On Multicast in Quantum Networks

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***Abstract*— We consider quantum multicast networks in which quantum states generated by multiple sources have to be simultaneously delivered to multiple receivers. We demonstrate that in addition to the apparent similarity to the multicommodity flow problems, quantum networks, to a certain extent, behave as classical communication networks. In particular, we show that lossless compression of special multicast quantum states is possible and significantly reduces the edge capacity requirements of the multicast.**

I. Introduction

Communication networks are, like their transportation or fluid counterparts, mathematically represented as directed or undirected graphs. In general, the results originally obtained for fluid flows in networks of pipes or vehicles on highways are immediately applicable to flows of data packets through communications networks. The opposite is not true, because classical (as opposed to quantum) information can be duplicated, merged, or in general, processed in a way that (non-quantum) physical entities can not. While a car at an intersection can proceed along only one highway, a bit available at a communication network router can be copied and sent out to multiple destinations simultaneously, and while two cars reaching a Y-junction have to proceed one behind the other, two bits at a router can be combined by binary addition and sent out as a single bit carrying the information on whether the two input bits are the same or different.

The advantage that the properties of classical information offer is best appreciated by considering the network multicast problem. Consider a communication network represented as a directed graph $G = (V, E)$ with unit capacity edges, and h unit rate information sources $S_1, \ldots, S_h$ that simultaneously transmitting information to N receivers $R_1, \ldots, R_N$ located at distinct nodes. Assume that the min-cut between the sources and each receiver node is h. The Ford-Fulkerson's min-cut, max-flow theorem states that, if a single receiver could utilize the network resources by itself, it would be able to receive information at rate h. Recently, it has been realized that allowing network nodes to re-encode the information they receive (in addition to re-routing) enables each receiver to retrieve information at rate h, even when N receivers simultaneously share the network resources [1], [2]. This type of coding is now known as network coding. Moreover, it was shown that by linear network coding, the min-cut rate can be achieved in multicasting to multiple sinks [1], [2]. This is not always the case when network nodes are only allowed to forward the information they receive, and network coding in general offers throughput benefits as compared to routing.

A simple multicast scenario shown in Fig. 1 (known as the *butterfly* network) is often used to illustrate

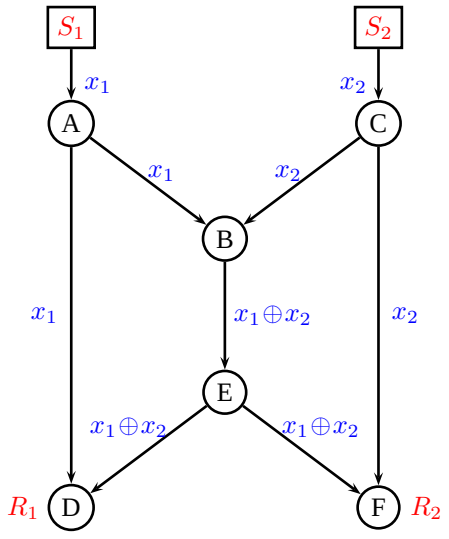


Fig. 1. Coding for a multicast (or for two unicasts) in a network with two sources and two receivers

throughput benefits of network coding. Note that nodes A and B broadcast the information they receive, while node C performs the binary addition of its input bits. The same figure can be used to illustrate how network coding enables two simultaneous unicasts where source S_1 wants to transmit information to receiver R_2 and source S_2 to receiver R_1.

An interesting question to ask is whether anything can be gained by allowing processing of quantum information at nodes in quantum networks. Consider the quantum counterpart of the network multicast scenario where the sources produce quantum states (*e.g.,* qubits as opposed to bits) which are then sent over quantum noiseless channels to be simultaneously delivered to the receivers. Since quantum states are represented by physical entities, the problem of quantum multicast at first seems nothing more than the multicommodity flow problem of shipping a collection of different commodities through a common network so that the total flow going through each edge in the network does not exceed its capacity. On the second thought, one realizes that although quantum states can not be cloned [3] or broadcast [4], they can be, in certain limited way, processed and compressed with no information loss as well as approximately cloned [5].

The only paper on quantum network coding [6] addresses the problem of two simultaneous unicasts on the butterfly network of Fig. 1, where source S_1 wants to deliver quantum states to receiver R_2 and source S_2 to receiver R_1. The obvious challenge is how to combine the information of both states into the qubit on the edge BE so that nodes D and F can recover their expected states with high fidelities. The paper provides protocols (based on approximate cloning) of which the recovery fidelities for both D and F (*i.e.,* receivers R_1 and R_2) are strictly larger than $1/2$.

In this paper we consider quantum multicast networks in which quantum states generated by multiple sources have to be simultaneously delivered to multiple receivers, and show that lossless compression of special multicast quantum information is possible, and significantly reduces the edge capacity requirements of the multicast. As expected, the required edge capacity turns out to be lower than in the equivalent multicommodity flow problems but higher than in the equivalent classical network multicast problems.

This paper is organized as follows: We start with a brief review of quantum information systems in Sec. II, and in Sec. III demonstrate that when the sources in the qunatum network network produce orthogonal states, generalization of classical network coding schemes to their quantum counterparts is straightforward. We then turn to single-source quantum multicast in Sec. IV and quantum network multicast in Sec. V.

II. Quantum Information Systems

A. Quantum States

Quantum states are, in the simplest case, mathematically represented as unit length column vectors in a d-dimensional Hilbert space $\mathcal{H}$. Such quantum states are called *pure*. When $d = 2$, quantum states are called *qubits*. A column vector is denoted by $|\varphi\rangle$, its complex conjugate transpose by $\langle\varphi|$. A pure state is mathematically described by its *density matrix* equal to the outer product $|\varphi\rangle\langle\varphi|$.

In a more complex case, all we know about a quantum state is that it is one of a finite number of possible pure states $|\varphi_i\rangle$ with probability p_i. Such quantum states are called *mixed*. A mixed state is also described by its density matrix which is equal to

$$\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|.$$

Note that a density matrix is a $d \times d$ Hermitian trace-one positive semidefinite matrix. A classical analog to a mixed state can be a multi-faced coin which turns up as any of its faces with the corresponding probability.

We often deal with sequences rather than with individual states. The quantum state corresponding to a sequence of length n has a $d^n \times d^n$ density matrix, equal to the tensor product of density matrices corresponding to the states in the sequence.

B. Quantum Operations

A quantum state ρ can be transformed to another state $\mathcal{E}(\rho)$ only by a physical process consistent with the lows of quantum theory. Such a process is, in the simplest case, mathematically described as a *unitary* evolution:

$$\mathcal{E}(\rho) = U\rho U^\dagger \quad \text{where} \quad UU^\dagger = I,$$

and, in a more general case, as an evolution by a *completely positive, trace-preserving* map:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{where} \quad \sum_k E_k^\dagger E_k = I.$$

It is envisioned that a quantum computer (like a classical) would implement such evolutions by using

universal quantum gates. An example of a two-qubit quantum gate is the XOR:

$$\mathsf{XOR} : |x, y\rangle \rightarrow |x, x \oplus y\rangle \quad U_{\mathsf{XOR}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

C. Quantum Measurements

A quantum measurement is a physical process applied to determine the state of the quantum system being measured. Only when the possible states, say $\{|\psi_j\rangle,\ j = 1, \ldots, J\}$, are orthogonal can a quantum measurement be designed to give an unambiguous answer.

The simplest model of quantum measurement is known as the von Neumann's measurement. Mathematically, this type of measurement is defined by a set of pairwise orthogonal projection operators $\{\Pi_i\}$ which form a complete resolution of the identity, that is, $\sum_i \Pi_i = I$. For input $|\psi_j\rangle$, the *classical* output $\Pi_i|\psi_j\rangle$ happens with probability $|\langle\psi_j|\Pi_i|\psi_j\rangle|^2$.

In a more general case, the pairwise orthogonal projection operators $\{\Pi_i\}$ are replaced by any positive-semidefinite operators $\{E_i\}$ which form a complete resolution of the identity. This type of measurement in known as positive operator-valued measure (POVM).

D. Cloning, Broadcasting, and Deleting

Quantum information cannot be cloned, broadcast or deleted in the sense made precise below, unless we are dealing with states with commuting density matrices.

1) The No-Cloning Principle: There is no physical process that leads to an evolution (*i.e.,* quantum process defined in Sec. II-B)

$$|\phi\rangle \otimes |s\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle$$

where $|\phi\rangle$ is an arbitrary state and $|s\rangle$ is a fixed state [3]. See also [7, p. 532] for a proof of this claim. Approximate cloning, on the other hand, is possible (*e.g.,* [5]), and it was instrumental in developing network coding protocols of [6].

2) The No-Deleting Principle: There is no physical process that leads to an evolution

$$|\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |s\rangle$$

where $|\phi\rangle$ is an arbitrary state and $|s\rangle$ is a fixed state.

3) The No-Broadcasting Principle: Suppose that quantum system A is in an unknown state ρ and quantum system B in some known or standard state ω. It was shown in [4] that there is no physical process that leads to an evolution

$$\rho \otimes \omega \rightarrow \tilde{\rho} \ \text{ s.t.} \operatorname{Tr}_A(\tilde{\rho}) = \rho \text{ and } \operatorname{Tr}_B(\tilde{\rho}) = \rho,$$

that is, so that both subsystem A and subsystem B evolve into state ρ.

III. Networks with Orthogonal States

Quantum information processing in networks in which sources produce orthogonal quantum states is a straightforward generalization of classical information processing. For example, in a quantum counterpart of the network coding scheme shown Fig. 1 where the sources produce mutually orthogonal quantum states $|0\rangle$ and $|1\rangle$, each node implements an XOR gate. Let $|b_1\rangle$ and $|b_2\rangle$ be the qubits produced by sources S_1 and S_2 respectively. The inputs $|x\rangle$ and $|y\rangle$ to the gate at each node and outputs sent along the outgoing edges are as follows:

NODE	INPUT	OUTPUT
A	$\|x\rangle = \|b_1\rangle$ $\|y\rangle = \|0\rangle$	AB : $\|b_1\rangle$ AD : $\|b_1\rangle$
B	$\|x\rangle = \|b_1\rangle$ $\|y\rangle = \|b_2\rangle$	BE : $\|b_1 \oplus b_2\rangle$
C	$\|x\rangle = \|b_2\rangle$ $\|y\rangle = \|0\rangle$	CB : $\|b_2\rangle$ CF : $\|b_2\rangle$
D	$\|x\rangle = \|b_1\rangle$ $\|y\rangle = \|b_1 \oplus b_2\rangle$	$\|b_1\rangle$ $\|b_2\rangle$
E	$\|x\rangle = \|b_1 \oplus b_2\rangle$ $\|y\rangle = \|0\rangle$	ED : $\|b_1 \oplus b_2\rangle$ EF : $\|b_1 \oplus b_2\rangle$
F	$\|x\rangle = \|b_2\rangle$ $\|y\rangle = \|b_1 \oplus b_2\rangle$	$\|b_2\rangle$ $\|b_1\rangle$

IV. Single-Source Quantum Multicast

We now consider a simple quantum multicast scenario as illustrated in Fig. 2. A quantum source of information S produces quantum states $|\phi\rangle \in \mathcal{H}$ which are to be reproduced at N receiving points $R_1, \ldots, R_N$ simultaneously. The source node is connected to another node A. Since A cannot copy or broadcast the quantum information it receives, state $|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$, prepared at the source, has to be made available at A. The question is whether that state has to be actually transmitted through the quantum channel from S to A if nodes S and A are

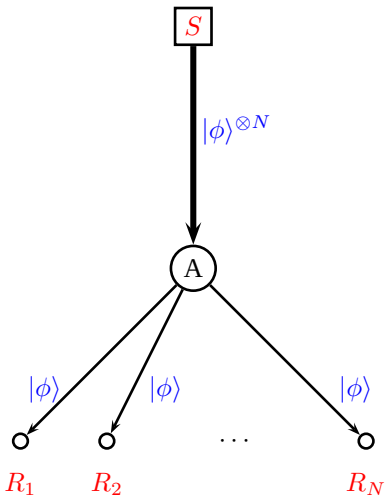


Fig. 2. Quantum multicast.

allowed to perform lossless compression and decompression of quantum information. Recall that if this network is classical, then only a single bit would have to be sent from S to A, which can then duplicate this bit and send it to the receivers. On the other hand, if N identical cars are to be delivered to N customers simultaneously over a network of highways with the same topology as the one shown in the figure, then clearly, N actual cars would have to be made at the source, and the highway between S and A would have to have N parallel lanes. We show that for the scenario of Fig. 2, a quantum link between S and A that can carry $\Theta(\log N)$ qubits is necessary and sufficient to accommodate multicast of any state $|\phi\rangle \in \mathcal{H}$ produced at the source.

The idea is to look into how states of the form $|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$ can be compressed. The smallest subspace of $\mathcal{H}^{\otimes N}$ that contains all vectors of the form $|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$ is known as the *symmetric subspace*. We denote this subspace by $\mathcal{SYM}(\mathcal{H}^{\otimes N})$. The symmetric subspace has played a role in several quantum information processing problems such as universal compression of quantum information [8] and stabilization of quantum computations [9] (see also [10, Ch. 7.3]). It is not hard to see (*e.g.*, [9]) that this subspace is precisely the subspace of $\mathcal{H}^{\otimes N}$ invariant under permutations of the qubits. We will next find a basis of the space $\mathcal{SYM}(\mathcal{H}^{\otimes N})$ and prove that its dimension is polynomial in N, and then show how any N-qubit state $|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$ can be compressed into a $\Theta(\log N)$-qubit state in $\mathcal{SYM}(\mathcal{H}^{\otimes N})$ and decompressed from it.

A. Types and Type Classes

We briefly review types and type classes since they will be instrumental in defining a basis for

$$\mathcal{SYM}(\mathcal{H}^{\otimes N}) = \langle\{|\phi\rangle^{\otimes N} : |\phi\rangle \in \mathcal{H}\}\rangle.$$

More details on this topic can be found in [11].

Let $\mathcal{X}$ be a finite set. Given a sequence $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$ and a letter $a \in \mathcal{X}$, let $\eta(a|\boldsymbol{x})$ denote the number occurrences of a in $\boldsymbol{x}$.

Definition 1: The *type* of a sequence $\boldsymbol{x} \in \mathcal{X}^N$ is the distribution $P_{\boldsymbol{x}}$ given by

$$P_{\boldsymbol{x}}(a) = \frac{1}{N}\eta(a|\boldsymbol{x}) \quad \text{for every} \quad a \in \mathcal{X}.$$

Conversely, the *type class* of a distribution P is the set T_P^N of all sequences of type P in $\mathcal{X}^N$:

$$\mathsf{T}_P^N = \{\boldsymbol{x} : \boldsymbol{x} \in \mathcal{X}^N \text{ and } P_{\boldsymbol{x}} = P\}.$$

The set consisting of all possible types of sequences $\boldsymbol{x} \in \mathcal{X}^N$ is denoted by $\mathcal{P}_n(\mathcal{X})$. It is easy to show by elementary combinatorics that

$$|\mathcal{P}_N(\mathcal{X})| = \binom{N + |\mathcal{X}| - 1}{|\mathcal{X}| - 1}. \tag{1}$$

Note that the number of types is polynomial in N.

B. A Basis for Vector Space $\langle\{|\phi\rangle^{\otimes N} : |\phi\rangle \in \mathcal{H}\}\rangle$

Let $|\phi\rangle$ be a vector in a d-dimensional Hilbert space $\mathcal{H}$ with a basis $\{|e_i\rangle,\ i = 1, \ldots, d\}$:

$$|\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle.$$

Let $\mathcal{X} = \{1, \ldots, d\}$ and $\boldsymbol{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$. Consider vector $|\phi\rangle^{\otimes n} \in \mathcal{H}^{\otimes n}$:

$$\begin{aligned} |\phi\rangle^{\otimes n} &= \Big(\sum_{i=1}^{d} \alpha_i |e_i\rangle\Big)^{\otimes N} \\ &= \sum_{\boldsymbol{x} \in \mathcal{X}^N} \alpha_{\boldsymbol{x}} |e_{\boldsymbol{x}}\rangle, \end{aligned}$$

where

$$\alpha_{\boldsymbol{x}} = \prod_{j=1}^{N} \alpha_{x_j} \text{ and } |e_{\boldsymbol{x}}\rangle = |e_{x_1}\rangle \otimes \cdots \otimes |e_{x_N}\rangle.$$

We further have

$$\begin{aligned} |\phi\rangle^{\otimes n} &= \sum_{\boldsymbol{x} \in \mathcal{X}^n} \alpha_{\boldsymbol{x}} |e_{\boldsymbol{x}}\rangle \\ &= \sum_{P \in \mathcal{P}_N(\mathcal{X})} \prod_{i=1}^{d} \alpha_i^{NP(i)} \sum_{\boldsymbol{x} \in \mathsf{T}_P^N} |e_{\boldsymbol{x}}\rangle \end{aligned}$$

Consequently, a vector of the form $|\phi\rangle^{\otimes N}$ has $|\mathcal{P}_N(\mathcal{X})|$ degrees of freedom, and the orthonormal basis vectors for $\langle\{|\phi\rangle^{\otimes N} : |\phi\rangle \in \mathcal{H}\}\rangle$ are indexed by types:

$$|E_P\rangle = \frac{1}{\sqrt{|\mathsf{T}_P^N|}} \sum_{\boldsymbol{x}\in\mathsf{T}_P^N} |e_{\boldsymbol{x}}\rangle, \quad P \in \mathcal{P}_N(\mathcal{X}). \qquad (2)$$

Since $|\mathcal{X}| = d$, from (1) we get that the number of these vectors is

$$\binom{N+d-1}{d-1}.$$

We have shown that all vectors in the space $\mathcal{H}^{\otimes N}$ (whose dimension is d^N) that have the form $|\phi\rangle^{\otimes N}$ actually belong to a subspace of $\mathcal{H}^{\otimes N}$ whose dimension is only polynomial in N (linear in the binary case $d = 2$). We next exploit this fact to show that instead of sending $|\phi\rangle^{\otimes N}$ along the noiseless quantum channel from S to A in Fig. 2, it is sufficient (and necessary) to send only $\Theta(\log N)$ qubits to communicate the same quantum information.

C. Lossless Compression into and Decompression from $\mathcal{SYM}(\mathcal{H}^{\otimes N})$

We describe a simple quantum algorithm to compress into and decompress from $\mathcal{SYM}(\mathcal{H}^{\otimes N})$ using arbitrary single-qubit and two-qubit gates, in addition to arbitrary reversible classical computation. Any of those gates can in turn be implemented by a circuit of gates from any *universal gate set* (*e.g.,* [7]). For the clarity of the presentation and because generalization is straightforward, from now on we consider the binary case only, *i.e.,* $\mathcal{X} = \{0, 1\}$.

For an N-bit binary string $x \in \{0,1\}^N$, let $|x|$ denote its Hamming weight (the number of 1s in the string). Since the type class of x is determined by its Hamming weight, the basis for $\mathcal{SYM}(\mathcal{H}^{\otimes N})$ (as defined by(2)) is

$$|E_i\rangle = \frac{1}{\sqrt{\binom{N}{i}}} \sum_{x\in\{0,1\}^N, |x|=i} |x\rangle, \qquad 0 \le i \le N.$$

We need only to construct a quantum circuit W that maps $|i\rangle \mapsto |E_i\rangle$. Then we can use the reverse of W to compress into $\mathcal{SYM}(\mathcal{H}^{\otimes N})$ and W itself to decompress. The circuit W would need ancilla states that are initialized as $|0\rangle$ and returned to the same state at the end of the computation.

We first apply the transformation

$$W_1 : \ |i\rangle \otimes |0\rangle^{\otimes N} \mapsto |i\rangle \otimes |E_i\rangle.$$

This can be done, for example, by implementing

$$|i\rangle \otimes |0\rangle^{\otimes N} \mapsto |i\rangle \otimes \Big(\frac{1}{\sqrt{\binom{N}{i}}} \sum_{0\le j\le \binom{N}{i}-1} |j\rangle\Big)$$

and then mapping j to the j'th element in $\{x \in \{0,1\}^N : |x| = i\}$ under some standard ordering. It is easy to implement

$$W_2 : |y\rangle \otimes |E_i\rangle \mapsto |i + y \mod (N+1)\rangle \otimes |E_i\rangle.$$

Hence

$$W_2^\dagger W_1(|i\rangle \otimes |0\rangle^{\otimes N}) = |0\rangle^{\lceil \log_2(N+1)\rceil} \otimes |E_i\rangle.$$

Swapping the qubits completes the construction of W.

V. Quantum Network Multicast

We now consider a quantum communication network represented as a directed graph $G = (V, E)$. There are h information sources $S_1, \ldots, S_h$, each producing a qubit per unit time, and N receivers $\mathcal{R} = \{R_1, \ldots, R_N\}$. For each receiver, there are h edge disjoint paths to it, one from each of the h sources. For receiver j, we denote these paths as (S_i, R_j), $i = 1, \ldots, h$. The h information sources need to multicast h qubits simultaneously to all N receivers at rate h. It is easy to see that the capacity at any cut which separates the sources from the receivers should be at least $h \log(N + 1)$. We will show that the multicast is possible if each edge of the network has the capacity of carrying $h \log(N + 1)$ qubits.

Definition 2: We say that source S_i is *present* at edge $e \in E$ if there is a receiver for which the path (S_i, R_j) form source S_i passes through edge e.

Let $\mathcal{I} = \{1, \ldots, h\}$ denote the set used to index the h sources, and $\mathcal{I}(e) \subseteq \mathcal{I}$ denote the index set comprising the labels of sources present at edge $e \in E$. Let ζ denote the maximum number of sources present at an edge in the network:

$$\zeta \triangleq \max_{e\in E} |\mathcal{I}(e)| \le h. \qquad (3)$$

Lemma 1: Let $r(i; e)$ denote the number of all receivers $R_j \in \mathcal{R}$ for which the paths (S_i, R_j) form source S_i pass through edge e. We have

$$\sum_{i\in\mathcal{I}(e)} r(i; e) \le N, \ \forall e \in E. \qquad (4)$$

Proof: The inequality (4) holds because there are N receivers and for each receiver, say R_j, the paths (S_i, R_j), $i \in \mathcal{I}$, are disjoint. ∎

Theorem 1: If in a quantum network with h sources of arbitrary qubits and N receivers the following three conditions hold:

1) the number of edges in the min-cut between the sources and each receiver is h,
2) each link has the capacity to carry $h \log(N+1)$-qubit states,
3) the nodes have the capability to losslessly compress and decompress quantum information,

then the sources can simultaneously multicast their qubits through the network to all receivers.

Proof: The first condition guarantees that, for each receiver, there are h edge disjoint paths to it, one from each of the h sources (see for example [12, p. 203]). If the quantum information is multicast from the sources to the receivers with no processing at the intermediate nodes (*e.g.*, compression/decompression), then edge e carries the quantum state $\Phi(e)$:

$$\Phi(e) = \bigotimes_{i \in \mathcal{I}(e)} |\phi_i\rangle^{\otimes r(i;e)}$$

where the qubits $|\phi_i\rangle$ are prepared by source S_i and destined to $r(i;e)$ receivers through edge e. States $\Phi(e)$ belong to the Hilbert subspace of dimension $\prod_{i \in \mathcal{I}(e)} (r(i;e)+1)$ which can be bounded as follows:

$$\prod_{i \in \mathcal{I}(e)} (r(i;e)+1) \leq \left[\frac{\sum_{i \in \mathcal{I}(e)} (r(i;e)+1)}{|\mathcal{I}(e)|} \right]^{|\mathcal{I}(e)|} \quad (5)$$

$$\leq (N+1)^{\zeta} \leq (N+1)^{h}. \quad (6)$$

where (5) follows from the geometric/arithmetic mean inequality, and (6) follows from (3) and (4). ■

Note that, even if all sources share the same link to all the receivers (*e.g.*, there are h sources in place of S in Fig. 2), the capacity of $h \log(N+1)$ is sufficient. This requirement is reduced when the qubits form the h sources are routed through the network in a way which (roughly speaking) favors more homogeneous edge flows. An example of qubit routing is shown for the network in Fig. 3 with $3k$ receivers. Note that this is the only valid routing for this network. The maximum required edge-capacity is $2 \log k$ qubits on the edge BE.

VI. Conclusion

Quantum multicast networks, in which quantum states generated by multiple sources have to be simultaneously delivered to multiple receivers, are shown to have similarities with both transportation multicommodity networks and classical communication networks. Allowing network nodes to merely losslessly compress and decompress special multicast states significantly reduces the edge capacity requirements of the multicast. Possible

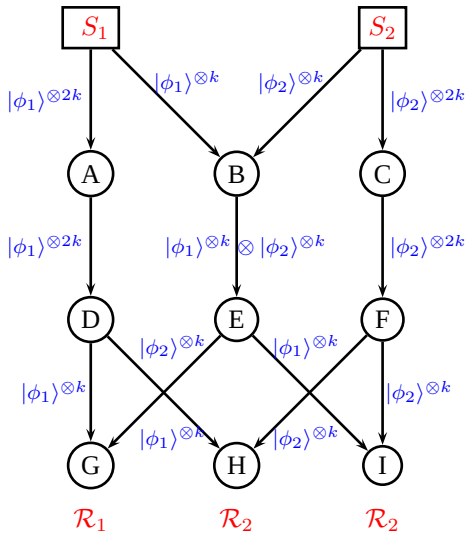


Fig. 3. Quantum network multicast from two sources S_1 and S_2 and three groups of receivers $\mathcal{R}_1 = \{R_1, \ldots, R_k\}$, $\mathcal{R}_2 = \{R_{k+1}, \ldots, R_{2k}\}$, and $\mathcal{R}_2 = \{R_{2k+1}, \ldots, R_{3k}\}$.

throughput improvements with other quantum processing analogous to network coding in classical networks should be investigated.

References

[1] R. Ahlswede, N. Cai, S-Y. R. Li, and R. W. Yeung, "Network information flow," *IEEE Trans. Inform. Theory*, pp. 1204–1216, July 2000.

[2] S-Y. R. Li, R. W. Yeung, and N. Cai, "Linear network coding," *IEEE Trans. Inform. Theory*, vol. 49, pp. 371–381, Feb. 2003.

[3] W. Wootters and W. Zurek, "A single quantum cannot be cloned," *Nature*, vol. 299, pp. 802–803, 1982.

[4] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, "Noncommuting mixed states cannot be broadcast," *Phys. Rev. Lett.*, vol. 76, pp. 2818–2821.

[5] V. Buzek and M. Hillery, "Quantum copying: beyond the no-cloning theorem," *Phys. Rev. A*, vol. 54, pp. 1844–1853, 1996.

[6] M. Hayashi, K. Iwama, H. Nishimura, R. Raymond, and S. Yamashita. [Online]. Available: http://arxiv.org/abs/quant-ph/0601088

[7] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[8] R. Jozsa, M. Horodecki, P. Horodecki, and R. Horodecki, "Universal quantum information compression," *Phys. Rev. Lett.*, vol. 81, pp. 1714–1717, 1998.

[9] A. Barenco, A. Berthiaume, D. Deutsch, A. Ekert, R. Jozsa, and C. Macchiavello, "Stabilization of quantum computations by symmetrization," *S.I.A.M. Journal on Computing*, pp. 1541–1557, 1997.

[10] J. Gruska, *Quantum Computing*. Mcgraw Hill, 2000.

[11] I. Csiszár and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Budapest, Hungary: Académiai Kiadó, 1986.

[12] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*. Amsterdam: North-Holland, 1979.