EXIT Functions of Hadamard Components in Repeat-Zigzag-Hadamard (RZH) Codes

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***Abstract*— We investigate the extrinsic information transfer (EXIT) functions of Hadamard codes in the context of repeat-zigzag Hadamard (RZH) codes with parallel decoding. The derived EXIT functions can serve as an effective tool for designing low-rate IRZH codes with parallel decoding in BIAWGN channels.**

I. INTRODUCTION

For communication systems operating in the low signal-to-noise ratio (SNR) regime (e.g., code-spread communication systems and power-limited sensor networks), low-rate coding schemes play a critical role. Hadamard codes have been shown to be a useful tool for improving the code performance in the low-rate region. Specifically, constructed from Hadamard code arrays, low-rate turbo-Hadamard codes [6] offer a bit-error-rate (BER) of 10^{-5} at $E_b/N_0 = -1.2$ dB, which is only around 0.4 dB away from the Shannon limit. More recently, built on zigzag codes, parallel-concatenated zigzag-Hadamard (PCZH) codes [3] and repeat-zigzag-Hadamard (RZH) codes [5] have been proposed, both of which exhibit much simpler encoder and decoder structures while still offering a similar performance as that of the turbo-Hadamard codes.

In [5], the turbo-like decoding and the LDPC-like parallel decoding are proposed. Although simulations have shown that for regular RZH codes, both decoding schemes work well and offer a similar BER performance, the LDPC-like decoding has certain advantage over the turbo-like decoding as follows: 1) its parallel nature is preferable in terms of hardware implementation; 2) its inner decoder is a Hadamard decoder, a fact that makes it possible to derive analytical expression of the extrinsic information transfer (EXIT) function [10] for the component decoder, which may significantly ease the irregular code design.

To elaborate the second point, we know that for LDPC-like parallel decoding, the connection bits of the ZH code are treated as degree-2 variable nodes, and the inner constituent decoder becomes a Hadamard decoder, having the channel observations of the parity bits and the *a priori* probabilities about the information/connection bits as the input, and the extrinsic information for the information/connection bits as the output. EXIT functions with multiple inputs and outputs are difficult to obtain using Monte Carlo simulations. In this paper, by extending the techniques introduced in [1], [9], we derive the EXIT functions of Hadamard codes with multiple inputs and outputs in the context of low-rate RZH codes over BIAWGN channels. With the EXIT functions, the BERs for a given code profile can be easily estimated and the differential evolution (DE) method [7] can be employed to optimize the degree profiles.

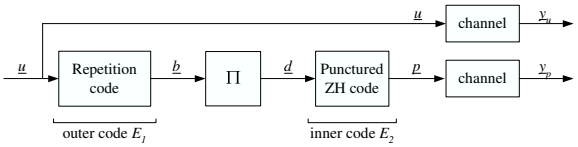


Fig. 1. Systematical RZH encoder.

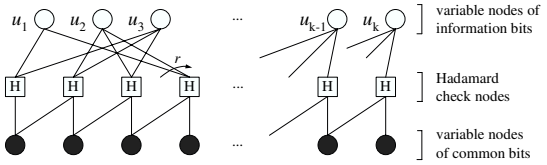


Fig. 2. Tanner graph of an un-punctured (n, k) RZH code.

The remainder of this paper is organized as follows. In Section II, we introduce the RZH codec. In Section III, we derive the EXIT functions over BIAWGN channel. In Section IV, the analytical EXIT functions are employed to optimize the IRZH code ensemble profiles and several design examples are given. Section V contains the conclusions.

II. REPEAT-ZIGZAG HADAMARD CODES

Repeat-ZH Codes and Parallel Decoding Scheme: In [5], we proposed an alternative code structure to parallel concatenated ZH codes and turbo-Hadamard codes, namely, the repeat-zigzag-Hadamard (RZH) codes. The structure of a systematic RZH code is shown in Fig. 1 where the outer code is a repetition code and the inner code is a punctured ZH code.

As shown in Fig. 2, an RZH code can be represented by its Tanner graph, from which a LDPC-like parallel decoding algorithm can be derived where the common bit nodes are treated as degree-2 information bit nodes, and all Hadamard check nodes and all information bit nodes are activated alternately and in parallel. Every time the Hadamard check nodes are activated, the J parallel Hadamard decoders firstly compute the *a posteriori* log-likelihood ratios (LLRs) of the information bits based on the channel observation $\{\underline{y}_j\}_j$ and the *a priori* LLRs passed from the information bit nodes, with which the extrinsic LLRs of the information bits can be obtained by subtracting the *a priori* LLRs from the *a posteriori* LLRs. The resulting extrinsic LLRs are then fed to the information bit

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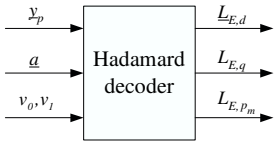


Fig. 3. Inputs/Outputs of the Hadamard decoder for RZH codes with parallel decoding.

nodes as the *a priori* information, with which the information bit nodes are activated in parallel and the extrinsic LLRs are computed for the next iteration.

Now let us focus on the Hadamard decoder in the context of parallel decoding for RZH codes. Let $\underline{y}_p$ be the channel observation of the parity bits $\underline{p}$, and $\underline{y}_{p_{[m]}}$ be the observation exclusive of y_{p_m}. Also let $\underline{a}$ be the *a priori* LLRs of the information bits $\underline{d}$, v_0 be the *a priori* LLR of connection bit q. Note that since the last parity bit p_m is identical to the connection bit of the next segment, it is connected to the degree-2 connection bit variable node in the Tanner graph as depicted in Fig. 2 and will have prior information passed from the degree-2 repetition decoder which can be denoted by v_1. Based on the channel observation of the parity bits, the prior information about the information bits, the connection bit and the last parity bit p_m, the soft-input soft-output (SISO) Hadamard decoder needs to compute the corresponding extrinsic LLRs for $\underline{d}$, q and p_m.

III. EXIT Functions on AWGN Channels

In this section, we consider the EXIT functions for Hadamard codes with multiple inputs on BIAWGN channels. Since it is difficult to directly compute the EXIT functions of the output from the MAP decoder, we make use of the pMAP decoder defined in [9] to derive an approximation for the extrinsic mutual information over BIAWGN channels. In what follows, we first introduce some basic properties of BIAWGN channel and the pMAP decoder, with which we will derive approximate EXIT functions over BIAWGN channels for general linear block codes and Hadamard codes.

We first characterize the BIAWGN channel, its extrinsic LLR and "soft bit" outputs. A BIAWGN channel is characterized by its variance σ^2. Let Y denote the BIAWGN channel observation for a transmitted BPSK modulated bit X, and let $L \triangleq \log \frac{\Pr(Y|X=+1)}{\Pr(Y|X=-1)}$, $T \triangleq \Pr(Y|X=+1) - \Pr(Y|X=-1)$ denote the extrinsic LLR and "soft bit" estimation of X based on the channel observation Y respectively. For a BIAWGN channel we have $L = \frac{2Y}{\sigma^2}$ and $T = \tanh(\frac{L}{2})$. Suppose that $+1$ is transmitted, then $(L|X=+1) \sim \mathcal{N}(m, 2m)$, where $m = \frac{2}{\sigma^2}$. Hence $(T|X=+1)$ has the following pdf

$$f_{T|X=+1}(t) = \frac{2}{1-t^2} f_L\left(\ln\frac{1+t}{1-t}\right), \tag{1}$$

where f_L denotes the pdf of $(L|X=+1)$. With (1), the $(2i)$-th moment of $(T|X=+1)$ as a function of m is given by $\Phi_i(m) \triangleq \mathbb{E}\left\{T^{2i} \mid X=+1\right\} = \int_{-1}^{+1} \frac{2t^{2i}}{(1-t^2)\sqrt{4\pi m}} \exp\left(-\frac{\left(\ln\frac{1+t}{1-t}-m\right)^2}{4m}\right) dt$. The mutual information between X and L as a function of m is given by $J(m) \triangleq I(X;L) = \int_{\mathcal{R}} \frac{1}{\sqrt{4\pi m}} \exp\left(-\frac{(\ell-m)^2}{4m}\right)\left(1-\log_2(1+e^{-\ell})\right) d\ell$, which is invertible and hence $m = J^{-1}(I)$.

Next we consider the extrinsic output of the MAP decoder and the pMAP decoder [9]. Given an (n,k) linear block code $\mathcal{C}$ over $GF(2)$, the dual code $\mathcal{C}^\perp$ is an $(n, n-k)$ linear block code. Let $\underline{c}_j^\perp = [c_{j,1}^\perp, c_{j,2}^\perp, \cdots, c_{j,n}^\perp]^T$ denote the j-th codeword of the $(n, n-k)$ dual code. Denote by $I_{i,1}^\perp = \{j | c_{j,i}^\perp = 1\}$ the set of indices of all codewords in the dual code with the i-th bit being 1 (or the set of indices of all parity-checks that the i-th bit participates in), and $I_{i,0}^\perp = \{j | c_{j,i}^\perp = 0\}$ the set of indices of all codewords in the dual code with the i-th bit being 0.

Suppose that the n transmitted BPSK modulated coded bits are $\underline{X} = [X_1, X_2, \cdots, X_n]$ and the prior soft estimations for these bits are $\underline{T} = [T_1, T_2, \cdots, T_n]$ which serve as the input of the MAP decoder. Based on $\underline{T}$, the extrinsic output of a MAP decoder for the i-th bit in terms of the LLR is given by

$$L_{EMAP,i} \triangleq \log \frac{\Pr(X_i = +1 \mid \underline{T}_{[i]})}{\Pr(X_i = -1 \mid \underline{T}_{[i]})}, \tag{2}$$

where $\underline{T}_{[i]}$ represents all the elements of $\underline{T}$ except T_i. The extrinsic output "soft bit" of the decoder is then defined as

$$T_{EMAP,i} \triangleq \Pr(X_i = +1 \mid \underline{T}_{[i]}) - \Pr(X_i = -1 \mid \underline{T}_{[i]}). \tag{3}$$

As in [2], the extrinsic MAP decoding can be implemented by using the dual code as follows

$$T_{EMAP,i} = \frac{\sum_{j \in I_{i,1}^\perp} \prod_{\ell=1,\ell\neq i}^{n} T_\ell^{c_{j,\ell}^\perp}}{\sum_{j \in I_{j,0}^\perp} \prod_{\ell=1,\ell\neq j}^{n} T_\ell^{c_{j,\ell}^\perp}}. \tag{4}$$

Also let $D_{EMAP,i}$ be the extrinsic output of MAP decoder after the following modification

$$D_{EMAP,i} \triangleq 1 + T_{EMAP,i}. \tag{5}$$

To facilitate the EXIT analysis, a suboptimal decoder called pMAP decoder is introduced in [9] with its extrinsic output given by

$$D_{EpMAP,i} \triangleq \prod_{S \in P(I_{i,1}^\perp)\setminus\phi} \left(1 + \prod_{\ell=1,\ell\neq i}^{n} T_\ell^{\vee_{j\in S}(c_{j,l}^\perp)}\right)^{(-1)^{|S|+1}}$$

where $P(I_{i,1}^\perp)$ denotes the set of all subsets of $I_{i,1}^\perp$ and "$\vee$" denotes the logical "or" operation.

In what follows, we will employ the pMAP decoder to obtain an approximate expression for the EXIT functions of linear block codes over a BIAWGN channel. Since the basic idea is to decompose the EXIT function in the form of a series EXIT functions of multiple single parity-check codes, we will start from the EXIT function of a single parity-check code.

Note that our goal is to find the extrinsic functions for different bits given that the *a priori* mutual information from the extrinsic and the communication channels for different bits is different. Keeping this in mind, we first compute the extrinsic information at the output of a single parity-check code.

Theorem 3.1: The extrinsic information of the i-th bit at the output of MAP decoder of an $(n, n-1)$ single parity-check code is given by:

$$I_{E,i} = \frac{1}{\ln 2}\sum_{j=1}^{\infty}\frac{1}{(2j)(2j-1)}\left(\prod_{\ell=1,\ell\neq i}^{n}\mathbb{E}\left\{T_\ell^{2j}\right\}\right). \quad (6)$$

Proof: The dual code of the $(n, n-1)$ single parity-check code is a repetition code of length n. Hence from (4) we have $T_{EMAP,i} = \prod_{\ell=1,\ell\neq i}^{n} T_\ell$, then using Proposition 2.6 in [9], it is easy to prove (6). ∎

As shown in [9], for the single parity-check codes, pMAP and MAP decoders are identical in any memoryless channel. Further more, we have the following result.

Theorem 3.2: Let $\mathcal{C}$ be an (n,k) linear block code such that in BIAWGN channel $D_{EMAP} = D_{EpMAP}$. Then the mutual information between X_i and $T_{EMAP,i}$ is given by

$$I_{E,i}^{BIAWGN} = \frac{1}{\ln 2}\sum_{S\in P(I_{i,1}^{\perp})\setminus\phi}(-1)^{|S|+1}\sum_{j=1}^{\infty}\frac{1}{(2j)(2j-1)}\cdot\left(\prod_{\ell\in V(S),\ell\neq i}\Phi_j(m_\ell)\right), \quad (7)$$

where $V(S) \triangleq \left\{\ell\in\{1,\ldots,n\}|\vee_{j\in S}\, c_{j,\ell}^{\perp} = 1\right\}$, and $m_\ell = J^{-1}(I_{A,\ell})$ with $I_{A,\ell}$ being the a priori mutual information about X_ℓ from the channel given $X_\ell = +1$ is transmitted.

Proof: According to Theorem 2.10 in [9] and (5), we have $I_{E_i}^{BIAWGN} = \mathbb{E}_{T_{EMAP,i}|\underline{c}_0}\{\log_2(1+T_{EMAP,i})\} = \mathbb{E}_{T_{EMAP,i}|\underline{c}_0}\{\log_2(D_{EMAP,i})\}$. Since $D_{EMAP} = D_{EpMAP}$, it is easy to show that

$$I_{E_i}^{BIAWGN} = \sum_{S\in P(I_{i,1}^{\perp})\setminus\phi}(-1)^{|S|+1}\cdot\mathbb{E}_{\underline{T}|\underline{c}_0}\left\{\log_2\left(1+\prod_{\ell=1,\ell\neq i}^{n}T_\ell^{\vee_{j\in S}(c_{j,\ell}^{\perp})}\right)\right\}.$$

Define $V(S) \triangleq \left\{\ell\in\{1,\ldots,n\}|\vee_{j\in S}\, c_{j,\ell}^{\perp} = 1\right\}$, then the $\mathbb{E}\{\cdot\}$ term in the above equation is the extrinsic information of a $(|V(S)|, |V(S)|-1)$ single parity-check code. With Theorem 3.1, it is easy to prove (7). Note that here we cannot expand the EXIT functions in the form of multiple EXIT functions of the same code in BEC channel, which is different from [9]. ∎

For high rate codes, we have $D_{EMAP} \cong D_{EpMAP}$ when the all-zero codeword is transmitted over the BIAWGN channel, and Theorem 3.2 can be used to approximately compute the extrinsic information at the output of the MAP decoder for a linear block code. Note that Hadamard codes are low rate codes. As in [9], for a low rate code, the extrinsic information can be approximated by

$$I_{E,i}^{BIAWGN} \cong 1 - I_{E,i}^{\perp,BIAWGN}(1-I_{A,1},\ldots,1-I_{A,n}), \quad (8)$$

where $I_{E,i}^{\perp,BIAWGN}(1-I_{A,1},\ldots,1-I_{A,n})$ is the extrinsic information of its high-rate dual code and can be computed by Theorem 3.2 with $m_\ell = J^{-1}(1-I_{A,\ell})$.

With the above derived results, next we consider the EXIT functions of Hadamard codes in the context of RZH codes with parallel decoding. We first consider the average mutual information about the information bits $\underline{d}$ which is given by $I_{E,d}^{BIAWGN} = \frac{1}{r}\sum_{i=1}^{r} I_{E,d_i}^{BIAWGN}$, where I_{E,d_i}^{BIAWGN} can be computed by (8).

Let $\underline{c}_j = [c_{j,1}, c_{j,2}, \ldots, c_{j,n}]^T$ denote the j-th codeword of the Hadamard code over $GF(2)$. For $\underline{c}_j$, we have $c_{j,2^{i-1}+1} = d_{j,i}$ for $i = 1,\ldots,r$. Denote by $I_{i,1}^{H,d} = \{j|c_{j,2^{i-1}+1} = 1\}$, for $i = 1,\ldots,r$, the set of indices of all Hadamard codewords with the i-th systematic bit equal 1, and $P(I_{i,1}^{H,d})$ the set of all subsets of $I_{i,1}^{H,d}$. Since a Hadamard codeword can be divided into systematic bits, parity bits and connection bits, we have $\underline{c}_j = \{\underline{d}_j, \underline{p}_{j,[m]}, q_j, p_{j,m}\}$ where $\underline{p}_{j,[m]}$ denotes the parity bits exclusive of $p_{j,m}$. For $S \in P(I_{i,1}^{H,d})$, the following definitions will be used:

$$V(S,\underline{d}) \triangleq \{\ell\in\{1,\ldots,r\} \mid \vee_{j\in S} d_{j,\ell} = 1\}, \quad (9)$$

$$V(S,\underline{p}_{[m]}) \triangleq \{\ell\in\{1,\ldots,m-1\} \mid \vee_{j\in S} p_{j,\ell} = 1\}, \quad (10)$$

$$V(S,q) \triangleq \{\ell\in\{1\} \mid \vee_{j\in S} q_{j,\ell} = 1\}, \quad (11)$$

$$V(S,p_m) \triangleq \{\ell\in\{m\} \mid \vee_{j\in S} p_{j,\ell} = 1\}, \quad (12)$$

where "$\vee$" is the logical "or" operation. It is clear that $V(S,q)$ is either $\{1\}$ or ϕ; similar for $V(S,p_m)$. Further define $A_{g,h,j,k}^{d_i,odd} \triangleq |\{S\in P(I_{i,1}^{H,d}) \mid |V(S,q)| = g, |V(S,\underline{d})| = h, |V(S,\underline{p}_{[m]})| = j, |V(S,p_m)| = k, |S| \text{ is odd}\}|$ and $A_{g,h,j,k}^{d_i,even} \triangleq |\{S\in P(I_{i,1}^{H,d}) \mid |V(S,q)| = g, |V(S,\underline{d})| = h, |V(S,\underline{p}_{[m]})| = j, |V(S,p_m)| = k, |S| \text{ is even}\}|$. Then we have the following results for Hadamard codes that will be subsequently used.

Lemma 3.1: For Hadamard codes $\mathcal{C}$ with order r, we have $A_{g,h,j,k}^{d_1,odd} = A_{g,h,j,k}^{d_2,odd} = \cdots = A_{g,h,j,k}^{d_r,odd} \triangleq A_{g,h,j,k}^{d,odd}$, and $A_{g,h,j,k}^{d_1,even} = A_{g,h,j,k}^{d_2,even} = \cdots = A_{g,h,j,k}^{d_r,even} \triangleq A_{g,h,j,k}^{d,even}$.

Proof: Since $A_{g,h,j,k}^{d_\ell,odd}$ and $A_{g,h,j,k}^{d_\ell,even}$ are based on the subcode $\mathcal{C}_i = \{\underline{c}_j | c_{j,2^{i-1}+1} = 1, \underline{c}_j \in \mathcal{C}\}$, $i = 1,\ldots,r$, we only need to show that for $\ell \neq i$, there is a $\mathcal{C}_\ell^*$ which is constructed from $\mathcal{C}_\ell$ by only switching coded bits within the systematic bit positions and/or the parity bit positions, such that $\mathcal{C}_i = \mathcal{C}_l^*$. Let the generation matrix of $\mathcal{C}$ in systematic form be $\underline{G}_{H,r}$ with $\underline{g}_0$ and $\underline{g}_i$ ($i = 1,\ldots,r$) being the rows of $\underline{G}_{H,r}$ corresponding to the connection bit and the i-th information bit, respectively. Suppose that the input of the Hadamard encoder is $\underline{d} = [d_0 = q, d_1, \ldots, d_r]^T$, then the codewords belong to $\mathcal{C}_i$ are given by $\{\underline{g}_i^T + \sum_{\ell\neq i} d_\ell \underline{g}_\ell^T\}$ for all possible $\{d_\ell\}_{\ell\neq i}$. Since $\underline{G}_{H,r}$ is in systematic form, $\underline{g}_i$ can be written as $[\underline{u}_i, \underline{p}_i]$ where $\underline{u}_i = [u_0, u_1, \ldots, u_r]$ with $u_\ell = 0$ for $\ell \neq i$ and $u_i = 1$. By switching the rows $\underline{g}_i$ and $\underline{g}_\ell$, we obtain $\underline{G}_{H,r}^{(1)}$ with $\underline{g}_i^{(1)} = \underline{g}_\ell$ and $\underline{g}_\ell^{(1)} = \underline{g}_i$. It is clear that the subcode $\mathcal{C}_i^{(1)}$ generated by $\underline{G}_{H,r}^{(1)}$ is identical to $\mathcal{C}_i$. We further construct $\underline{G}_{H,r}^{(2)}$ from $\underline{G}_{H,r}^{(1)}$ by letting $\underline{g}_i^{(2)} = \underline{g}_i^{(1)} + (\underline{g}_\ell^{(1)} + \underline{g}_i^{(1)}) = \underline{g}_\ell^{(1)} = \underline{g}_i$ and $\underline{g}_\ell^{(2)} = \underline{g}_\ell^{(1)} + (\underline{g}_\ell^{(1)} + \underline{g}_i^{(1)}) = \underline{g}_i^{(1)} = \underline{g}_\ell$, by which we have $\underline{G}_{H,r}^{(2)} = \underline{G}_{H,r}$ and hence $\mathcal{C}_i^{(2)} = \mathcal{C}_i$. The effect of transforming $\underline{G}_{H,r}^{(1)}$ to $\underline{G}_{H,r}^{(2)}$ is only to change the order of the coded bits. Since both codes are systematic codes, there is one

order change between systematic bit position i and ℓ (note that there are only two 1's in $\underline{u}_i + \underline{u}_\ell$), and all the other changes are between parity bit positions. It is clear that $\mathcal{C}_i^{(2)} = \mathcal{C}_i$ is the $\mathcal{C}_\ell^*$ we are looking for and this concludes the proof. ■

Note that in case that the size of the code book is not huge, it is easy to compute $A_{g,h,j,k}^{d,even}$ and $A_{g,h,j,k}^{d,odd}$ directly from their definitions given the codewords are known.

With the above results, the average mutual information about the information bits $\underline{d}$ can be computed by the following result.

Theorem 3.3: If the extrinsic and communication channels are AWGN channels with respective mutual information $I_{A,d}$, $I_{A,p}$, $I_{A,q}$ and I_{A,p_m}, we have

$$I_{E,d}^{BIAWGN} \cong 1 - \frac{1}{\ln 2} \sum_{g=0}^{1} \sum_{h=1}^{r} \sum_{j=0}^{m-1} \sum_{k=0}^{1} \left(A_{g,h,j,k}^{d,odd} - A_{g,h,j,k}^{d,even} \right) \cdot \sum_{i=1}^{\infty} \frac{1}{(2i)(2i-1)} \Phi_i^g(m_q) \Phi_i^{h-1}(m_d) \Phi_i^j(m_p) \Phi_i^k(m_{p_m}),$$

where $m_q = J^{-1}(1 - I_{A,q})$, $m_d = J^{-1}(1 - I_{A,d})$, $m_p = J^{-1}(1 - I_{A,p})$ and $m_{p_m} = J^{-1}(1 - I_{A,p_m})$. Note that here for simplicity, I_{A,p_m} denotes the overall mutual information about p_m from both the communication channel and the extrinsic channel.

Proof: Using Theorem 3.2, (8) and Theorem 3.1, we obtain the above result. ■

Next we consider the connection bit q. Similarly, we denote by $I_1^{H,q} = \{j | c_{j,1} = 1\}$ the set of indices of all Hadamard codewords with the connection bit equal 1, and $P(I_1^{H,q})$ the set of all subsets of $I_1^{H,q}$. Let $S \in P(I_1^{H,q})$, then we can obtain $V(S, \underline{d})$, $V(S, \underline{p}_{[m]})$, and $V(S, p_m)$ as in (9), (10) and (12). Define $A_{h,j,k}^{q,odd} \triangleq |\{S \in P(I_1^{H,q}) \mid |V(S,\underline{d})| = h, |V(S,\underline{p}_{[m]})| = j, |V(S,p_m)| = k, |S| \text{ is odd}\}|$ and $A_{h,j,k}^{q,even} \triangleq |\{S \in P(I_{i,1}^{H,q}) \mid |V(S,\underline{d})| = h, |V(S,\underline{p}_{[m]})| = j, |V(S,p_m)| = k, |S| \text{ is even}\}|$. Then similar to Theorem 3.3, the mutual information of q can be computed by the following result.

Theorem 3.4: If the extrinsic and communication channels are AWGN channels with respective mutual information $I_{A,d}$, $I_{A,p}$ and I_{A,p_m}, then we have

$$I_{E,q}^{BIAWGN} \cong 1 - \frac{1}{\ln 2} \sum_{h=0}^{r} \sum_{j=0}^{m-1} \sum_{k=0}^{1} \left(A_{h,j,k}^{d,odd} - A_{h,j,k}^{d,even} \right) \cdot \sum_{i=1}^{\infty} \frac{1}{(2i)(2i-1)} \Phi_i^h(m_d) \Phi_i^j(m_p) \Phi_i^k(m_{p_m}),$$

where $m_d = J^{-1}(1 - I_{A,d})$, $m_p = J^{-1}(1 - I_{A,p})$ and $m_{p_m} = J^{-1}(1 - I_{A,p_m})$. Again, here I_{A,p_m} represents the overall mutual information about p_m from both the communication channel and the extrinsic channel.

Finally we consider the last parity bit p_m. Since p_m has a channel observation y_m, the EXIT function of the last parity bit p_m is different from that of q. Again, let $\mathcal{C}^*$ be a new code formed by all pairs $(\underline{d}, \underline{p}, q, p_m)$, then we can define $I_1^{H,p_m} = \{j | p_{j,m} = 1\}$ the set of indices of all codewords in $\mathcal{C}^*$ with $p_{j,m} = 1$, and $P(I_1^{H,p_m})$ the set of all subsets of I_1^{H,p_m}. Then for $S \in P(I_1^{H,p_m})$, similar to the case for q, we can obtain $V(S,q)$, $V(S,\underline{d})$, and $V(S,\underline{p})$, with which we can define $A_{g,h,j}^{p_m,odd} \triangleq |\{S \in P(I_1^{H,p_m}) \mid |V(S,q)| = g, |V(S,\underline{d})| = h, |V(S,\underline{p})| = j, |S| \text{ is odd}\}|$ and $A_{g,h,j}^{p_m,odd} \triangleq |\{S \in P(I_1^{H,p_m}) \mid |V(S,q)| = g, |V(S,\underline{d})| = h, |V(S,\underline{p})| = j, |S| \text{ is even}\}|$, by which we can compute the EXIT function for p_m as follows.

Theorem 3.5: If the extrinsic and communication channels are AWGN channels with respective mutual information $I_{A,q}$, $I_{A,d}$ and $I_{A,p}$, then we have

$$I_{E,p_m}^{BIAWGN} \cong 1 - \frac{1}{\ln 2} \sum_{g=0}^{1} \sum_{h=0}^{r} \sum_{j=0}^{m} \left(A_{g,h,j}^{p_m,odd} - A_{g,h,j}^{p_m,even} \right) \cdot \sum_{i=1}^{\infty} \frac{1}{(2i)(2i-1)} \Phi_i^g(m_q) \Phi_i^h(m_d) \Phi_i^j(m_p),$$

where $m_q = J^{-1}(1 - I_{A,q})$, $m_d = J^{-1}(1 - I_{A,d})$ and $m_p = J^{-1}(1 - I_{A,p})$.

IV. Application to IRZH Code Design

In this section we consider the application of Hadamard code EXIT functions in the design of IRZH codes where the outer code is a mixture of repetition codes with different rates.

For given r and R_c, the optimal IRZH ensemble parameters $\{a_i\}$ are the one that minimize E_b/N_0, subject to a vanishing BER P_b. In practice, the vanishing BER condition is often replaced by a BER threshold ε for a given iteration number N_{it}, i.e., $P_b(N_{it}) < \varepsilon$, which leads to the following optimization problem

$$\min \ E_b/N_0 \text{ s.t. } \sum_i a_i = 1, a_i \geq 0, \ \forall i \text{ and } P_b(N_{it}) < \varepsilon.$$

To solve this problem, we resort to the differential evolution (DE) algorithm [7], which is a powerful population-based genetic algorithm and was introduced in [8] for finding the optimal degree profile of the irregular LDPC codes. The design approach closely follows that for irregular LDPC codes. During the optimization, we need to estimate the BER for a given code profile, and this can be done through mutual information evolution as discussed in [4].

In what follows, we provide several examples for RZH code analysis and degree profile optimization by using the EXIT function results obtained in this paper.

Example 1: (IRZH code ensemble optimization with $r = 4$ and $\bar{d}_v = 7$ over BIAWGN channel.) Note that for the analysis and design of LDPC codes, it is a common practice to use the EXIT function of the inner check node over BEC to approximate that over BIAWGN channel. And results show that such an approximation works very well for irregular LDPC code design. Here we will show that for RZH codes, however, such a BEC approximation provides less satisfactory results. On the other hand, the approximate EXIT functions

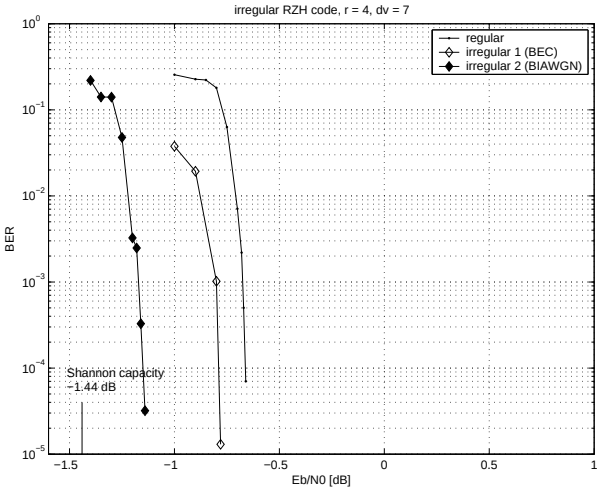


Fig. 4. BER performance of irregular RZH codes over BIAWGN channel.

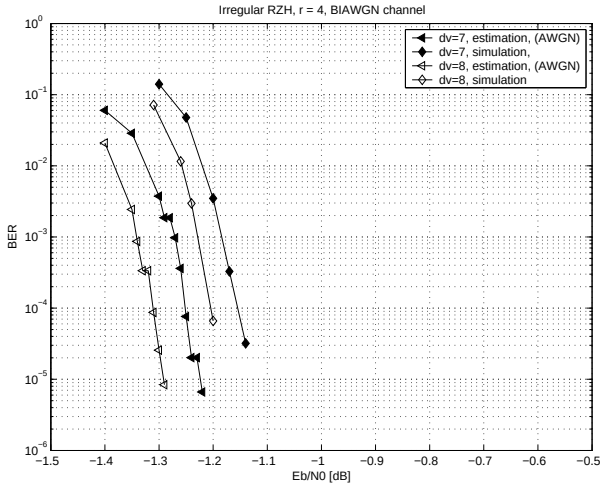


Fig. 5. BER performance of irregular RZH codes over BIAWGN channel.

for BIAWGN channel derived in Section 4, when employed to design IRZH codes, lead to codes with performance much closer to the capacity.

Consider a regular systematic RZH code with $r = 4$ and $\bar{d}_v = 7$ corresponding to a rate of $R_c = 0.0494$. Fig. 4 depicts the simulated BER performance of this code. The simulated SNR threshold (measured at $P_b = 10^{-4}$) is around -0.66 dB which is 0.78 dB away from the Shannon capacity -1.44 dB.

Next we consider irregular code design. Using BEC approximation, we are able to obtain a code at -0.96 dB with profile $f(x) = 0.547\,x^3 + 0.00430\,x^5 + 0.00760\,x^7 + 0.188\,x^9 + 0.0919\,x^{11} + 0.0984\,x^{13} + 0.0162\,x^{15} + 0.0148\,x^{17} + 0.0166\,x^{19} + 0.00380\,x^{21} + 0.00760\,x^{25} + 0.00380\,x^{29} + 0.00100\,x^{121}$. The simulated threshold is -0.79 dB with a gain of 0.14 dB over the regular code. By using the BIAWGN EXIT functions, we are able to optimize a code at -1.25 dB with profile $f(x) = 0.4353\,x^3 + 0.0072\,x^4 + 0.0016\,x^5 + 0.0099\,x^8 + 0.1759\,x^9 + 0.1829\,x^{10} + 0.0800\,x^{11} + 0.1071\,x^{12}$, which is only 0.19 dB away from the capacity. The simulated SNR threshold is -1.15 dB, which is only 0.29 dB away from the capacity and has a gain of 0.36 dB over the code optimized by BEC method and a gain of 0.49 dB over the regular one.

Example 2: (IRZH code ensemble optimization of with $r = 4$ and $\bar{d}_v = 8$ over BIAWGN channel) By setting $r = 4$ and $\bar{d}_v = 8$, the systematic RZH code has a rate of 0.0435. The Shannon limit for this code is -1.46 dB. Again, with EXIT functions over BIAWGN channel, we are able to optimize a code at -1.34 dB with profile $f(x) = 0.4044\,x^3 + 0.0026\,x^4 + 0.0048\,x^5 + 0.0245\,x^6 + 0.0453\,x^8 + 0.0669\,x^9 + 0.0296\,x^{10} + 0.0203\,x^{11} + 0.0985\,x^{12} + 0.3030\,x^{13}$, which is only 0.12 dB away from the capacity. BER performance of this code is depicted in Fig. 5. It is seen that the simulated SNR threshold for this code is -1.2 dB, which is only 0.26 dB away from the capacity and 0.4 dB away from the ultimate Shannon limit. Hence the EXIT functions for BIAWGN channels derived in this paper provide a good approximation in the low-rate region and can serve as a effective tool for the low-rate IRZH code design.

V. Conclusions

We have investigated the EXIT functions of Hadamard codes in the context of RZH codes with parallel decoding, and applied the obtained EXIT functions to the design of low-rate capacity-approaching irregular RZH codes. With these EXIT functions, BERs for a given code profile can be easily estimated and differential evolution (DE) technique can be employed to optimize the degree profile for the given design parameters, e.g., the coding rate and the Hadamard order. It is seen that the obtained EXIT functions can serve as a practical tool for designing IRZH codes with parallel decoding. The techniques introduced in this paper can also be easily extended to the design of general low-density parity-check codes where the check nodes are replaced by other linear block codes, e.g., Hadamard codes or Hamming codes.

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