FRAMES IN QUANTUM AND CLASSICAL INFORMATION THEORY

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A FRAME

- A sequence $\{x_i\}$ of vectors in a Hilbert space with the property that there are constants $A, B \geq 0$ such that

$$A\|x\|^2 \leq \sum_i |\langle x, x_i\rangle|^2 \leq B\|x\|^2$$

  for all x in the Hilbert space.

- Examples?
  - basis
  - finite case

SYNCHRONOUS CDMA SYSTEMS

K users and processing gain N

- Each user has a signature $N \times 1$ length-$\sqrt{N}$ complex vector.
- Let $\boldsymbol{s_i}$ be the signature and p_i the power of user i for $1 \leq i \leq K$.
- The received vector is given by

$$\boldsymbol{r} = \sum_{i=1}^{K} \sqrt{p_i} b_i \boldsymbol{s_i} + \boldsymbol{n}$$

where

- b_i is the information symbol, for user i, $E[b_i] = 0$, $E[b_i^2] = 1$;
- $\boldsymbol{n}$ is the (Gaussian) noise vector; $E[\boldsymbol{n}] = \boldsymbol{0}$, $E[\boldsymbol{n}\boldsymbol{n}^\dagger] = \sigma^2 I_N$.

SYNCHRONOUS CDMA SYSTEMS

The Sum of Rates

- User i transmits sequences $b_{i_1} \ldots b_{i_n}$ drawn from a code $\mathcal{C}_i$.
- $\mathcal{C}_i \subset \{-1, 1\}^n$ contains 2^{nR_i} codewords.
- We say that user i transmits at rate R_i.
- An important performance measure is the sum of rates:

$$\sum_{i=1}^{K} R_i$$

when user i transmits at rate R_i reliably.

SYNCHRONOUS CDMA SYSTEMS

The Sum Capacity

- Let user signatures and powers be given:

$$S = [\boldsymbol{s}_1, \ldots \boldsymbol{s}_K] \text{ and } P = \text{diag}\{p_1, \ldots, p_K\}$$

- The sum capacity:

$$C_{\text{sum}} = \frac{1}{2} \log[\det(I_N + \sigma^{-2} S P S^{\dagger})]$$

A SOURCE OF INFORMATION

Classical

- Discrete: produces sequences of letters.
- Letters belong to a finite alphabet $\mathcal{X}$.
- Memoryless: each letter is produced independently.
- Probability of letter a is P_a.
- Example: coin tossing with $\mathcal{X} = \{H, T\}$.

A SOURCE OF INFORMATION

Quantum

- Letters are transmitted as d-dimensional unit-length vectors.
- $|e_0\rangle$ and $|e_1\rangle$ are the basis vectors of 2D space $\mathcal{H}_2$:

$$|e_0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |e_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Example: Alphabet $\mathcal{X} = \{0, 1, 2, 3\}$ mapped onto 4 qubits

$$|\psi_0\rangle = \alpha_0|e_0\rangle + \beta_0|e_1\rangle \quad |\psi_1\rangle = \alpha_1|e_0\rangle + \beta_1|e_1\rangle$$
$$|\psi_2\rangle = \alpha_2|e_0\rangle + \beta_2|e_1\rangle \quad |\psi_3\rangle = \alpha_3|e_0\rangle + \beta_3|e_1\rangle.$$

QUANTUM DISCRETE MEMORYLESS SOURCE

The Density Matrix and Von Neumann Entropy

- Source density matrix:

$$\rho = \sum_{a \in \mathcal{X}} P_a |\psi_a\rangle\langle\psi_a|.$$

- Von Neumann entropy of the source:

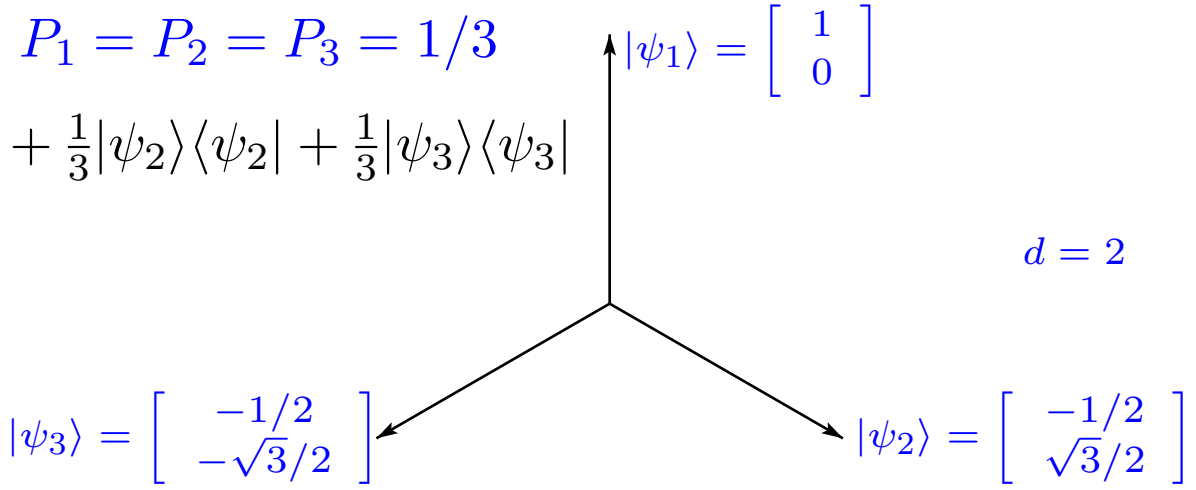
$$\begin{aligned} S(\rho) &= -\operatorname{Tr} \rho \log \rho \\ &= -\sum_i \lambda_i \log \lambda_i, \end{aligned}$$

where λ_i are the eigenvalues of ρ.

QUANTUM DISCRETE MEMORYLESS SOURCE

MB Example

$$
\begin{aligned}
\mathcal{X} &= \{1,2,3\} \quad P_1 = P_2 = P_3 = 1/3 \\
\rho &= \tfrac{1}{3}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{3}|\psi_2\rangle\langle\psi_2| + \tfrac{1}{3}|\psi_3\rangle\langle\psi_3| \\
&= \tfrac{1}{2}I \\
S(\rho) &= 1
\end{aligned}
$$



QUANTUM DISCRETE MEMORYLESS SOURCE

Vector Sequences

- Sequences of length n are d^n-dimensional vectors.
- Source vector-sequence (state):

$$|\Psi_{\boldsymbol{x}}\rangle = |\psi_{x_1}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle, \qquad x_i \in \mathcal{X}.$$

- Among all states that come from the source, we can distinguish

$$2^{n(S(\rho)-\varepsilon_n)}$$

reliably.

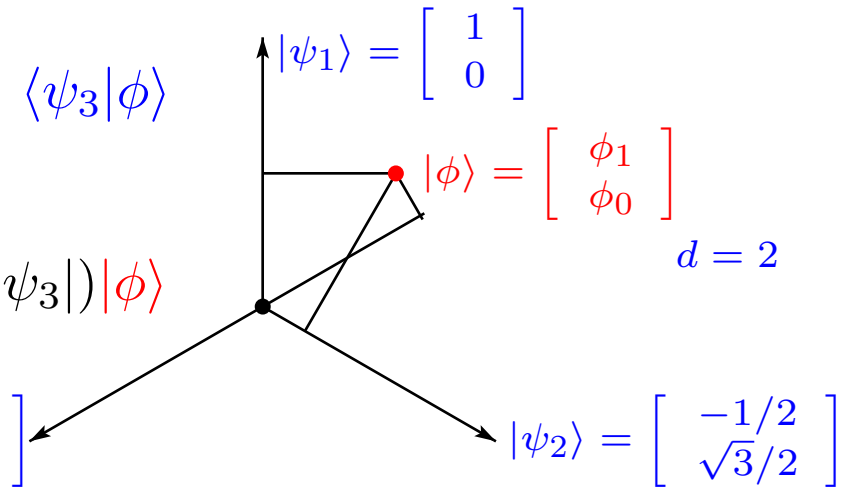
SENDING PACKETS OVER LOSSY NETWORKS

MB Example

Send $|\phi\rangle$ by sending ϕ_0 and ϕ_1

Send $|\phi\rangle$ by sending $\langle\psi_1|\phi\rangle$, $\langle\psi_2|\phi\rangle$, $\langle\psi_3|\phi\rangle$

$$|\phi\rangle = \tfrac{2}{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)|\phi\rangle$$



$$|\phi\rangle = \tfrac{2}{3}(\langle\psi_1|\phi\rangle|\psi_1\rangle + \langle\psi_2|\phi\rangle|\psi_2\rangle + \langle\psi_3|\phi\rangle|\psi_3\rangle)$$

$$|\phi\rangle = \tfrac{2}{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)|\phi\rangle$$

A COMMON MODEL
The Object of Interest

- An ensemble:
  - K d-dimensional unit-length vectors $|\psi_i\rangle$
  - K real numbers p_i such that $p_1 + \cdots + p_K = 1$.

- Two matrices:
  - F is the $d \times K$ matrix whose columns are $\sqrt{p_i}|\psi_i\rangle$. Thus

$$FF^\dagger = \sum_{i=1}^{K} p_i |\psi_i\rangle\langle\psi_i|.$$

  - $FF^\dagger$ is the density matrix (frame operator)
  - $F^\dagger F$ is the Gram matrix.

A FRAME

- An ensemble $\{\sqrt{p_i}|\psi_i\rangle\}$ of vectors in a Hilbert space with the property that there are constants $A, B \geq 0$ such that

$$A\langle\varphi|\varphi\rangle \leq \sum_i p_i |\langle\varphi|\psi_i\rangle|^2 \leq B\langle\varphi|\varphi\rangle$$

  for all $|\varphi\rangle$ in the Hilbert space.

- Equivalently,

$$AI_d \leq FF^\dagger \leq BI_d$$

A COMMON MODEL

Information Measures

- The Von Neumann entropy:

$$S = -\operatorname{Tr} FF^\dagger \log FF^\dagger$$

- The sum capacity:

$$C_{\text{sum}} = \frac{1}{2} \log[\det(I_d + d\sigma^{-2} FF^\dagger)]$$

- Both are maximized by

$$FF^\dagger = \frac{1}{d} I_d.$$

HOW TO MIX A DENSITY MATRIX

$$FF^\dagger = \sum_{i=1}^K p_i |\psi_i\rangle\langle\psi_i|$$

- Two problems:
  - classification of ensembles having a given density matrix,
  - characterization of PDs consistent with a given density matrix.

- Let $\{p_i\}$ be a PD, and $p_1 \geq p_2 \geq \cdots \geq p_K$.

- Let ρ be a density matrix, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$ its eigenvalues.

- There exist vectors $|\psi_i\rangle$ such that $\rho = \sum_{i=1}^K p_i |\psi_i\rangle\langle\psi_i|$ iff

$$\sum_{i=1}^n p_i \leq \sum_{i=1}^n \lambda_i \quad \text{for all } n < d.$$

HOW TO MIX A DENSITY MATRIX

$$FF^\dagger = \tfrac{1}{d} I_d$$

- For $\rho = \frac{1}{d} I_d$, condition

$$\sum_{i=1}^{n} p_i \leq \sum_{i=1}^{n} \lambda_i \quad \text{for all } n < d.$$

  becomes $p_1 \leq 1/d$.

- In CDMA, user i is said to be oversized if

$$p_i > \frac{\sum_{j=i+1}^{K} p_j}{d-i}$$

INTERFERENCE MEASURE IN CDMA

- Total Square Correlation: $\mathrm{TSC} = \sum_{i=1}^{K} \sum_{j=1}^{K} p_i p_j |\langle \psi_i | \psi_j \rangle|^2$.
- The Welch's lower bound to TSC:

$$\mathrm{TSC} \geq \frac{1}{d}$$

  with equality iff $\sum_{i=1}^{K} p_i |\psi_i\rangle\langle\psi_i| = \frac{1}{d} I_d$.

- WBE sequences
  - minimize the TSC
  - maximize the sum capacity and Von Neumann entropy
- What does reducing TSC mean?

$\frac{1}{d} I_d$ IN OTHER COMMUNITIES

- A frame is an ensemble $\{\sqrt{p_i}|\psi_i\rangle\}$ of vectors in a Hilbert space with the property that there are constants $A, B \geq 0$ such that

$$AI_d \leq FF^\dagger \leq BI_d$$

- When $A = B = 1/d$, we have a tight frame.

- In multiple description coding, tight frames are of interest because they offer computational and performance benefits.

- Naimark's Theorm.

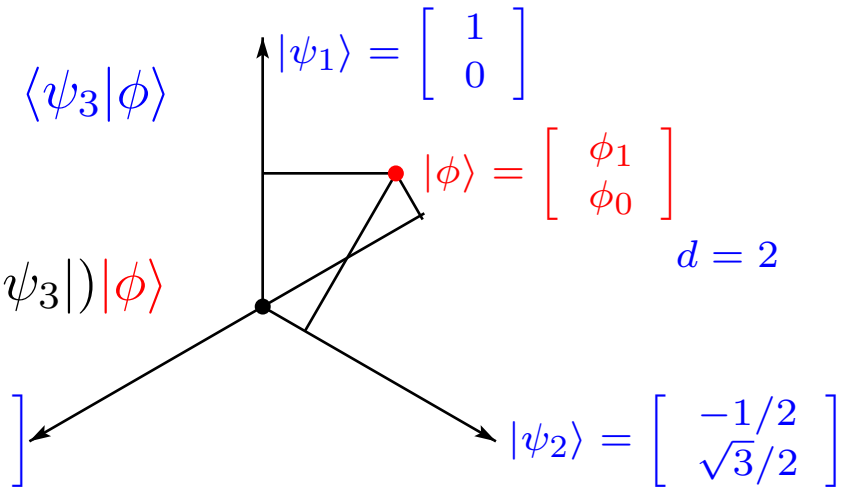
TRANSMISSION OVER LOSSY NETWORKS

MB Example

Send $|\phi\rangle$ by sending ϕ_0 and ϕ_1

Send $|\phi\rangle$ by sending $\langle\psi_1|\phi\rangle$, $\langle\psi_2|\phi\rangle$, $\langle\psi_3|\phi\rangle$

$$|\phi\rangle = \tfrac{2}{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)|\phi\rangle$$



$$|\phi\rangle = \tfrac{2}{3}(\langle\psi_1|\phi\rangle|\psi_1\rangle + \langle\psi_2|\phi\rangle|\psi_2\rangle + \langle\psi_3|\phi\rangle|\psi_3\rangle)$$
$$|\phi\rangle = \tfrac{2}{3}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)|\phi\rangle$$

TSC IN OTHER COMMUNITIES

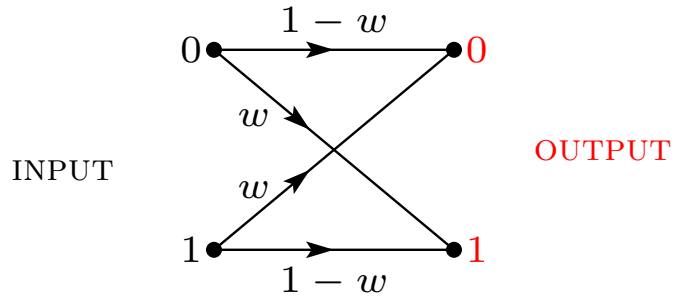
- Frame potential:

$$\sum_{i=1}^{K}\sum_{j=1}^{K} p_i p_j |\langle\psi_i|\psi_j\rangle|^2$$

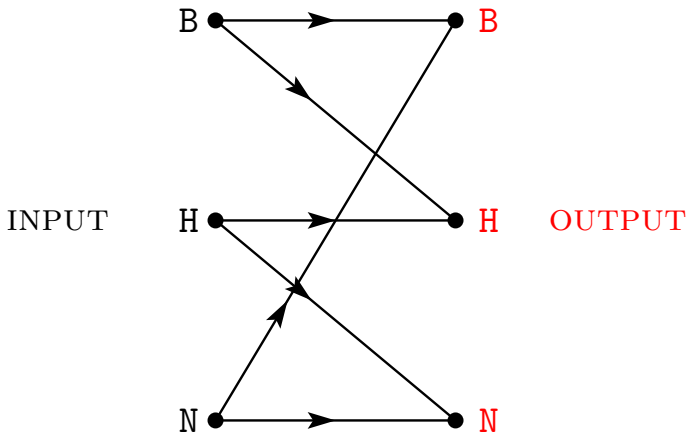
- Tight frames minimize the frame potential.
- How to mix a density matrix with no majorization?
- TSC in QC transmission, spherical codes.

SOME COMMUNICATION CHANNELS

- Binary Symmetric Channel:



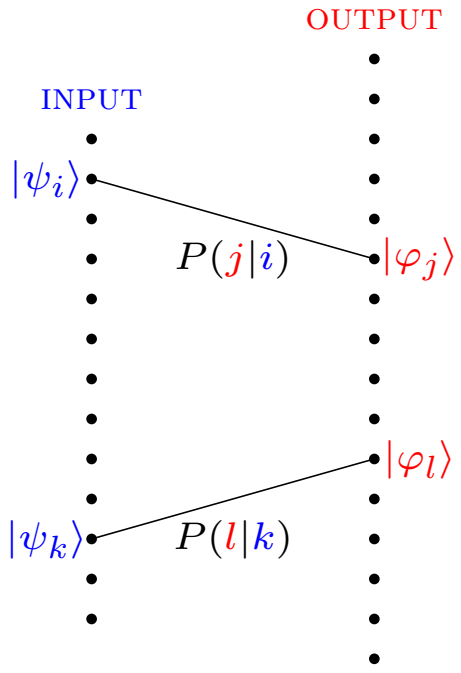
- Noisy Typewriter:



A CQ COMMUNICATION CHANNEL

A Probabilistic Device

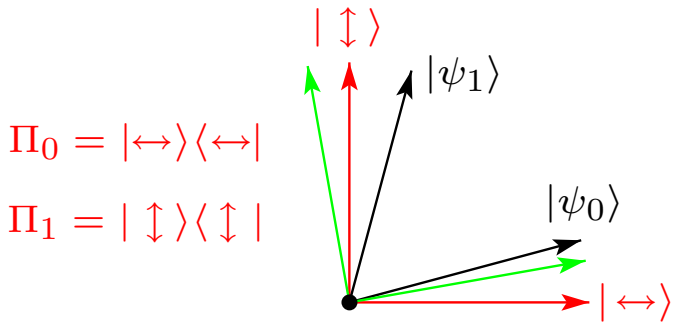
- Inputs: vectors $|\psi_i\rangle$, $i \in \mathcal{X}$.
- Outputs: vectors $|\varphi_j\rangle$ determined by the chosen measurement.
- Transition probabilities determined by the chosen measurement.

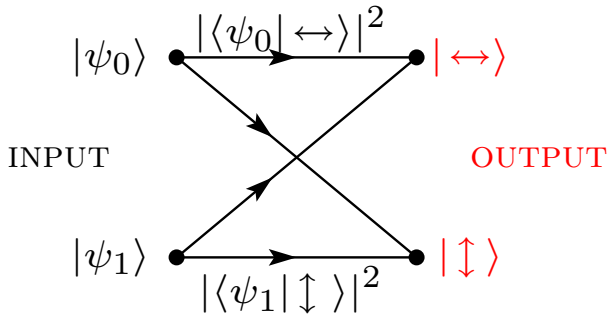


QUANTUM MEASUREMENT

Von Neumann's Measurement

- A set of pairwise orthogonal projection operators $\{\Pi_i\}$.
- They form a complete resolution of the identity: $\sum_i \Pi_i = I$.
- For input $|\psi_j\rangle$, output $\Pi_i|\psi_j\rangle$ happens with probability $\langle\psi_j|\Pi_i|\psi_j\rangle$.
- Example:

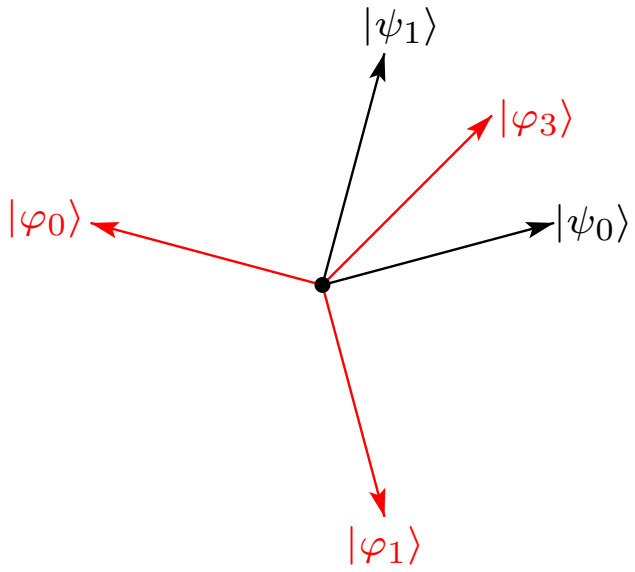


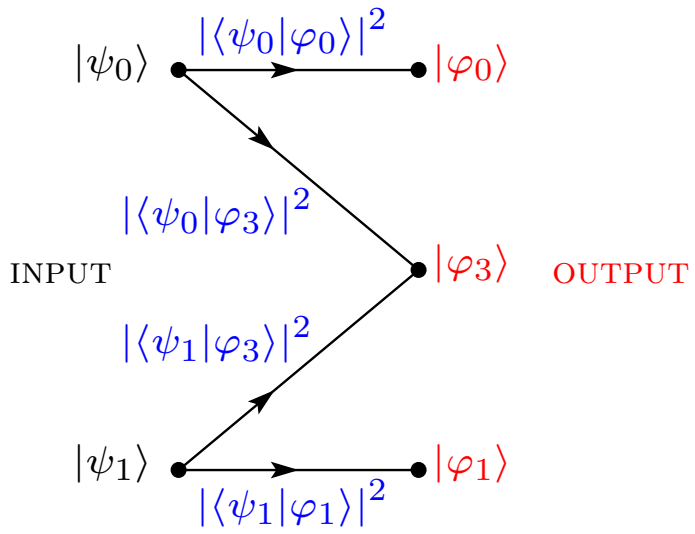


QUANTUM MEASUREMENT

Positive Operator-Valued Measure

- Any set of positive-semidefinite operators $\{E_i\}$.
- They form a complete resolution of the identity: $\sum_i E_i = I$.
- For input $|\psi_j\rangle$, output $E_i|\psi_j\rangle$ happens with probability $\langle\psi_j|E_i|\psi_j\rangle$.
- Example:





OPTIMAL QUANTUM MEASUREMENTS

Some Open Problems

- POVMs minimizing the detection error-probability.
- POVMs attaining the accessible information (number of elements)
- Example:

$$|\widetilde{\psi_1}\rangle = \begin{bmatrix} \sqrt{1-\alpha} \\ 0 \\ \sqrt{\alpha} \end{bmatrix}$$

$$|\psi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\widetilde{\psi_3}\rangle = \begin{bmatrix} -\sqrt{1-\alpha}/2 \\ -\sqrt{3}\sqrt{1-\alpha}/2 \\ \sqrt{\alpha} \end{bmatrix}$$

$$|\psi_3\rangle = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix}$$

$$|\widetilde{\psi_2}\rangle = \begin{bmatrix} -\sqrt{1-\alpha}/2 \\ \sqrt{3}\sqrt{1-\alpha}/2 \\ \sqrt{\alpha} \end{bmatrix}$$

$$|\psi_2\rangle = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

DISTANCES BETWEEN DENSITY MATRICES

ρ and σ

- Trace distance:

$$D(\sigma,\omega) = \frac{1}{2}\,\mathrm{Tr}\,|\sigma-\omega|,$$

  $|A|$ denotes the positive square root of $A^\dagger A$.

- Uhlman Fidelity:

$$F(\sigma,\omega) = \left\{\mathrm{Tr}\left[(\sqrt{\sigma}\omega\sqrt{\sigma})^{1/2}\right]\right\}^2.$$

- $1 - F(\sigma,\omega) \leq D(\sigma,\omega) \leq \sqrt{1-F(\sigma,\omega)^2}$

- Frobenius (Hilbert-Schmidt)?

DISTANCES BETWEEN PD'S

An Example

- $\mathcal{A}_N = \{a_1, \ldots, a_N\}$
- $N = 2^K$ and $n = 2^k$, with $k/K = c < 1$.
- Distributions P and Q:

$$P(a_i) = \frac{1}{N} Q(a_i) = \begin{cases} 1/n, & 1 \le i \le n \\ 0 & n+1 \le i \le N \end{cases}$$

- $P(\{a_{n+1}, \ldots, a_N\}) \to 1$ as $k, K \to \infty$.
- $\frac{1}{2} \sum_i |P(a_i) - Q(a_i)| \to 1$ and $\sum_i |P(a_i) - Q(a_i)|^2 \to 0$.

QUANTUM DISCRETE MEMORYLESS SOURCE

Vector Sequences

- Source vector-sequence (state)

$$|\Psi_{\boldsymbol{x}}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle, \qquad x_i \in \mathcal{X},$$

  appears with probability $P_{\boldsymbol{x}} = P_{x_1} \cdot P_{x_2} \cdot \ldots \cdot P_{x_n}$.

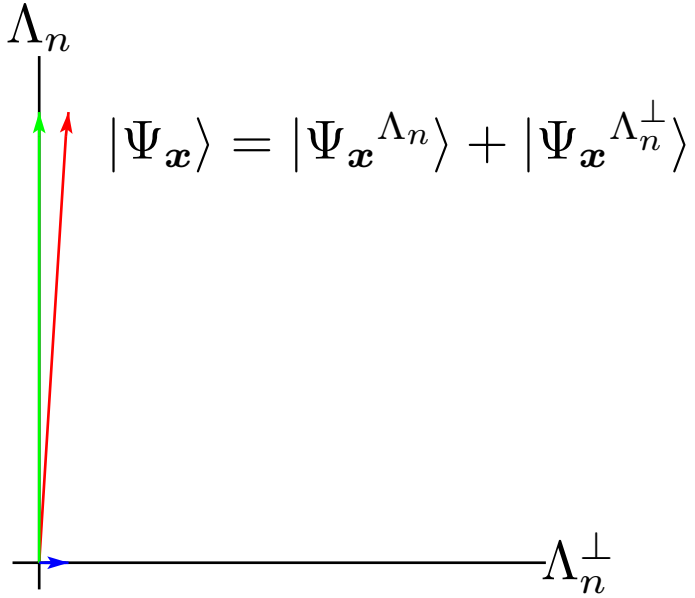
- Typical states $|\Psi_{\boldsymbol{x}}\rangle \in \mathcal{H}^{2^n}$ correspond to typical sequences $\boldsymbol{x}$.

- There are approximately $2^{nH(P)}$ typical states.

QUANTUM DISCRETE MEMORYLESS SOURCE

Typical Subspace

- Typical states $|\Psi_{\boldsymbol{x}}\rangle \in \mathcal{H}^{2^n}$ "live" in the typical subspace.
- Typical subspace Λ_n of $\mathcal{H}^{2^n}$:



- The dimension of Λ_n is approximately $2^{nS(\rho)}$.

ASYMPTOTIC PROPERTIES OF $\rho^{\otimes n}$

- We represent the source density matrix

$$\rho = \sum_{a \in \mathcal{X}} P(a) |\psi_a\rangle\langle\psi_a|$$

in terms of its eigenvectors and eigenvalues:

$$\rho = \lambda_0 |\varphi_0\rangle\langle\varphi_0| + \lambda_1 |\varphi_1\rangle\langle\varphi_1|.$$

- Note that $\lambda = \{\lambda_0, \lambda_1\}$ is a PD on $\{0, 1\}$ and $\langle\varphi_0|\varphi_1\rangle = 0$.

- Consider

$$\rho^{\otimes n} = \sum_{\boldsymbol{x} \in \mathcal{X}^n} P(\boldsymbol{x}) |\Psi_{\boldsymbol{x}}\rangle\langle\Psi_{\boldsymbol{x}}| = \sum_{\boldsymbol{z} \in \{0,1\}^n} \lambda(\boldsymbol{z}) |\Phi_{\boldsymbol{z}}\rangle\langle\Phi_{\boldsymbol{z}}|.$$

ASYMPTOTIC PROPERTIES OF $\rho^{\otimes n}$

- Consider two density matrices:

$$\rho_\lambda = \sum_{\boldsymbol{z} \in \mathsf{T}_\lambda^n} \lambda(\boldsymbol{z}) |\Phi_{\boldsymbol{z}}\rangle\langle\Phi_{\boldsymbol{z}}| \text{ and } \rho_P = \sum_{\boldsymbol{x} \in \mathsf{T}_P^n} P(\boldsymbol{x}) |\Psi_{\boldsymbol{x}}\rangle\langle\Psi_{\boldsymbol{x}}|.$$

- Claim: $D(\rho_\lambda, \rho_P) \to 0$ as $n \to \infty$.

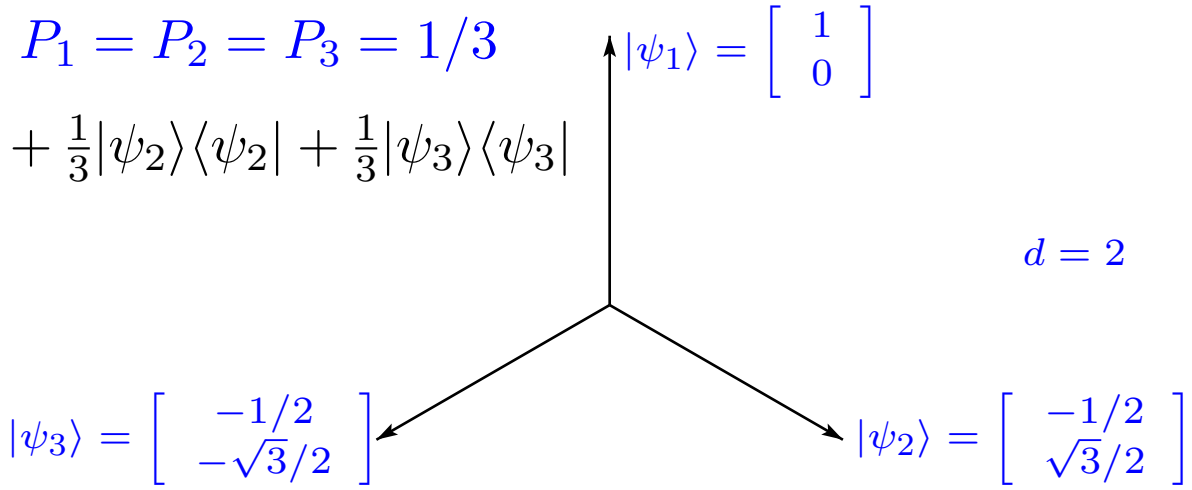
- Claim: There exist a code $\mathcal{C}_n \subseteq \mathcal{X}^n$, $|\mathcal{C}_n| = 2^{n(S(\rho) - \varepsilon_n)}$ s.t.

$$D(\sum_{\boldsymbol{x} \in \mathcal{C}_n} P(\boldsymbol{x}) |\Psi_{\boldsymbol{x}}\rangle\langle\Psi_{\boldsymbol{x}}|, I) \to 0 \text{ as } n \to \infty.$$

COMBINATORICS AND GEOMETRY

The MB Example

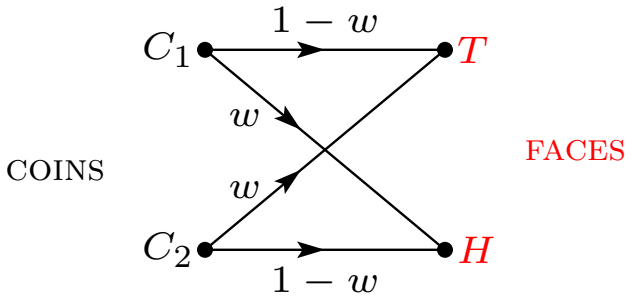
$$\begin{aligned} \mathcal{X} &= \{1,2,3\} \quad P_1 = P_2 = P_3 = 1/3 \\ \rho &= \tfrac{1}{3}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{3}|\psi_2\rangle\langle\psi_2| + \tfrac{1}{3}|\psi_3\rangle\langle\psi_3| \\ &= \tfrac{1}{2}I \\ S(\rho) &= 1 \end{aligned}$$



- There are 3^n typical vectors forming a frame in H^{2^n}.
- About 2^n of those vectors form an orthogonal basis of H^{2^n}.
- Which ones?

SOURCES OF MIXED STATES

- Produce sequences of sources, $e.g.$, coins:



- Example: $P(C_1) = P(C_2) = 1/2$ and $w = 2/3$.

- Sequences:

C_1	C_1	C_2	C_1	C_2	C_2
H	H	H	T	T	T
H	T	H	H	T	T
H	T	T	H	T	H
$\vdots$					

CORRELATED SOURCES

Description

- Source 1 produces a letter $x \in \mathcal{X}$ with probability p_x.
- Given that source 1 produced letter x,
  source 2 produces a letter $y \in \mathcal{Y}$ with probability w_{xy}.
- Letters x are mapped into states $|\psi_x\rangle\langle\psi_x|$.
- Letters y are mapped into states $|\psi_y\rangle\langle\psi_y|$.
- Given that source 1 produced letter x,
  the density matrix of source 2 is

$$\rho_x = \sum_{y \in \mathcal{Y}} w_{xy} |\psi_y\rangle\langle\psi_y|.$$

CORRELATED SOURCES

Problems

- CQ transmission:
  If we have access to quantum outputs of source 2,
  how many $\boldsymbol{x} \in \mathcal{X}^n$ can we tell apart?

- Compressing mixed state sources:
  If we have access to classical outputs of source 1,
  how can we emulate quantum source 2 at a distant location?

- Slepian-Wolf family of problems