Bounds on the Covering Radius of Linear Codes

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Abstract

Asymptotically bounding the covering radius in terms of the dual distance is a well-studied problem. We will combine the polynomial approach with estimates of the distance distribution of codes to derive new results for linear codes.

1 Introduction

Let $F = \mathbb{F}_2^n$ and let $C \subset F$ be a binary linear code of length n. The covering radius of C is defined as

$$r(C) = \max_{x \in F} \min_{y \in C+x} \mathrm{wt}(x),$$

where $\mathrm{wt}(\cdot)$ denotes the Hamming weight. Bounding the covering radius of codes is one of the main extremal problems of coding theory. Let $C' = \{x \in F : (x, c) = 0 \ \forall c \in C\}$ be the dual code of C and $(A'_i, 0 \le i \le n)$ be its weight distribution, where $A'_i = |\{x \in C' : \mathrm{wt}(x) = i\}|$. The minimal $i \ge 1$ such that $A'_i > 0$ is called the dual distance of C, denoted d'.

The most developed direction is deriving bounds on r in terms of the strength of C as a design in F or, in other words, in terms of the distance d'. This problem received considerable attention through the last decade, see [2], [3, Ch. 12], [4], [6]–[8], [11]–[19], and this is the problem studied in the present paper. Let $r(d') = \max r(C)$ where the maximum is taken over all codes of dual distance at most d'. We will be interested in asymptotic upper bounds on $\rho = r/n$ valid for any sequence of codes C_n of growing length and dual distance $d \le \delta' n$. A review of the methods used for obtaining such bounds is given in [2], [7], see also [11]. A large part of ideas concentrates around an application of Delsarte's polynomial method put forward by Tietäväinen in [19]. The result in that paper has the following form:

$$\rho(\delta') \le \varphi(\delta'/2), \tag{1}$$

where $\varphi(x) = \frac{1}{2} - \sqrt{x(1-x)}$. Ultimately the best upper bounds on $\rho(\delta')$ known are obtained based on the following theorem. By $K_i(x)$ we denote the Krawtchouk polynomial of degree i.

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Theorem 1 [3, p. 230] *Let s be an integer, and let $f(x) = \sum_{i=0}^{n} f_i K_i(x)$ be a polynomial such that $f(i) \le 0, i = s+1, \cdots, n$, and*

$$f_0 > \sum_{j=d'}^{n} |f_j| A'_j \tag{2}$$

Then the covering radius of a linear code C with dual distance d' is at most s.

So far applications of this theorem were based on bounding A'_j above by the maximal size $A(n, d', j)$ of a code of length n, distance d' and constant weight j. This enables one to derive an upper bound of the form $\sum_{j=d'}^{n} |f_j| A'_j \le F$; then to establish (2) for a given polynomial $f(x)$ one only needs to verify that $F < f_0$. Asymptotically this approach amounts to using bounds on the rate $R(\delta', \xi)$ of constant-weight codes with relative distance δ' and relative weight $\xi = j/n$. In particular [2] relies upon the "JPL" bound [13], and [7] uses an improvement of this bound from [9], [14] to improve the bound on $\rho(\delta')$ for $0.04 \le \delta' \le 0.20$. Papers [2], [7] use the polynomial

$$W_t(x) := \frac{(K_{t+1}(x) + K_t(x))^2}{a - x}, \tag{3}$$

where a is the smallest root of the numerator and t an appropriately chosen parameter. This polynomial was first suggested in [13] for bounding the size of codes. Jointly papers [2] and [7] together with [6] contain the best bounds known to-date. The first two bounds are cited in Theorem 4 below and the remarks following its proof; the third one is too cumbersome to reproduce here (see also Theorem 2 in [2]).

In this paper we suggest to replace estimates of $R(\delta', \xi)$ with other bounds on the weight distribution of codes. One option is, for a given choice of the polynomial $f(x)$, to bound the sum $\sum_{j=d'}^{n} |f_j| A'_j$ as a whole. A method for deriving universal bounds of this type was suggested in a recent work [1]. In the same paper we also obtained estimates of individual coefficients A_i which can be used in Theorem 1. Both approaches enable us to improve the cited results. We establish the following new bounds.

Theorem 2 *For $0 \le \delta' \le 1/2$*

$$\rho(\delta') \le \varphi(H^{-1}(1 - H(\varphi(\delta')))) \tag{4}$$

$$\rho(\delta') \le 2\varphi(\delta'), \tag{5}$$

where $H(x) = -x \log_2(x) - (1-x)\log_2(1-x)$.

(Logarithms are base 2 throughout). These results improve the bound of [6] for all δ' and the bound of [2] in the interval $\delta' \in [\delta'_1, 1/2)$, where $\delta'_1 \approx 0.27$. For $\delta' \le 0.267...$, (4) is better than (5). For $0 < \delta' \le \delta'_1$ the results of [2],[7] remain the best known. The improvement of the previous results given by Theorem 2 is rather substantial, as shown in Table 1.

Apart from proving Theorem 2 we also present a new proof of the results of [2], [7] which enables us to formulate them in a more explicit manner than in the original works.

Table 1: Bounds on $\rho(\delta')$

δ'	0.28	0.32	0.36	0.4	0.44	0.48
(5)	0.102	0.067	0.04	0.020	0.0072	0.00080
[2]	0.105	0.0769	0.051	0.029	0.0125	0.00178
[6]	0.149	0.096	0.057	0.029	0.0103	0.0011

2 Bounds on the weight distribution

In this section we collect the results from [2] that we need. We also prove an extended version of one of the estimates on the weight distribution from [2].

A. KRAWTCHOUK POLYNOMIALS (for the proofs see, for instance, [3]). Let $\alpha(i) = 2^{-n}\binom{n}{i}$. We have $\int K_t K_s d\alpha = \binom{n}{t}\delta_{t,s}$ and thus $\|K_t\|^2 = \binom{n}{t}$. This implies that

$$(K_t(i))^2 \le \binom{n}{t} 2^n / \binom{n}{i}.$$

The Krawtchouk coefficients of any polynomial $f(x) = \sum f_j K_j(x)$ can be computed from $\|K_j\|^2 f_j = \int f K_j d\alpha$. In particular,

$$K_a(x)K_b(x) = \sum_{c=0}^{n} p_{a,b}^c K_c(x), \tag{6}$$

where

$$p_{a,b}^c = \binom{c}{(b-a+c)/2}\binom{n-c}{(b+a-c)/2}\mathbf{1}_{\{a+b-c\in 2\mathbb{Z}\}} \tag{7}$$

Note that $p_{a,b}^c = 0$ for $a+b<c$ and $p_{a,b}^0 = \delta_{a,b}\binom{n}{a}$.

With $t=\tau n$ and $i=\mu n$ we see that $n^{-1}\log_2|K_t(i)| \le E_1(\tau,\mu)+o(1)$, where

$$E_1(u,v) = \frac{1}{2}(H(u)-H(v)+1).$$

The zeros of $K_t(x)$ are located inside the segment $[n\varphi(\tau), n(1-\varphi(\tau))]$ and for the minimum zero we have

$$n\varphi(t/n) \le x_t \le n\varphi(t/n) + t^{1/6}\sqrt{n-t}. \tag{8}$$

Note that the function $\varphi(x)$ is monotone decreasing for $0\le x\le 1/2$. It is also an involution and so $\varphi^2 = \mathrm{id}$. Let

$$I(\tau,\mu) = \int_0^\mu \log\frac{s+\sqrt{s^2-4y(1-y)}}{2-2y}dy$$

where $s = 1 - 2\tau$. It is known [5] that $n^{-1}\log_2 K_{\tau n}(\mu n) = E_2(\tau,\mu) + o(1)$, where

$$E_2(\tau,\mu) := (H(\tau) + I(\tau,\mu)) \quad (0 \le \mu \le \varphi(\tau)). \tag{9}$$

An upper bound on the exponent of $K_t(i)$ which has a simpler form and is not as crude as E_1 was derived in [10]. It has the form $n^{-1}\log_2 K_{\tau n}(\mu n) \le E_3(\tau,\mu) + o(1)$, where

$$E_3(\tau,\mu) := \frac{1}{2}\Big[\mu \log \frac{\varphi(\tau)}{1-\varphi(\tau)} + \log(1-\varphi(\tau)) + H(\tau) + 1\Big] \quad (0 \le \mu \le \varphi(\tau)). \tag{10}$$

We note that $E_3(\tau,\mu) \ge E_2(\tau,\mu) \ge E_1(\tau,\mu)$ for $0 \le \mu \le \varphi(\tau)$ with equality if and only if $\mu = \varphi(\tau)$. Moreover, for this μ also

$$(E_1(\tau,\mu))'_\mu = (E_2(\tau,\mu))'_\mu = (E_3(\tau,\mu))'_\mu.$$

B. Weight distribution. Let $a_\xi(C) = (\log_2 A_{\xi n})/n$ be the exponent of the weight coefficient of a linear code C of length n. We denote

$$a_\xi(\delta) = \max_{C:\, d(C)=\delta n} a_\xi(C).$$

Theorem 3 *For any sequence of codes of relative distance δ*

$$a_\xi \lesssim \begin{cases} H(\xi) + H(\varphi(\delta)) - 1 & \delta \le \xi \le 1-\delta \\ -2I(\varphi(\delta),\xi) & 1-\delta \le \xi \le 1. \end{cases} \tag{11}$$

(See Fig. 1).

Proof (outline).

(a) The starting point is the following result of [1]. Let $g(x)$ be a function and $Z(x)$ a polynomial over $\mathbb{R}$. Suppose that $Z(x) = \sum_i z_i K_i(x)$ and that $z_i \le 0, 0 \le i \le n$. If $Z(i) \ge g(i), d \le i \le n$ then

$$\sum_{i=d}^{n} g(i) A_i \le z_0 |C| - Z(0). \tag{12}$$

(b) Take $\tau = \varphi(\delta)$ and $t = \tau n$. As in [1], we take

$$g(i) = (K_w(i))^2,$$

$$Z(i) = (K_w(i))^2 - \frac{t+1}{2}\frac{\binom{n}{w}}{\binom{n}{t}} W_t(i),$$

for a suitably chosen w. It is proved in [1], Prop.2 that for large n and $\frac{w}{n} = \omega < \frac{t}{n} = \tau$, this choice satsifies the conditions of (a). Computing the right-hand side of (12) we get

$$\sum_{i=d}^{n} (K_w(i))^2 A_i \le \frac{n^2(t+1)}{2at^2}\binom{n}{w}\binom{n}{t} - \binom{n}{w}^2. \tag{13}$$

(c) Let $\delta \le \xi \le 1/2$ and take $\omega = \varphi(\xi)$. Then we have $\omega < \tau = \varphi(\delta)$. Taking logarithms in (13) and bounding the exponent of $K_w(i)$ by $E_1(\omega, \xi)$ we obtain the first inequality in (11).

(d) Let $1/2 \le \xi \le 1 - \delta$. Then we take the same polynomials $g(i)$ and $Z(i)$ with $\tau = \varphi(\delta)$ but choose $\omega = 1 - \varphi(\xi)$. The argument of (c) is then repeated which is possible since $|K_k(x)| = |K_k(n-x)|$.

(e) Now let $1 - \delta \le \xi \le 1$. Since we want to ensure that $\omega < \tau$, we replace the above choise of ω by a number arbitrariliy close to τ. Then a better estimate of $K_w(i)$ is given in (9), and we get the second inequality of the statement.

□

Note that (13) implies the following estimate of the left-hand side: for any sequence of codes with distance d

$$n^{-1} \log \sum_{i=d}^{n} (K_w(i))^2 A_i \lesssim H(\omega) + H(\tau) \quad (\omega \le \tau). \tag{14}$$

One more result from [1] that we use is as follows:

$$n^{-1} \log \sum_{i=d}^{n} p_{t,t}^i A_i \le 2H(\tau) - H(\delta/2) \quad (\delta/2 \le \tau). \tag{15}$$

3 Bounds on the covering radius

We will use the estimates of the previous section for the weight distribution $(A'_i, d' \le i \le n)$ of the code C'. By abuse of notation we let $a_\xi = n^{-1} \log A'_i$ and denote by $a_\xi(\delta')$ a generic upper bound on a_ξ that holds for any family of codes with relative distance δ'. The following theorem gives results of [2], [7] in an analytic form, replacing a numerical procedure employed there.

Theorem 4 *Let τ be the minimal number such that*

$$\max_{\delta' \le \xi \le 2\tau} \left\{ (1-\xi) H\Big(\frac{\tau - \xi/2}{1-\xi}\Big) - H(\tau) + \xi + a_\xi(\delta') \right\} < 0, \tag{16}$$

Then $\rho \le \varphi(\tau)$.

This theorem will follow immediately if we establish the asymptotic behavior of the Krawtchouk coefficients of $f(x)$ from (3).

Lemma 5 *Let $f(x)$ be the polynomial (3) and let $f(x) = \sum_{j=0}^{n} f_j K_j(x)$ be its Krawtchouk expansion. Then*

$$\log f_j \sim \log p_{t,t}^j. \tag{17}$$

Suppose that $j = \xi n$ and $t = \tau n$. Then

$$n^{-1} \log_2 f_j = (1-\xi) H\Big(\frac{\tau - \xi/2}{1-\xi}\Big) + \xi + o(1)$$

PROOF. The expression for f_j has the form [13]:

$$f_j = \frac{2}{(t+1)K_t(a)}\binom{n}{t}\sum_{i=0}^{t}\frac{K_i(a)}{\binom{n}{i}}(p_{t,i}^j + p_{t+1,i}^j), \tag{18}$$

and $f_j \geq 0$ for all $0 \leq j \leq n$. Note that for any i if $p_{t,i}^j \neq 0$ then $p_{t,i+1}^j = 0$ and vice versa. Estimating (18) below by the last term, we get

$$f_j \geq \frac{2}{t+1}p_{t,t}^j(1+O(n))$$

which implies the expression of the Lemma as a lower bound. Let us prove that this is also an upper bound. Since[1]

$$\binom{n}{a}K_t(a) = \binom{n}{t}K_a(t)$$

we can rewrite (18) as

$$f_j = \frac{2}{(t+1)K_t(a)}\frac{\binom{n}{t}}{\binom{n}{a}}\sum_{i=0}^{t}K_a(i)(p_{t,i}^j + p_{t+1,i}^j). \tag{19}$$

Note that the main term of the minimal zero x_a of $K_a(x)$ behaves as $n\varphi(a)$, and by definition $a = n\varphi(t) + o(n)$. Hence $x_a \sim t$, and so the sum on i ranges over the segment $0 < i < x_a$. Now let us estimate $K_a(i)$ from above by $\exp(nE_3(\varphi(\tau), \xi))$ (see 10) and take logarithms. Putting $i = \mu n$ and recalling that φ is an involution we can bound the exponent of the summation term in (19) as follows:

$$\frac{1}{2}\Big[x\log\frac{\tau}{1-\tau} + \log(1-\tau) + H(\varphi(\tau)) + 1\Big] + \xi H\big(\frac{\tau-\mu+\xi}{2\xi}\big)$$
$$+(1-\xi)H\big(\frac{\tau+\mu-\xi}{2(1-\xi)}\big). \tag{20}$$

The derivative of this expression on μ has the form

$$M(\mu) = \log\frac{(\xi+\tau-\mu)(2-\xi-\tau-\mu)\tau}{(\xi-\tau+\mu)(\tau+\mu-\xi)(1-\tau)}.$$

Note that $0 \leq \mu \leq \tau \leq 1/2$. We easily check that $M(\tau) \geq 0$ for any $0 \leq \xi \leq 2\tau$. Next we prove that $M(\mu)$ has no zeros for $\tau - \xi \leq \mu \leq \tau$. If it does then the values of μ that satisfy the expression

$$\frac{(\xi+\tau-\mu)(2-\xi-\tau-\mu)\tau}{(\xi-\tau+\mu)(\tau+\mu-\xi)(1-\tau)} = 1,$$

[1] Strictly speaking, for this to be true, a must be an integer. However since our aim is asymptotic results, the effect of rounding is insignificant

which are

$$\mu_{1,2} = \frac{\pm\sqrt{\xi^2 - 4(1-\tau)\tau(\xi^2 - \tau(1-\tau))} - \tau}{1-2\tau}$$

fall in the segment $\tau - \xi \leq \mu \leq \tau$. However, $\mu_2 < 0$ and it is checked directly that $\mu_1 > \tau$. Hence the term in (20) attains its maximum on μ for $\mu = \tau$ and we finally obtain

$$n^{-1} \log f_j \lesssim E_3(\varphi(\tau), \tau) + n^{-1} \log p^j_{t,t}.$$

Finally since $E_3(\tau, \varphi(\tau)) = E_1(\tau, \varphi(\tau))$, we can substitute E_1 together with the logarithm of $p^j_{t,t}$. This implies that the expression in the statement of the lemma is also an upper bound on the exponent of f_j and completes the proof. □

PROOF of Theorem 4. For $j = \xi n$ we bound above the exponent of A'_j by $a_\xi(\delta')$. Asymptotically the sum $\sum_j f_j A'_j$ is dominated by the largest term, j_0 say, such that its exponent attains maximum on $\xi = j/n$ for $\delta' \leq \xi \leq 2\tau$. It is also immediate from Lemma 5 that

$$\log f_0 = n(H(\tau) + 2E_1(\tau, \varphi(\tau)) + o(n).$$

Comuting $\log f_0 - \log(f_{j_0} A_{j_0})$, we obtain the expression under the maximum in (16). Together with Theorem 1 this establishes the claim. □

Obtaining bounds on $\rho(\delta')$ with this theorem is a matter of choosing a suitable bound $a_\xi(\delta')$. An obvious idea is to use upper bounds on constant weight codes:

$$a_\xi(\delta') \leq R(\delta', \xi).$$

Substituting the bound on $R(\delta', \xi)$ from [13] we obtain the result of [2][2]. An improved bound from [14] gives the result of [7].

A better result for large δ' is obtained if we combine Theorem 4 with Theorem 3. The result can be expressed in a closed form. We need to substitute the bound (11) into (16) and optimize on $\xi, \delta' \leq \xi \leq 1$. Depending on whether $\xi \leq 1 - \delta'$ or not the expression whose maximum on ξ is sought is different. First we remark that this maximum is always attained for $\delta' \leq \xi \leq 1 - \delta'$. Indeed, suppose that $1 - \delta' \leq \xi \leq 1$, then substituting the second inequality in (11) we observe that the the part of the expression that depends on ξ equals $(n^{-1} \log p^{\xi n}_{t,t}) - 2I(\varphi(\delta), \xi)$. The first logarithm is the falling function of ξ since its derivative equals

$$\frac{d}{d\xi}((1-\xi)H\big(\frac{\tau - \xi/2}{1-\xi}\big) + \xi) = \frac{1}{2} \log \frac{(2\tau - \xi)(2 - 2\tau - \xi)}{(1-\xi)^2},$$

which has no zeros and is negative for $0 \leq \xi \leq 2\tau$. The function $-2I(\varphi(\delta'), \xi)$ is also directly checked to be falling on ξ. Thus the maximum in this case is attained for $\xi = 1 - \delta'$ but for this ξ

[2] In fact, [2] does a little more: the authors there substitute the bound from [13] and some recurrence relations on the function $R(\delta', \xi)$ to improve the result for small δ'.

the bound is the same as in the first case of (11). It remains to analyze the case $\delta' \le \xi \le 1-\delta'$. Substituting the first upper bound (11) into the expression to be maximized in (16), we obtain

$$(1-\xi)H\big(\frac{\tau-\xi/2}{1-\xi}\big) - H(\tau) + \xi + H(\xi) - H(\varphi(\delta)) + 1.$$

This function has a unique maximum on ξ for $\xi = 2\tau(1-\tau)$. Substituting this value of ξ, we get, upon simplification, the expression

$$H(\tau) - 1 + H(\varphi(\delta)).$$

The minimum τ for which this is negative is thus arbitrarily close to $\tau_0 := H^{-1}(1 - H(\varphi(\delta)))$. Thus $\rho(\delta') \le \varphi(\tau_0)$, which proves the first part of Theorem 2.

To prove the second part of this theorem, let us take in Theorem 1 the polynomial $f(x)$ given by $f(i) = 2^n p_{t,t}^i$, where $t/n = \tau = \varphi(\delta')$. By (6)-(7) we get $f(i) = 0$ $(2t+1 \le i \le n)$ and $f_i = (K_t(i))^2$. Now from (14) we have

$$n^{-1} \log \sum_{i=d}^{n} (K_t(i))^2 A_i \le 2H(\tau)(1 + o(1)).$$

Since $n^{-1} \log f_0 \sim 2H(\tau)$, this choice of $f(x)$ satisfies for large n the conditions of Theorem 1. So $\rho \le 2\tau = 2\varphi(\delta')$, as was to be proved.

Remark. If we take $f(x) = W_t(x)$ then by (6) $\log f_i \sim \log p_{t,t}^i$, and so

$$f_0 \sim \binom{n}{t} = \exp(nH(\tau) + o(n)).$$

Hence together with (15) we see that (2) is satifsied if τ is arbitrarily close to but less than $\delta'/2$. The first zero of $f(x)$ behaves as $n\varphi(\delta'/2)$ and for greater x, $f(x)$ stays nonpositive; hence $\varphi(\delta'/2)$ is an upper bound on ρ. This gives another proof of (1).

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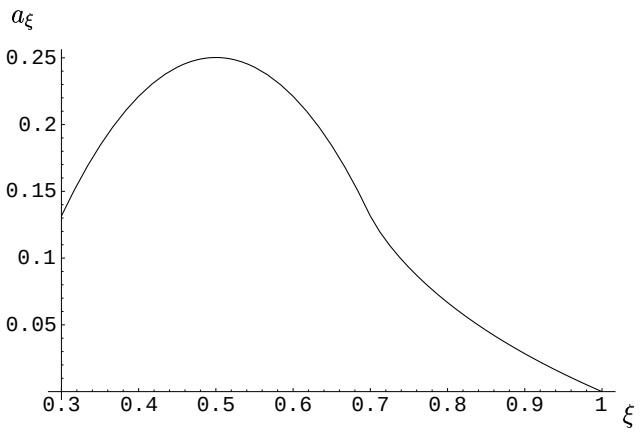


Figure 1: Upper estimate of the exponent a_ξ of the distance spectrum for a family of codes with distance $\delta = 0.3$

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