The Capacity Region of a Collection of Multicast Sessions in an Undirected Ring Network

S. M. Sadegh Tabatabaei Yazdi
EECS Dept.
University of Michigan
Ann Arbor, MI USA
sadegh@umich.edu

Serap A. Savari
EECS Dept.
University of Michigan
Ann Arbor, MI USA
savari@eecs.umich.edu

Kelli Carlson (Talaska)
Mathematics Dept.
University of Michigan
Ann Arbor, MI USA
kellicar@umich.edu

Gerhard Kramer
Bell Labs
Alcatel-Lucent
Murray Hill, NJ USA
gkr@bell-labs.edu

Abstract

A finite set of inequalities is given that characterizes the routing rate region for an undirected ring network in which the source and destination vertices of each communication session form a string of adjacent vertices. The result uses an extension of the Japanese theorem for communication problems with multiple multicast sessions and an interpretation of the extension in terms of the collection of minimum length routing trees for the various multicast sessions. It is further demonstrated that routing is rate optimal in this case using new extensions to progressive d-separating edge set bounds.

1. Introduction

Various high-demand and real-time applications requiring reliable high-bandwidth and long-distance communications systems have resulted in an increasing use of fiber-optic communication networks (see, e.g., [2], [5], [15], [16]). Metropolitan area networks and wide area networks are often based on SONET/SDH rings (see, e.g., [2], [5]); in practice, these ring networks can be either unidirectional or bidirectional.

We consider undirected ring networks, in which the sum of the flow carried along both directions of an edge is bounded by its capacity. The multicommodity flow capacity for such networks has been derived by Vachani et.al. in [16] and earlier by Okamura and Seymour [11] as the special case of a more general result for planar graphs. A collection of multicommodity (unicast) flows is feasible if and only if the total rate across every cut in the network is bounded above by the capacity of the cut edges. The "cut" property in the multicommodity flow setting for undirected ring networks does not apply to multiple broadcast sessions [13].

For the multiple session multicast problem, Tabatabaei Yazdi et.al. in [18] characterize the routable rate-tuples on the hyperplanes of the Japanese Theorem of [6], [12] and show that certain of these hyperplanes are redundant. We provide the relevant background information from this paper in Section II. That paper used this result as a step in determining the capacity region of a multiple session multicast problem on undirected ring networks in which the sessions are either broadcasts or unicasts and in showing that routing is rate optimal in that setting.

In Section III, we study a different special case of the multiple session multicast problem on undirected ring networks. We call a multicast session in a network a *line session* if the source and all destinations of the session form a sequence (in any order) of adjacent vertices in the network. This gives a generalization of broadcast sessions. In the network communication problem where every multicast session is a line session, we explicitly derive a finite set of bounds defining the routing capacity region. In Section IV, we broaden a result of [7], [14], using a new extension of progressive d-separating edge set (*PdE*) bounds [8], [9] to show that routing is rate optimal in the case where all sessions are line sessions.

2. Routing Capacity Region in Networks

2.1. Preliminaries

Suppose we have an undirected ring network $\mathcal{N}$ in which the vertex set $\{1,\ 2,\ \ldots,\ n\}$ is configured in clockwise order Let C_i represent the capacity of edge i, i.e., the edge between vertices i and $i+1$ for $i \in \{1,\ 2,\ \ldots,\ n-1\}$ and between vertices n and 1 for $i = n$; we assume that C_i is rational for each i. We consider the communication problem in which there is a session A_j with rate R_{A_j} for each type of multicast scenario; here, $A_j \equiv s \rightarrow \{t_1, t_2, \cdots t_{i_j}\}$ means that the source for session A_j is s and the set of destinations

is $\{t_1, t_2, \cdots t_{i_j}\}$. In a routing approach, each session A_j is routed along a set of minimum length routing trees which connect the source to the various destinations.

The Japanese Theorem of [6], [12] gives the routing capacity region in an arbitrary network with multiple unicast sessions in terms of an infinite set of linear constraints. From [18], we have the following extension of the Japanese Theorem for the case of ring networks. Here, we assign edge distance a_i to edge i. For any path or tree T, we define its length to be the sum of the distances of the edges in T. Let ℓ_i be the minimum length of a tree spanning the source and destinations of A_i.

Proposition 2.1 (Extended Japanese Theorem for Rings) *Suppose there are a total of K sessions. The collection of non-negative rates $(R_{A_1}, R_{A_2}, \cdots, R_{A_K})$ is routable if and only if for every possible set of non-negative integer edge distances $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ the following inequality holds:*

$$\sum_{i=1}^{K} \ell_i R_{A_i} \leq \sum_{i=1}^{n} a_i C_i. \tag{1}$$

This result is a consequence of Farkas' Lemma and extends to networks other than rings [18].

2.2. The Reduced Set of Bounds

The routing rate region defined by (1) can be interpreted as the intersection of a finite number of halfspaces. For each possible set of edge distances $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ we wish to determine whether the hyperplane $H_{\mathcal{A}}$ defined by

$$\sum_{i=1}^{K} \ell_i R_{A_i} = \sum_{i=1}^{n} a_i C_i \tag{2}$$

is in fact a defining hyperplane for the routing capacity region. The following lemma and proposition from [18] give a way to establish the redundancy of a large collection of hyperplanes. The result indicates that the true significance of the edge distances in the extended Japanese theorem lies in the partial ordering by length of the routing trees used by the various multicast sessions. We note that the remaining results in this section are valid for any networks [18].

Lemma 2.2 *Suppose we have K multicast sessions $A_1, \ldots, A_K$. Let $\mathcal{T}_i$ denote the collection of routing trees minimally spanning the source and all destination vertices of the session A_i; i.e., if we are considering a collection of k vertices $1 \leq v_1 < v_2 < \cdots < v_k \leq n$, then a minimal spanning tree is the subgraph consisting of the original network except for the vertices $v_j + 1, \ldots, v_{j+1} - 1$ and edges $v_j, \ldots, v_{j+1} - 1$ for some $j \in \{1, \ldots, k\}$ (with $v_{k+1} = v_1$). Necessary and sufficient conditions for $(R_{A_1}, \ldots, R_{A_K})$ to lie on the hyperplane corresponding to $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ are as follows.*

1. *For each i, session A_i is routed only along those routing trees in $\mathcal{T}_i$ which have minimum length with respect to the edge distances $\mathcal{A} = (a_1, a_2, \cdots, a_n)$.*
2. *Every edge with a non-zero distance is fully utilized.*

We offer a new proof for the next result.

Proposition 2.3 *Consider two sets of edge distances, $\mathcal{A} = (a_1, a_2, \cdots, a_n)$ and $\mathcal{B} = (b_1, b_2, \cdots, b_n)$. If $b_i = 0$ whenever $a_i = 0$ and if for every multicast session, all minimum length routing trees with respect to $\mathcal{A}$ are also minimum length routing trees for $\mathcal{B}$, then the halfspace (1) corresponding to $\mathcal{A}$ gives a redundant bound for the routing capacity region. (Note that there may exist multicast sessions for which $\mathcal{B}$ has multiple minimum length routing trees and only some of these routing trees are also of minimum length for $\mathcal{A}$.)*

Proof:

- Let N_i be the number of possible routing trees for session A_i; that is, the number of minimal spanning routing trees for the vertex set consisting of the source and all destinations for A_i.
- Enumerate the routing trees for each session A_i. Let L_{ij} denote the length of the jth tree for session A_i.
- Let $\ell_i = \min\{L_{ij} \ : \ 1 \leq j \leq N_i\}$ be the minimum among routing tree lengths for session A_i with respect to edge distances $\mathcal{A}$.
- Let B_{ij} denote the flow along the jth tree for session A_i in a routing solution; that is, for each i, we have $\sum_{j=1}^{N_i} B_{ij} = R_{A_i}$.
- Let $M_k, \ k \in \{1, \ldots, K\}$ denote the number of minimum length routing trees for session A_k with respect to edge distances $\mathcal{A}$. Assume without loss of generality that for a fixed k, L_{kj} is non-decreasing with j so that $L_{k1} = \cdots = L_{kM_k} = \ell_k$.

Suppose that $(R_{A_1} \ldots, R_{A_K})$ is a routable rate-tuple lying on the hyperplane $H_{\mathcal{A}}$. By Condition 1) in Lemma 2.2 it follows that

$$\sum_{k=1}^{K} \ell_k R_{A_k} = \sum_{k=1}^{K} \ell_k \sum_{j=1}^{M_k} B_{kj} = \sum_{k=1}^{K} \sum_{j=1}^{M_k} L_{kj} B_{kj} = \sum_{i=1}^{n} a_i C_i.$$

Define $\delta_{i,(k,j)}$ to be one if edge i is in the j^{th} minimum length routing tree for session A_k and 0 otherwise. Then by writing each L_{kj} as the sum of the edge distances a_i

for which edge i occurs in the j^{th} minimum length routing tree for session A_k, we see that for $k \in \{1,\ldots,K\}, j \in \{1,\ldots,M_k\}$ we have

$$\ell_k = L_{kj} = \sum_{i=1}^{n} \delta_{i,(k,j)} a_i. \tag{3}$$

Condition 2) in Lemma 2.2 states that every edge with a non-zero distance is fully utilized; i.e., if $a_i > 0$ for $i \in \{1,\ldots,n\}$, then

$$\sum_{k=1}^{K}\sum_{j=1}^{M_k} \delta_{i,(k,j)} B_{kj} = C_i. \tag{4}$$

Set S to be the $n \times n$ diagonal matrix with diagonal entries $s_i = \sum_{k=1}^{K}\sum_{j=1}^{M_k} \delta_{i,(k,j)} B_{jk} - C_i$, and set D to be the $n \times n$ diagonal matrix with diagonal entries $d_i = \frac{b_i}{a_i}$ when $a_i \neq 0$ and $d_i = 0$ when $a_i = 0$. Viewing $\mathcal{A}$ as a column vector, it follows from (4) that $S\mathcal{A} = 0$, and hence $DS\mathcal{A} = 0$ as well. Diagonal matrices commute, so $SD\mathcal{A} = 0$. Since $D\mathcal{A} = \mathcal{B}$, we have $S\mathcal{B} = 0$, and thus,

$$0 = \sum_{i=1}^{n} s_i b_i = \sum_{k=1}^{K}\sum_{j=1}^{M_k} \left(\sum_{i=1}^{n} \delta_{i,(k,j)} b_i \right) B_{kj} - \sum_{i=1}^{n} b_i C_i. \tag{5}$$

Suppose ℓ_k' is the length of the minimal length routing tree and $L_{kj}^{'}$ is the length of the j^{th} routing tree with respect to the edge set $\mathcal{B}$ for session A_k. By assumption $L_{k1}^{'} = \cdots = L_{kM_k}^{'} = \ell_k^{'}$, and the counterpart to (3) is

$$\ell_k^{'} = L_{kj}^{'} = \sum_{i=1}^{n} \delta_{i,(k,j)} b_i, \tag{6}$$

Substituting (6) into (5) we obtain

$$0 = \sum_{k=1}^{K}\sum_{j=1}^{M_k} \ell_k' B_{kj} - \sum_{i=1}^{n} b_i C_i = \sum_{k=1}^{K} \ell_k' R_{A_k} - \sum_{i=1}^{n} b_i C_i,$$

and so $(R_{A_1},\ldots,R_{A_K})$ does lie on the hyperplane $H_{\mathcal{B}}$.

Thus, if a routable rate-tuple $(R_{A_1}\ldots,R_{A_K})$ is on the hyperplane $H_{\mathcal{A}}$, then it is also on the hyperplane $H_{\mathcal{B}}$. Since the routing rate region can be described in terms of its defining hyperplanes, the bound given by $H_{\mathcal{A}}$ is redundant assuming we already have the bound given by $H_{\mathcal{B}}$.

□

3. Routing Capacity for Undirected Ring Networks with Line Sessions Only

In this section we will use Proposition 2.3 to derive a finite set of inequalities characterizing the routing capacity region of the multicast problem in a ring network in which each session is a line session; i.e., if the source and all destinations of the session form a sequence (in any order) of adjacent vertices in the network. Consider the n vertex ring of Figure **??**, and suppose that all sessions in the communication problem are line sessions. In the last section we proved that given two sets of edge distances $\mathcal{A}$ and $\mathcal{B}$ which satisfy the conditions of Proposition 2.3, the bound corresponding to $\mathcal{A}$ is redundant. Here we give a constructive proof that for an arbitrary set of integer edge distances $\mathcal{A} = (a_1,\cdots,a_n)$, there exists a set of edge distances $\mathcal{B} = (b_1,\cdots,b_n)$ which satisfies the following conditions:

1. For $1 \leq i \leq n$, $b_i \in \{0,1\}$.
2. If $a_i = 0$, then $b_i = 0$.
3. For each line session on the ring, each minimum length tree with respect to the edge distances in $\mathcal{A}$ is also a minimum length tree with respect to the edge distances in $\mathcal{B}$.

In an undirected ring network with only line sessions, Proposition 2.3 then implies that given the inequality corresponding to $\mathcal{B}$ in (1), the inequality corresponding to $\mathcal{A}$ is redundant. Thus, the collection of binary edge distances suffices for characterizing the routing capacity region in this case. To show this, we give the following algorithm for constructing a set of binary edge distances $\mathcal{B}$ which eliminates an arbitrary set of edge distances $\mathcal{A}$.

Algorithm: Let $P = \max\{a_1,\cdots,a_n\} > 0$ and take $m_1 < m_2 < \cdots < m_N$ to be the set of indices of all edges of maximum distance in $\mathcal{A}$, so that $a_{m_1} = a_{m_2} = \cdots = a_{m_N} = P$. Without loss of generality, we can assume that $m_1 = 1$ and so $a_1 = P > 0$. Let $S_p^q = \sum_{i=p}^{q} a_i,\ 1 \leq p \leq q \leq n$. The following algorithm will give the desired binary edge distance set $\mathcal{B}$.

1. Set $b_j = 0$ for all j with $a_j = 0$.
2. Set $i = 1$.
3. Complete the following steps:
   (a) Set $b_i = 1$.
   (b) If there exists an index j such that (7) holds, it must be unique. Increase i by j and return to Step 3.
   If no such j exists, go on to Step 4.
   $$S_{i+1}^{i+j-1} < P, \text{ but } S_{i+1}^{i+j} \geq P, \text{ and } i+j \leq n \tag{7}$$
4. Set each remaining edge distance in $\mathcal{B}$ to 0.

In a longer version of this paper we will show that for any line session, this algorithm preserves the collection of minimum length routing trees with respect to the edge distances $\mathcal{A}$ in switching to appropriate binary edge distances $\mathcal{B}$. Thus, each set of edge distances with any distance greater than 1 is redundant by Proposition 2.3, so the inequalities corresponding to the binary edge distances are sufficient to characterize the routing capacity region. We now have a (relatively small) finite set of inequalities defining the routing capacity region of feasible rate-tuples. In general, all binary edge distance sets with at least two edge distances of 1 will in fact give defining hyperplanes of the feasible region.

4. Network Coding Bound for Undirected Ring Networks with Line Sessions Only

In this section, we will show that any routing bound corresponding to a binary set of edge distances $\mathcal{B}$ is simultaneously an upper bound for capacity using network coding [1]. We conclude that routing is rate optimal for this special case of the multiple session multicast problem on an undirected ring. We assume that all line sessions are potentially present in an n vertex ring, and we start by deriving the bound corresponding to the case where there are exactly two edges with length 1. Let i and j (with $i \leq j$) be the two edges with $b_i = b_j = 1$, so that all other edge distances are zero. We define two sets of vertices, $G_1 = \{j+1, \cdots, n, 1, \cdots, i\}$ and $G_2 = \{i+1, \cdots, j\}$. Given a particular line session, if the source and destination vertices all lie in G_1 or all lie in G_2, then the minimum length tree for this session is zero. Otherwise, the minimum length tree will include exactly one of edge i or edge j. We obtain the following routing bound.

$$\sum_{A_k \cap G_1 \neq \emptyset, A_k \cap G_2 \neq \emptyset} R_{A_k} \leq C_i + C_j, \tag{8}$$

where the sum is over all line sessions A_k whose vertices are not all in G_1 and not all in G_2. The inequality (8) is the same as the bidirected cut set bound in [7] corresponding to the cut set including edges i and j, and thus the bound applies to network coding as well as routing.

Next consider the case where exactly three edges i, j, and κ (with $i < j < \kappa$) each have edge distance equal to one. Define the three sets of vertices $G_1 = \{i+1, \cdots, j\}, G_2 = \{j+1, \cdots, \kappa\}, G_3 = \{\kappa+1, \cdots, n, 1, \cdots, i\}$. It is straightforward to verify that the routing bound corresponding to this case is as follows.

$$\sum_{\substack{a,b,c \in \{1,2,3\}, \\ a \neq b \neq c}} \sum_{\substack{A_k \cap G_a \neq \emptyset, \\ A_k \cap G_b \neq \emptyset, \\ A_k \cap G_c = \emptyset}} R_{A_k} + 2 \sum_{\substack{a,b,c \in \{1,2,3\}, \\ a \neq b \neq c}} \sum_{\substack{A_k \cap G_a \neq \emptyset, \\ A_k \cap G_b \neq \emptyset, \\ A_k \cap G_c \neq \emptyset}} R_{A_k}$$
$$\leq C_i + C_j + C_\kappa. \tag{9}$$

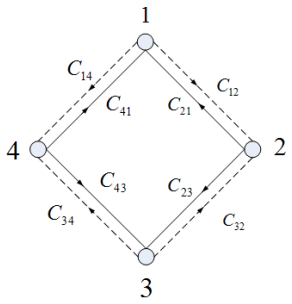


Figure 1. A four vertex ring network.

To show that (9) holds for network coding, we use the bidirected cut set bounds from [7] . Decompose each of the undirected capacities C_i, C_j, C_κ into two unidirectional capacities: $C_i = C_{13} + C_{31}$, $C_j = C_{12} + C_{21}$, and $C_\kappa = C_{23} + C_{32}$, so that C_{pq} denotes the portion of edge capacity which is directed from G_p to G_q. Then (9) can be obtained by summing the three bidirected cut set bounds derived from the pairs of directed edges, $((i, i+1), (j+1, j))$, $((j, j+1), (\kappa+1, \kappa))$, and $((i+1, i), (\kappa, \kappa+1))$.

For the case where there are four or more edges with edge distance 1 in the binary edge distance set $\mathcal{B}$, it turns out that the bidirected cut set bounds can not provide us with tight enough bounds in the case when all sessions are lines. In this case we use the progressive d-separating (PdE) edge set bounds introduced in [8] and [9]), along with an extension of this construction which was first used in [18] to obtain a tighter set of constraints on the edge capacities.

Consider the four vertex ring depicted in Figure 2, where we have separated each undirected edge capacity into two unidirectional capacities. For this ring there are potentially 24 different line sessions. For example, the ones which have their source at vertex 1 are $1 \rightarrow \{2\}$, $1 \rightarrow \{2,3\}$, $1 \rightarrow \{2,3,4\}$, $1 \rightarrow \{4\}$, $1 \rightarrow \{4,3\}$, and $1 \rightarrow \{2,4\}$. The other sessions have identical structures, but with different source vertices; they can be obtained by cyclic permutation of the vertices. We define the following notation in this setting.

- Let A_k denote an arbitrary line session.
- Let W_{A_k} denote the message of the session A_k.
- When j is a destination for session A_k, let $W^j_{A_k}$ denote the estimate at vertex j of the message W_{A_k}.
- Let $\mathcal{W}^j = \{W^j_{A_k} | j \text{ is a destination of } A_k\}$.
- Let $\mathcal{W}_i = \{W_{A_k} | i \text{ is the source of } A_k\}$.
- Let $\mathcal{R}_i = \sum_{\{A_k | i \text{ is the source of } A_k\}} R_{A_k}$.

- Let (i,j) be the directed edge from vertex i to vertex j after all edges have been split in two oppositely-directed edges between vertex i and vertex j.
- For each directed edge (i,j), let X_{ij} be the bit sequence transmitted from vertex i to vertex j on (i,j).
- Let C_{ij} be the entropy of X_{ij}, also denoted by $H(X_{ij})$.

For the PdE bounds we specify a set ε_d of edges to "cut". We choose $\varepsilon_d = \{(1,2),(3,2),(1,4),(3,4)\}$. Further, consider the set of source indices $S_d = \{1,3\}$ and an ordering of these indices via a one to one mapping $\pi(\cdot)$, from $\{1,2\}$ to $S_d = \{1,3\}$. It is possible to show that we can obtain the following PdE bound through a straightforward application of the verification steps in [9].

$$\mathcal{R}_1 + \mathcal{R}_3 \leq C_{12} + C_{14} + C_{32} + C_{34} \tag{10}$$

The tighter bound we will use for the rest of our argument is not (10) itself, but a conditional version of this bound which is also true as a consequence of passing the verification steps. (See the Appendix in [9].)

$$\mathcal{R}_1 + \mathcal{R}_3 \leq H(X_{12}X_{14}X_{32}X_{34}|\mathcal{W}_2\mathcal{W}_4) \tag{11}$$

Since different messages corresponding to line sessions in the network are independent of each other, we have:

$$\begin{aligned}
&H(X_{12}X_{14}X_{32}X_{34}|\mathcal{W}_2\mathcal{W}_4)\\
=&H(X_{12}X_{14}X_{32}X_{34}\mathcal{W}_2\mathcal{W}_4) - H(\mathcal{W}_2\mathcal{W}_4)\\
=&H(X_{12}X_{32}\mathcal{W}_2) + H(X_{14}X_{34}\mathcal{W}_4)\\
&-I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4) - H(\mathcal{W}_2\mathcal{W}_4)\\
\leq&H(X_{12}) + H(X_{14}) + H(X_{32}) + H(X_{34})\\
&-I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4).
\end{aligned} \tag{12}$$

By combining (11) and (12) we find that

$$\begin{aligned}
\mathcal{R}_1 + \mathcal{R}_3 &+ I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4)\\
&\leq C_{12}+C_{14}+C_{32}+C_{34}.
\end{aligned} \tag{13}$$

The bound (13) improves (10) by subtracting a "common" information term in (12). By the Data Processing Inequality [4, §2.8] and the fact that the estimated messages at vertices 2 and 4 are functions of their received messages and their incoming bit sequences, we have

$$\begin{aligned}
&I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4)\\
=&I(X_{12}X_{32}\mathcal{W}_2\mathcal{W}^2; X_{14}X_{34}\mathcal{W}_4\mathcal{W}^4)\\
\geq&I(\mathcal{W}_2\mathcal{W}^2; \mathcal{W}_4\mathcal{W}^4).
\end{aligned} \tag{14}$$

Now we consider an upper bound for the term $I(\mathcal{W}_2\mathcal{W}^2; \mathcal{W}_4\mathcal{W}^4)$ in (14). We define the following sets of messages. Let Y_2 be the binary sequence representing the messages with their source at vertex 2 and a destination at vertex 4. Let Y_4 be the binary sequence representing the messages with their source at vertex 4 and a destination at vertex 2. Finally, let Y be the binary sequence representing the messages for which vertices $2,4$ are both destinations. Further we define Y_2^4, Y_4^2, Y^2, Y^4 to be the binary estimates of these messages at the destinations indicated by their superscripts. From the inequality $I(AB;CD) \geq I(A;C)$, we have

$$\begin{aligned}
&I(\mathcal{W}_2\mathcal{W}^2; \mathcal{W}_4\mathcal{W}^4)\\
\geq&I(Y_2Y_4^2Y^2; Y_4Y_2^4Y^4)\\
=&\underbrace{H(Y_2Y_4^2Y^2)}_{E_1} - \underbrace{H(Y_4Y_2^4Y^4|Y_2Y_4^2Y^2)}_{E_2}.
\end{aligned} \tag{15}$$

Now define e_i^j to be the bit sequence of error between Y_i and Y_i^j, i.e., the bitwise binary XOR of Y_i and Y_i^j denoted by $Y_i \oplus Y_i^j$. Similarly, let e^j be the bit sequence of error between Y and Y^j. Then we have

$$\begin{aligned}
E_1 + H(e^2e_4^2) &= H(Y_2(Y_4 \oplus e_4^2)(Y \oplus e^2)) + H(e^2e_4^2)\\
&\geq H(Y_2(Y_4 \oplus e_4^2)(Y \oplus e^2)|e^2e_4^2) + H(e^2e_4^2)\\
&= H(Y_2Y_4Ye^2e_4^2)\\
&\geq H(Y_2Y_4Y),
\end{aligned}$$

where the first inequality follows because conditioning cannot increase entropy. We thus have

$$E_1 \geq H(Y_2Y_4Y) - H(e^2e_4^2). \tag{16}$$

We can put a lower bound on E_2 as follows:

$$\begin{aligned}
E_2 &= H(Y_4Y_2^4Y^4|Y_2Y_4^2Y^2)\\
&\leq H(Y_4Y_2^4Y^4e_2^4e_4^2e^2e^4|Y_2Y_4^2Y^2)\\
&= H(e_2^4e_4^2e^2e^4|Y_2Y_4^2Y^2)\\
&\leq H(e_2^4e_4^2e^2e^4).
\end{aligned} \tag{17}$$

The second equality in (17) follows since $Y_4 = Y_4^2 \oplus e_4^2$, $Y_2^4 = Y_2 \oplus e_2^4$, and $Y^4 = (Y^2 \oplus e^2) \oplus e^4$. From (14)-(17) we have

$$\begin{aligned}
&I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4)\\
\geq&E_1 - E_2\\
\geq&H(Y_2Y_4Y) - H(e^2e_4^2) - H(e_2^4e_4^2e^2e^4),
\end{aligned} \tag{18}$$

but for reliable transmission we can make the probability of error occurrence and consequently the error entropy arbitrarily small. Moreover we can replace $H(Y_2Y_4Y)$ with the sum of rates of respective messages, which gives us the

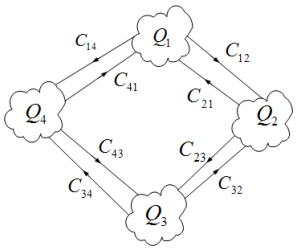


Figure 2. A general ring with four sets of vertices.

following.

$$\begin{aligned} & I(X_{12}X_{32}\mathcal{W}_2; X_{14}X_{34}\mathcal{W}_4) \\ \geq & H(Y_2Y_4Y) \\ = & \sum_{i\in\{1,3\}} R_{i\to\{2,4\}} + \sum_{\substack{i,j\in\{2,4\}\\ k\in\{1,3\}}} R_{i\to\{j,k\}} \\ + & \sum_{\substack{i\in\{1,2,3,4\}\\ \{j,k,\ell\}=\{1,2,3,4\}-i}} R_{i\to\{j,k,\ell\}}, \end{aligned} \tag{19}$$

where the notation $R_{i\to\mathcal{S}}$ denotes the rate of the line session with source vertex i and set $\mathcal{S}$ of destination vertices. Then, by combining (13) and (19) we will get the following bound:

$$\begin{aligned} & \sum_{i\in\{1,3\}} R_{i\to\{2,4\}} + \sum_{\substack{i,j\in\{2,4\}\\ k\in\{1,3\}}} R_{i\to\{j,k\}} \\ + & \sum_{\substack{i\in\{1,2,3,4\}\\ \{j,k,\ell\}=\{1,2,3,4\}-i}} R_{i\to\{j,k,\ell\}} + \mathcal{R}_1 + \mathcal{R}_3 \\ \leq & C_{12}+C_{14}+C_{32}+C_{34} \end{aligned} \tag{20}$$

For larger ring networks, a similar relationship can be established when vertices 1, 2, 3, and 4 are replaced by four sets of neighboring vertices Q_1, Q_2, Q_3, and Q_4, respectively.

Proposition 4.1 *For the ring in Figure 2 we have the inequality*

$$\sum_{A_j\in U} R_{A_j} + 2\sum_{A_j\in V} R_{A_j} \leq C_{12}+C_{32}+C_{14}+C_{34}, \tag{21}$$

where U includes all sessions which have:

- *the source in Q_1 and at least one destination in Q_2, Q_3, or Q_4, but not destinations in both Q_2 and Q_4,*
- *the source in Q_3 and at least one destination in Q_1, Q_2, or Q_4, but not destinations in both Q_2 and Q_4,*
- *the source in Q_2 and at least one destination in Q_4, or*
- *the source in Q_4 and at least one destination in Q_2,*

and V includes all sessions which have:

- *the source in Q_1 and at least one destination in each of Q_2 and Q_4, or*
- *the source in Q_3 and at least one destination in each of Q_2 and Q_4,*

and C_{ij} denotes the entropy of the bit sequence on the edge with its respective vertices in Q_i and Q_j, directed from Q_i to Q_j.

Proof: Consider a network in which the number of vertices and the capacities C_{12}, C_{21}, C_{14}, C_{41}, C_{32}, C_{23}, C_{34}, and C_{43} are the same as in our original network in Figure 2, but all other edge capacities are infinite. Assume that we ask for the same traffic demands in the new network as the original ring. Obviously, a network coding solution which achieves the demands of the original ring is also a solution for this new ring. On the other hand, within any of the four groups of vertices Q_1, Q_2, Q_3, or Q_4, all capacities are infinite, so each group can be treated as a single vertex. Hence, our previous bound (20) for the four vertex ring still holds for this modified ring with four group of vertices.

□

Now we will use inequality (21) as the basic tool to show that any network code must satisfy the routing bound corresponding to an arbitrary binary set of edge distances, $\mathcal{B}$, with at least four elements of distance 1. As the first step, collect each set of vertices which is connected by a path of length zero into a single supervertex. Note that a supervertex may consist only of a single vertex. Label the supervertices in clockwise order from 1 to F, where F is the number of edges with distance one. Now we call an edge on the ring that connects two supervertices to each other a superedge, and it is obvious that the length of each superedge is one and each edge with distance one is a superedge. For a line session A_k, we call a supervertex s a supersource if s includes the source vertex of session A_k, and similarly we call a supervertex d a superdestination for A_k if d includes at least one destination vertex of A_k, but not the source for A_k. At this point it is obvious that any line session on the ring is still a line session with respect to the ring of supervertices and superedges with supersource and superdestinations. Consider inequality (21) and let $E(i,j)$ denote the inequality derived by setting Q_2 and Q_4 to be two supervertices on the ring with indices i and j, with $i<j$, respectively. According to (21), the coefficient of an arbitrary line session rate R_{A_k} on the left hand side of $E(i,j)$, depending on the respective sets U and V, will be as follows.

1. If neither i nor j is the supersource or a superdestination for A_k, then we have R_{A_k} with coefficient 0. (As the session is a line session, we should have all source and destination supervertices on one side of the line which connects i, j on the ring, which means that we have neither a subset of U nor one of V.)

2. If one of i or j is the supersource and the other is not a superdestination for A_k, then we have R_{A_k} with coefficient 0.

3. If one of i or j is the supersource and the other is a superdestination for A_k, then we have R_{A_k} with coefficient 1.

4. If one of i or j is a superdestination and the other is neither the supersource nor a superdestination for A_k, then we have R_{A_k} with coefficient 1.

5. If both i and j are superdestinations for A_k, then we have R_{A_k} with coefficient 2.

One can summarize these five cases with the following criterion. On the left hand side of $E(i, j)$, the coefficient of R_{A_k} is the number of superdestinations among the two supervertices i and j.

We are ready to derive the final set of inequalities using inequalities of the form $E(i, j)$. First consider the case with an even number of supervertices, so that $F = 2m$ for some positive integer m. The routing bound for this case can be written in the following form:

$$\sum_{A_k \text{is a line session}} \ell_{A_k} R_{A_k} \leq C_{12} + C_{21} + C_{23} + C_{32} + \cdots + C_{F1} + C_{1F}, \quad (22)$$

where ℓ_{A_k} is the minimum routing tree length for session A_k. With respect to the edge distance set $\mathcal{B}$, we can see that ℓ_{A_k} is equal to the number of superdestinations for A_k. As before, $C_{ij} = H(X_{ij})$. We can group the capacities on the right hand side of (22) into m groups: $(C_{F1} + C_{21} + C_{(m)(m+1)} + C_{(m+2)(m+1)}) + (C_{12} + C_{32} + C_{(m+1)(m+2)} + C_{(m+3)(m+2)}) + \cdots + (C_{(m-1)(m)} + C_{(m+1)(m)} + C_{(F-1)(F)} + C_{(F+1)(F)})$. The total capacity of each of these groups $(C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(i+m-1)(i+m)} + C_{(i+m+1)(i+m)})$ is equal to the capacity which appears on the right hand side of $E(i, i + m)$. So by summing the inequalities $E(1, m + 1), E(2, m + 2), \cdots E(m, F)$, we will have on the left hand side of the resulting inequality the sum of all capacities on the righthand side of (22). Further, each of the supervertices i appears exactly once in some inequality $E(i, i + m)$. This means on the left hand side of the resulting inequality we will have the coefficient of R_{A_k} as given by our criterion, the number of superdestinations for A_k, or equivalently, ℓ_{A_k}. Therefore the resulting inequality is the same as (22), and thus routing bound is also a network coding bound in this case.

Next, consider the case with an odd number of supervertices, so that $F = 2m + 1$ for some integer m. In this case the routing bound can be written as follows, when both sides are multiplied by a factor of 2.

$$\sum_{A_k \text{is a line session}} 2\ell_{A_k} R_{A_k} \leq 2(C_{12} + C_{21} + C_{23} + C_{32} + \cdots + C_{F1} + C_{1F}) \quad (23)$$

Again we have ℓ_{A_k} equal to the number of superdestinations for A_k. We group the capacities on the right hand side of (23) into $2m + 1$ groups: $m + 1$ of the form $(C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(i+m-1)(i+m)} + C_{(i+m+1)(i+m)})$ and m of the form $(C_{(i-1)(i)} + C_{(i+1)(i)} + C_{(i+m)(i+m+1)} + C_{(i+m+2)(i+m+1)})$. The capacity of each of these groups appears on the right hand side of the inequalities $E(i, m+i)$ or $E(i, m + i + 1)$ respectively. Further, each super destination for an arbitrary line session appears in exactly two of these inequalities. Thus by summing these inequalities, we will have the coefficient of R_{A_k} equal to twice the number of superdestinations for A_k or $2\ell_{A_k}$ on the left hand side of the resulting inequality, and on the right hand side, we will have the capacities on right hand side of (23), which shows that routing bound is again a network coding bound in this case. Thus, routing is rate optimal.

5. Acknowledgments

The work of S. M. S. Tabatabaei Yazdi was done at EPFL, Lausanne, Switzerland. The work of S. A. Savari was partially supported by NSF Grant No. CCF-0430201 and K. Carlson (Talaska) was sponsored by the same grant. The work of G. Kramer was partially supported by the Board of Trustees of the University of Illinois Subaward No. 04-217 under NSF Grant No. CCR-0325673.

References

[1] R. Ahlswede, N. Cai, S.-Y .R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1204–1216, July 2000.

[2] U. Black, *Optical Networks: Third Generation Transport Systems*, New Jersey: Prentice Hall PTR, 2002.

[3] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey.* SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial & Applied Mathematics, 1999.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, New York: Wiley, 1991.

[5] R. C. Elsenpeter and T. J. Velte, *Optical Networking: A Beginner's Guide*, Berkeley: McGraw-Hill/Osborne, 2002.

[6] M. Iri, "On an extension of the maximum-flow minimum-cut theorem to multicommodity flows," *J. Operations Research Soc. of Japan*, vol. 13, no. 3, pp. 129–135, Jan. 1970.

[7] G. Kramer and S. A. Savari, "On networks of two-way channels," in A. Ashikhmin, A. Barg, eds., *DIMACS Workshop on Algebraic Coding Theory and Information Theory, (December 15-18, 2003, Rutgers University, Piscataway, NJ)*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 68, pp. 133-143, American Mathematical Society, Rhode Island, 2005.

[8] G. Kramer and S. A. Savari, "Progressive d-separating edge set bounds on network coding rates," *Proc. 2005 IEEE Int. Symp. Inform. Theory*, (Adelaide, Australia), pp. 1588-1592, Sept. 4-9 2005.

[9] G. Kramer and S. A. Savari, "Edge-cut bounds on network coding rates," *J. Network and Systems Management*, vol. 14, no. 1, pp. 49-67, March 2006.

[10] Z. Li and B. Li, "Network coding: the case of multiple unicast sessions," *Proc. 42nd Annual Allerton Conf. on Communication, Control and Computing*, (Monticello, IL), September, 2004.

[11] H. Okamura and P. D. Seymour, "Multicommodity flows in planar graphs," *J. Combin. Theory*, Series B 31, pp. 75–81, 1981.

[12] K. Onaga and O. Kakusho, "On feasibility conditions of multicommodity flows in networks," *IEEE Trans. Circuit Theory*, vol. CT-18, no. 4, pp. 425-429, July 1971.

[13] S. A. Savari and G. Kramer, "A Study of Ring Networks," Invited presentation, *2006 IEEE Communication Theory Workshop,* Dorado, Puerto Rico, May 24, 2006.

[14] S. A. Savari and G. Kramer, "The Multimessage Unicast Capacity Region for Bidirectional Ring Networks," *Proc. 2006 IEEE Int. Symp. Inform. Theory*, Seattle, WA, July 2006.

[15] A. Schrijver, P. Seymour, and P. Winkler, "The ring loading problem," *SIAM Review*, vol. 41, no. 4, pp. 777-791, 1999.

[16] R. Vachani, A. Shulman, P. Kubat, and J. Ward, "Multicommodity flows in ring networks," *INFORMS J. Computing*, vol. 8, no. 3, pp. 235-242, Summer 1996.

[17] Gunter M. Ziegler, *Lectures on Polytopes*, New York: Springer-Verlag, 1995.

[18] S. M. S. Tabatabaei Yazdi, S. A. Savari, F. Hassanzadeh, and G. Kramer, "A Multimessage Capacity Region for Undirected Ring Networks," To appear in *Proceedings of the 2007 IEEE International Symposium on Information Theory*.