}_{3,3}$ graph.

C_0^X. For our coding scheme the size of the smallest possible vector X_0^t can be found from the maximum number P^* of edge disjoint cycles in the demand graph G. Let Π^* represent a permutation that minimizes $\sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)}$. We will show a class of demand graphs for which $P^* = \sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi^*(i)\to\Pi^*(j)}$. In these cases it follows that our coding and decoding schemes are optimal. The following theorem provides the maximum number of edge disjoint cycles in planar directed multigraphs:

Theorem 2 (Lucchesi, 1976[11]): Let $G(V, E)$ be a planar directed multigraph. We define a set $S \subset E$ to be a feedback edge set if $G \setminus S$ is an acyclic, directed multigraph. The maximum number P^* of edge disjoint cycles of G, is equal to the number of elements of the minimal feedback edge set S^*.

Notice that for every feedback edge set S, the multigraph $G \setminus S$ is acyclic, and hence there is a permutation Π of nodes in $G \setminus S$ such that the edges from $\Pi(i)$ to $\Pi(j)$ corresponding to $R_{\Pi(i)\to\Pi(j)}$ are in $G \setminus S$ if and only if $i > j$; i.e., the edges corresponding to $R_{\Pi(i)\to\Pi(j)}$ are in S if and only if $i < j$. This implies that S^* corresponds to the permutation Π^* that minimizes $\sum_{i=1}^{M}\sum_{j=i+1}^{M} R_{\Pi(i)\to\Pi(j)}$. It follows from Theorem 2 that $|S^*| = P^*$, and hence our scheme is optimal for planar demand graphs.

Since [11] also establishes that the minimal feedback edge set can be computed by a polynomial time algorithm, our coding algorithm is a polynomial time algorithm. Furthermore, Theorem 2 has been extended to a broader class of directed graphs (see, e.g., [14], [16], and [13]). The smallest directed graph that violates Theorem 2 is the nonplanar complete bipartite graph $\mathcal{K}_{3,3}$ depicted in Fig. 5. We observe that we can potentially improve our coding scheme if we use buffers and for each source-destination pair encode a sequence of bits corresponding to multiple time instants. Observe that if we replace each edge in our graph with n parallel edges, for large and even n the maximum number of edge disjoint cycles can be shown to be $1.5n$, and it is easy to verify that when $n = 1$, this maximum number is 1. Hence, our scheme requires $C_0^X = \frac{9n-1.5n}{n} = 7.5$ bits/sec. However, the best bound from (4) is that $C_0^X \geq 7$ bits/sec, and this corresponds to the permutation function $\Pi^* = (1, 3, 5, 2, 4, 6)$. Yet for this case there is a linear code for which $C_0^X = 7$ bits/sec. We will discuss optimal linear codes in a longer version of this paper.

V. Conclusions

In this paper we discussed network coding for node-constrained star networks. We proposed a binary, linear coding scheme based on the maximum number of edge disjoint cycles in the demand graph, and we showed that this scheme is optimal for planar demand graphs.

We remark that a direction for future work is to extend the coding schemes of [15] and this paper to a scheme that works for general tree networks.

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