NETWORK CODING
from Graph Theory to Algebraic Geometry

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CREDITS AND ACKNOWLEDGEMENTS

- Joint work with

Christina Fragouli, EPFL

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  Ellen Eischen, University of Michigan

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NSF

COMMUNICATION NETWORK

A Mathematical Model

- A network is represented as a directed acyclic graph $G = (V, E)$.
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- There are N receivers $R_1, \ldots, R_N$ located at N distinct nodes.
- Can all sources simultaneously transmit information to all receivers? (We are interested in the transmission at full rate.)

NETWORK MULTICAST

Rates

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  Yes, if in addition each node of G can re-encode information.

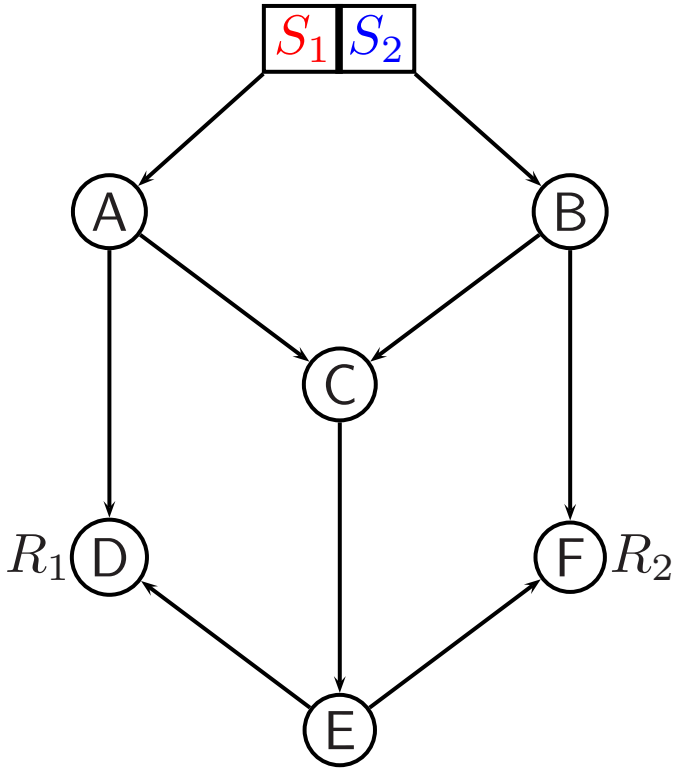
  [Alshwede, Cai, Li, Yeung] $\sim$ 2000

NETWORK MULTICAST

Example 1

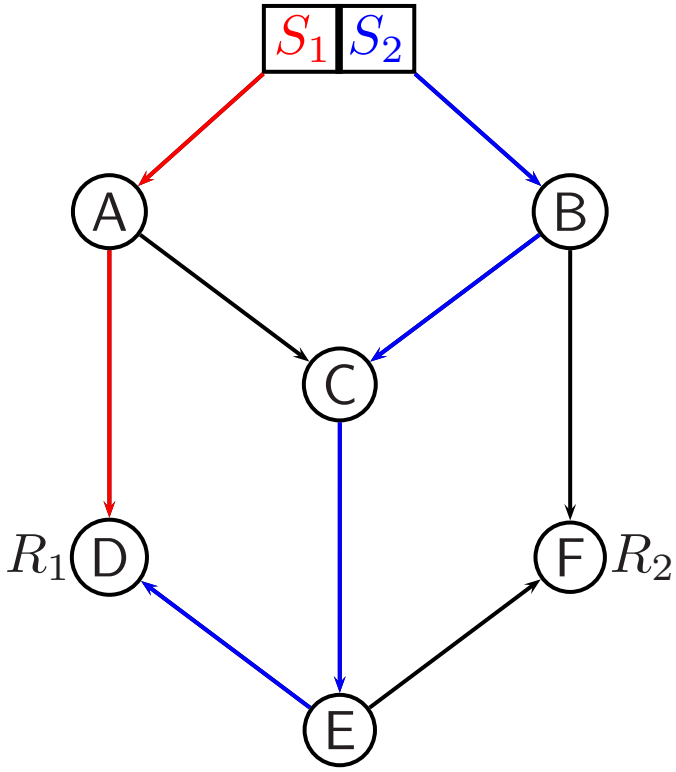
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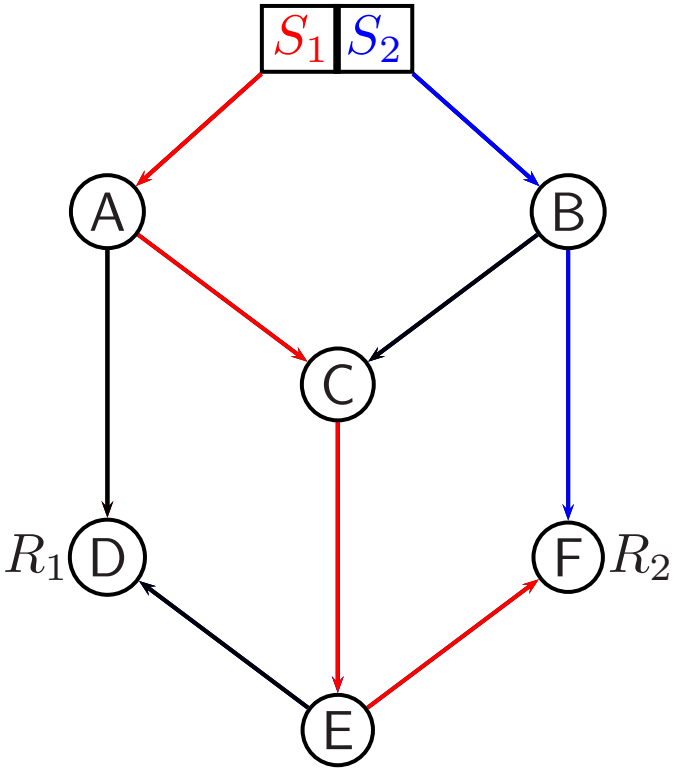
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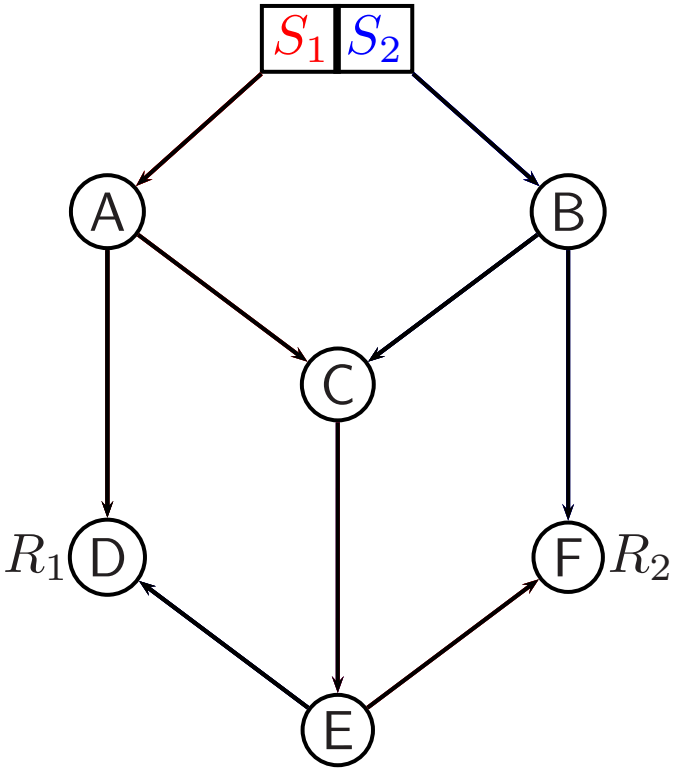
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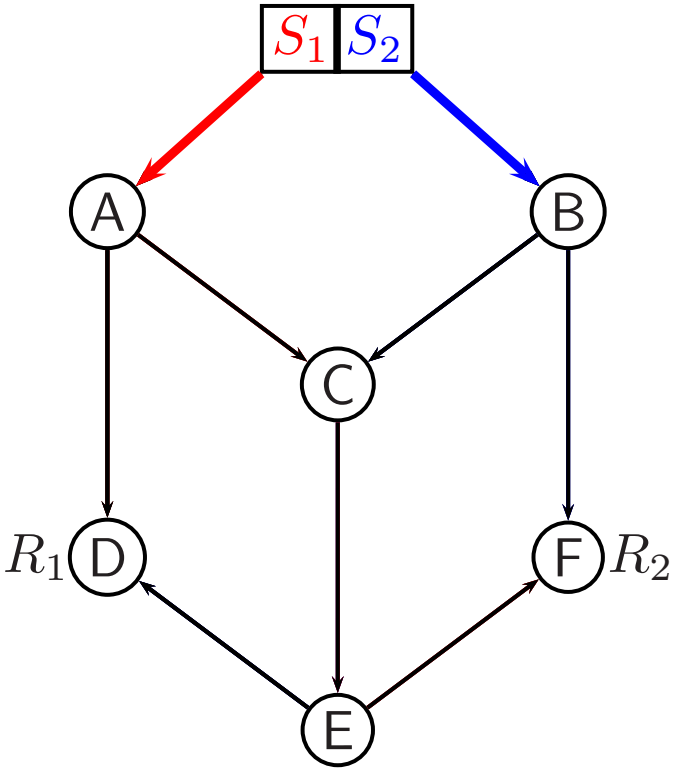
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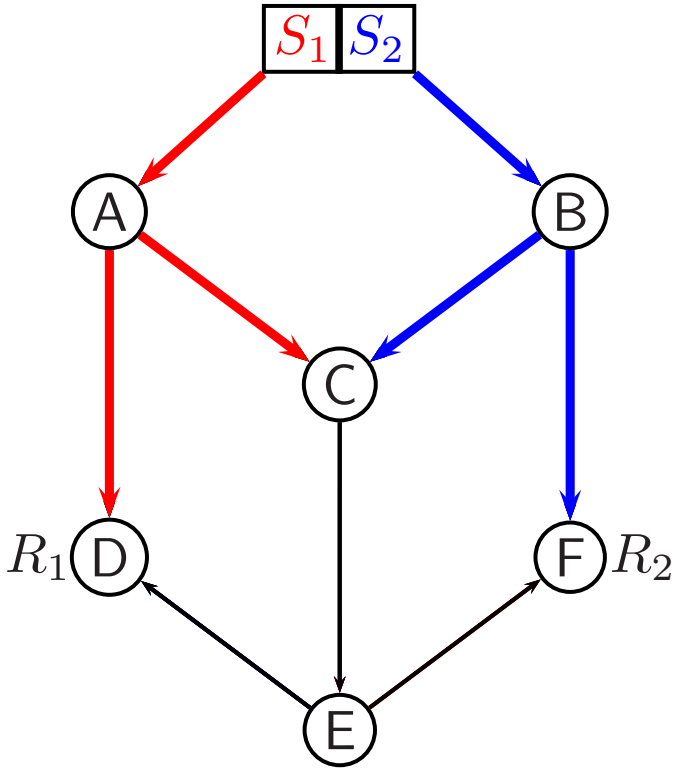
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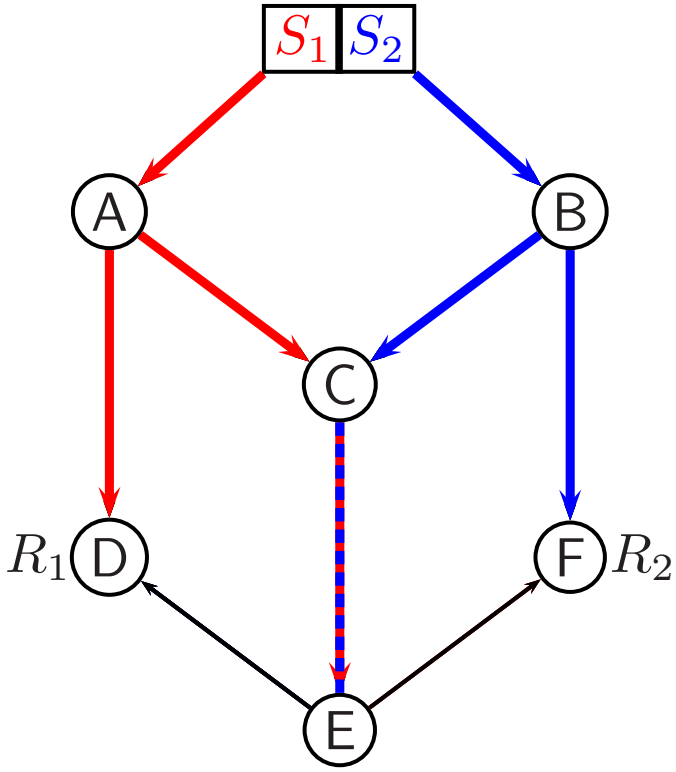
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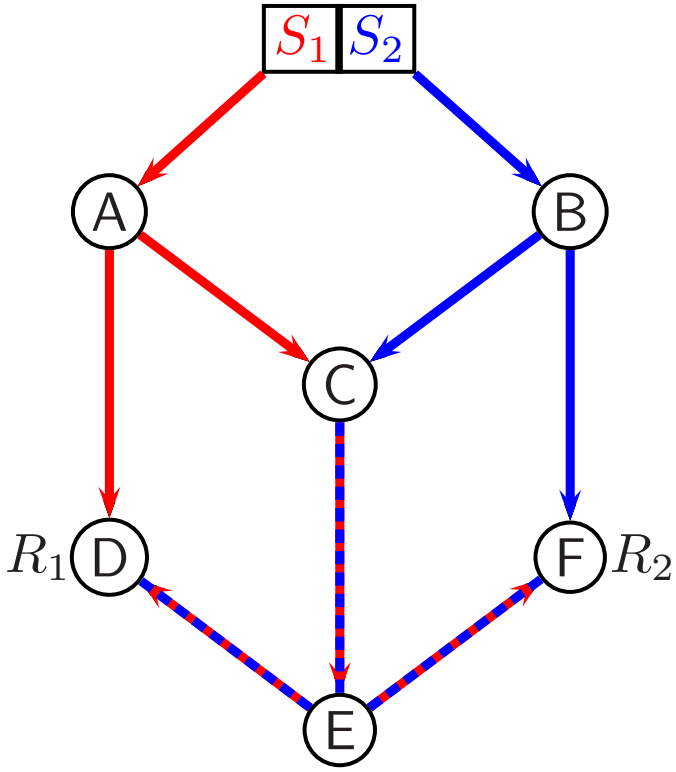
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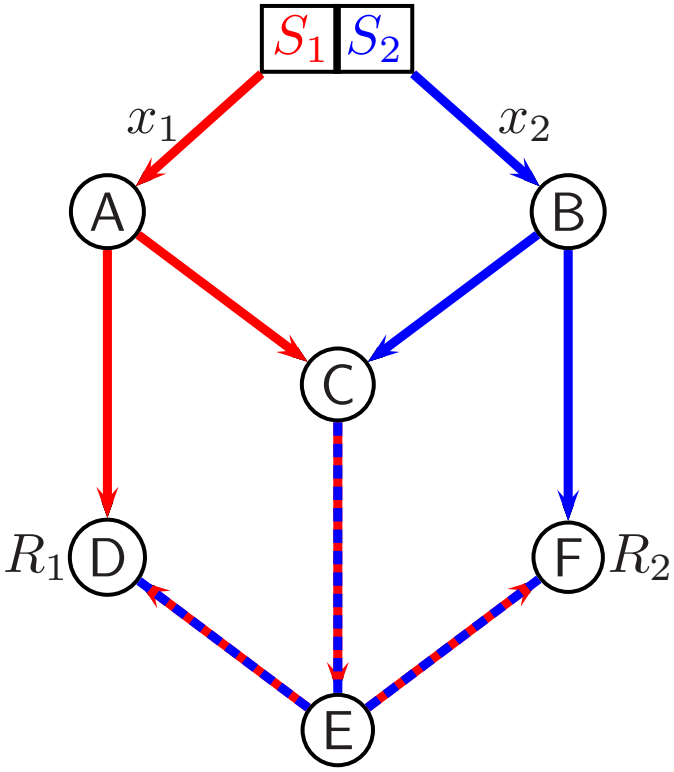
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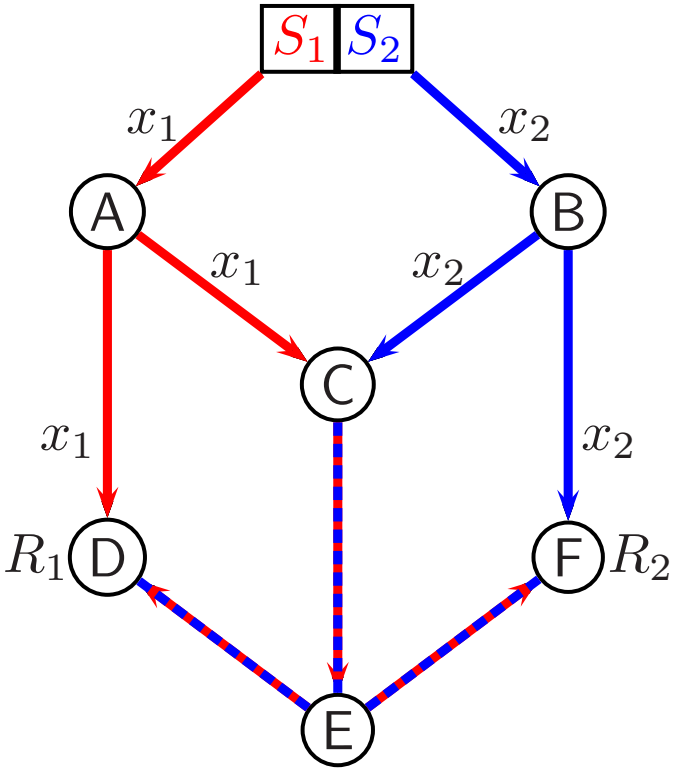
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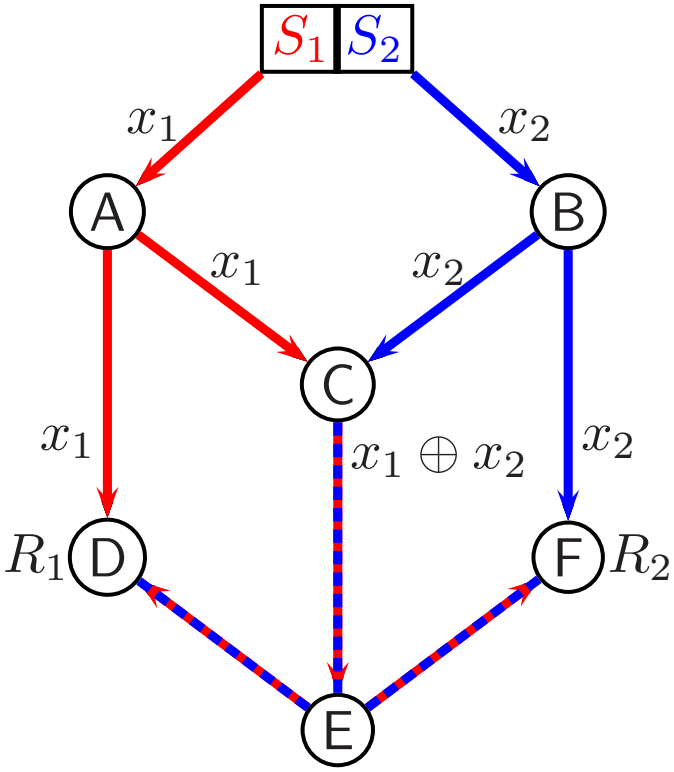
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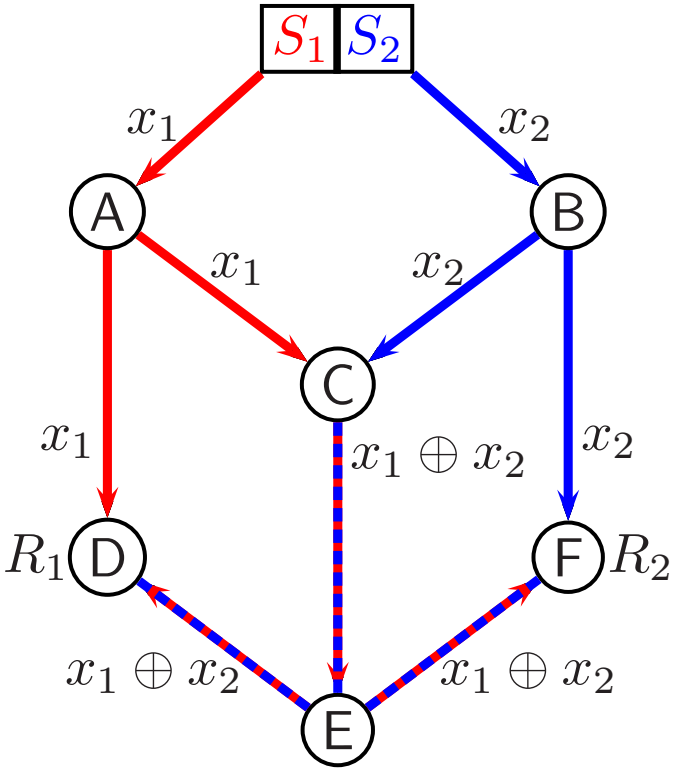
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CODES FOR NETWORK MULTICAST

Previous Work on the Basic Model

- An algebraic framework for network coding. [Koetter, Medard]
- Multicast by linear coding. [Li, Yueng, Cai]
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- Bounds on linear codes. [Tavory, Feder, Ron]

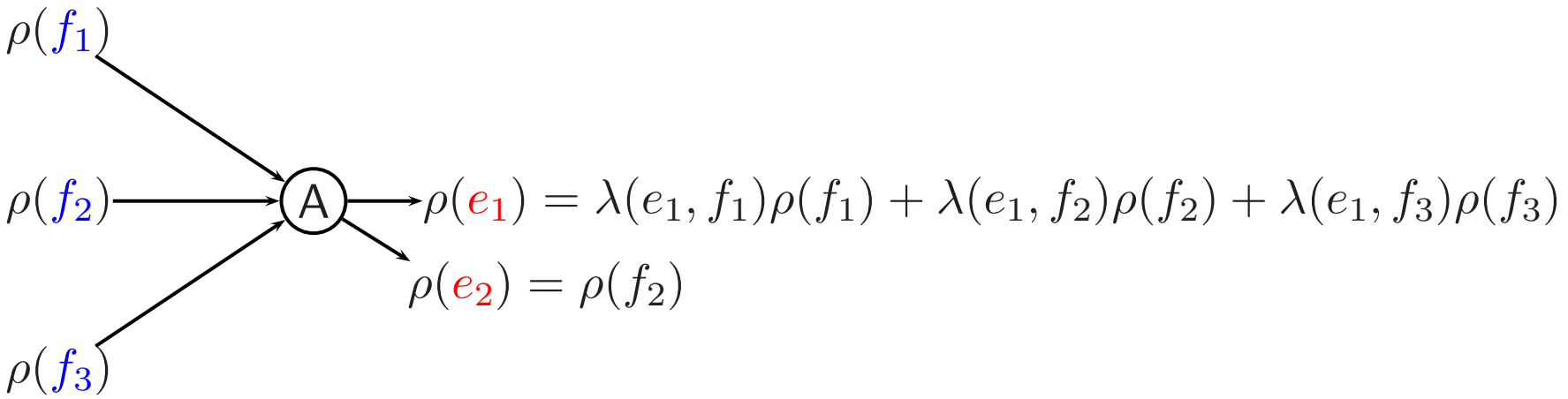
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- Bounds on linear codes. [Tavory, Feder, Ron]
- Decentralized design of linear codes. [Fragouli, Soljanin]

NETWORK MULTICAST
Linear Combining

- Source S_i emits σ_i which is an element of some finite field $\mathbb{F}_q$.
- Each node of G
  - receives an element of $\mathbb{F}_q$ from each input edge, and then
  - forwards a linear combination of its inputs to each output edge.



NETWORK MULTICAST

Coding Vector

- Each edge carries a linear combination of its parent node inputs.

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- Each edge carries a linear combination of its parent node inputs.
- Each edge e carries a linear combination of source symbols:

$$c_1(e)\sigma_1 + \cdots + c_h(e)\sigma_h = \left[c_1(e) \ldots c_h(e)\right] \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_h \end{bmatrix}$$

- $[c_1(e) \ldots c_h(e)] \in \mathbb{F}_q^n$ is the coding vector of edge e.

NETWORK MULTICAST

Decoding for Receiver j

- ρ_i^j is the symbol on the last edge of the path (S_i, R_j).
- $\boldsymbol{c}_i^j$ is the coding vector of the last edge on the path (S_i, R_j).
- $\mathbf{C}_j$ is the matrix whose i-th row is $\boldsymbol{c}_i^j$.

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- $\mathbf{C}_j$ is the matrix whose i-th row is $\boldsymbol{c}_i^j$.
- The receiver j has to solve the following system of linear equations:

$$\begin{bmatrix} \rho_1^j \\ \vdots \\ \rho_h^j \end{bmatrix} = \mathbf{C}_j \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_h \end{bmatrix}.$$

NETWORK MULTICAST
The Code Design Problem

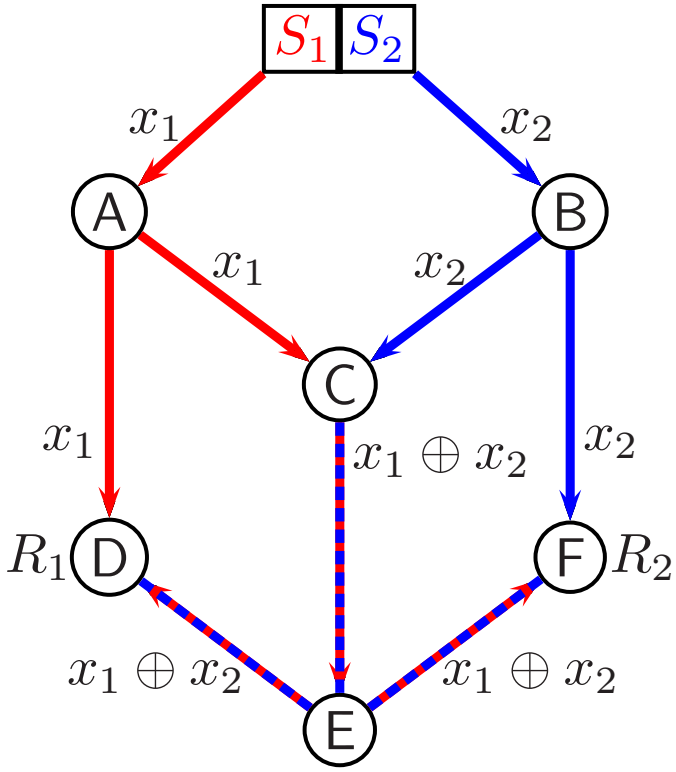
Select a coding vector for each edge e of the network so that

1. the matrix $\mathbf{C}_j$ is full rank for each receiver j and

2. the coding vector of e is in the linear span of the coding vectors of the input edges to the parent node of e.

NETWORK MULTICAST

Example 1



CODING VECTORS

Networks with Two Sources

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- Let $\mathcal{A}$ be the following set of vectors:

$$[0\,1],\ [1\,0],\ \text{and } [1\,\alpha^i] \text{ for } 0 \le i \le q-2,$$

where α is a primitive element of $\mathbb{F}_q$.

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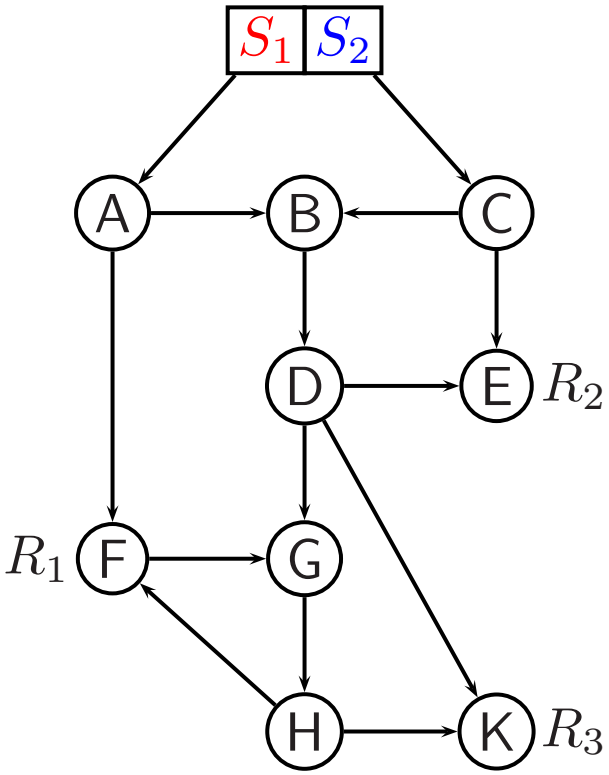
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- Consider any two different vectors in $\mathcal{A}$:
  - they are linearly independent, and
  - any vector in $\mathcal{A}$ is in their linear span.

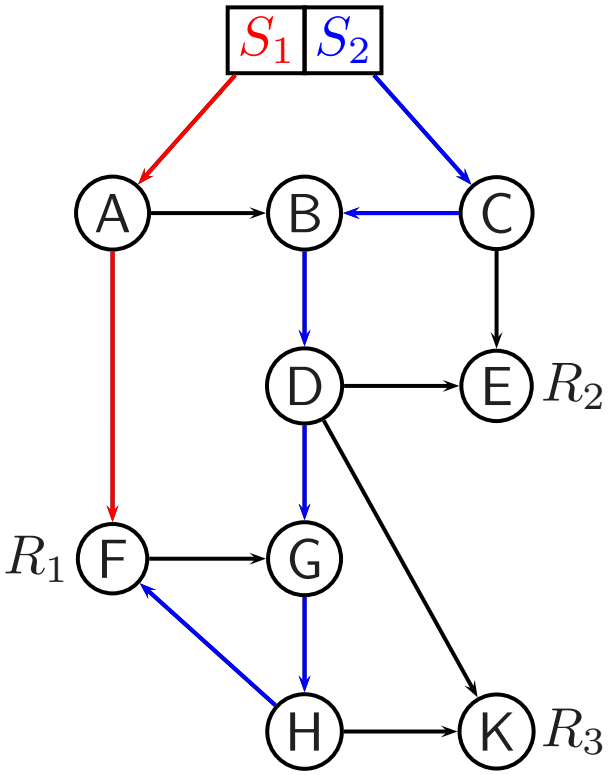
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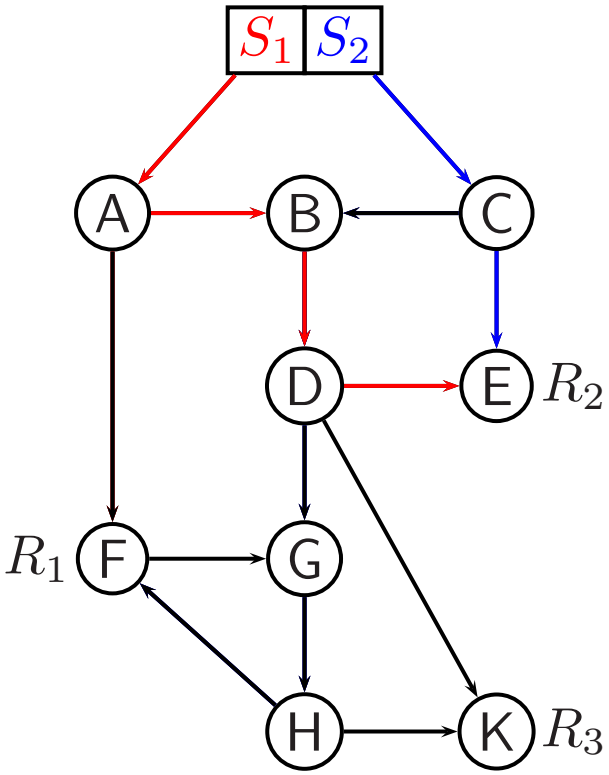
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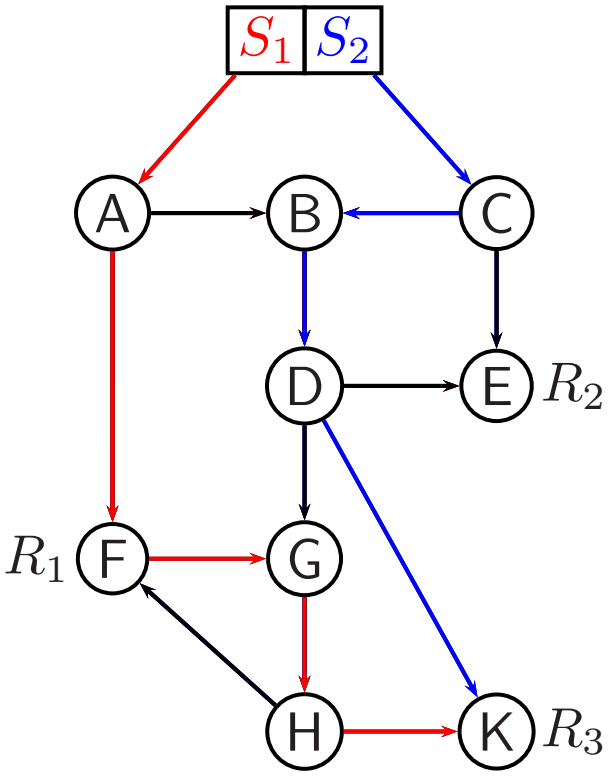
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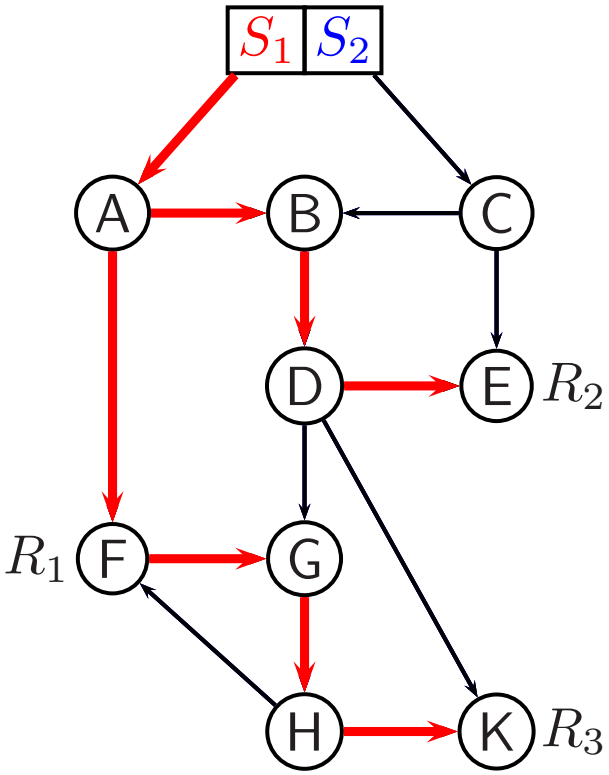
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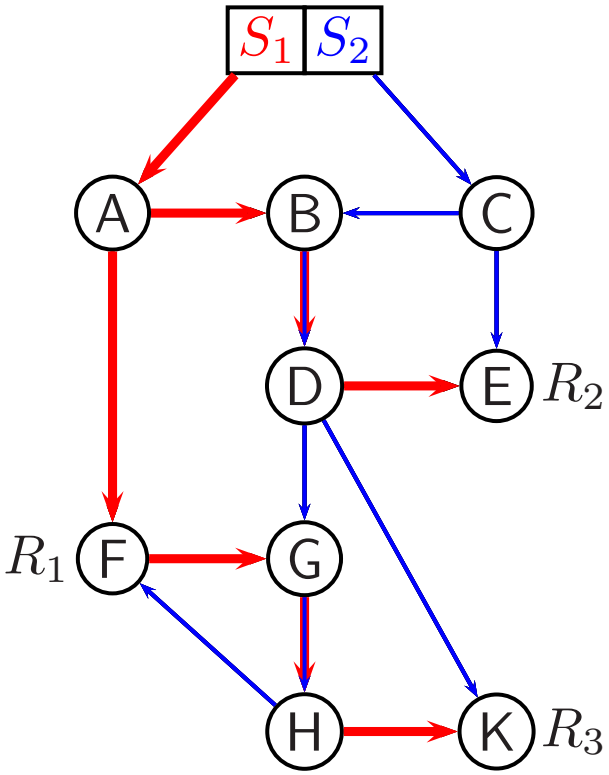
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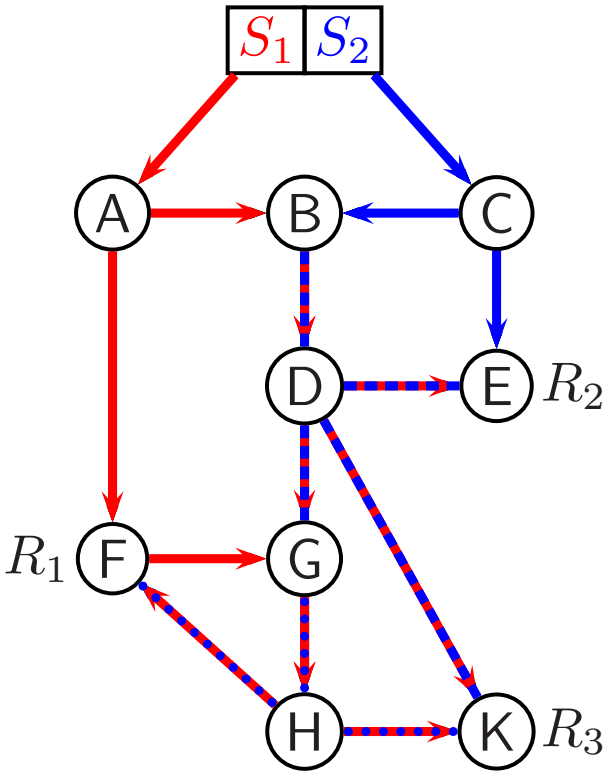
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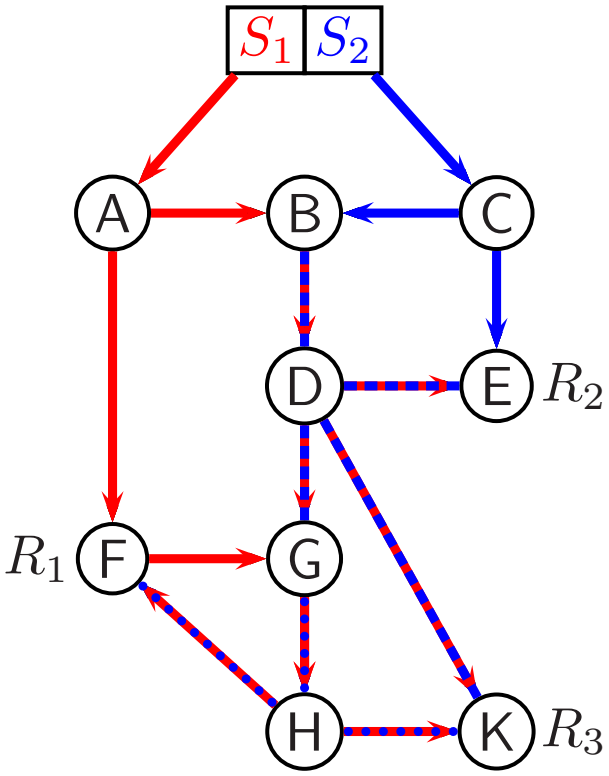
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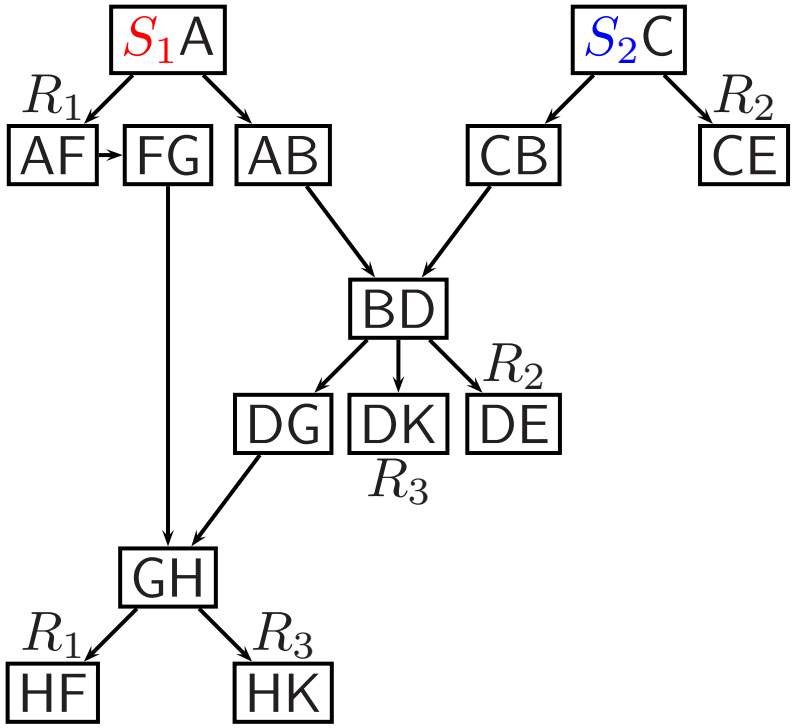
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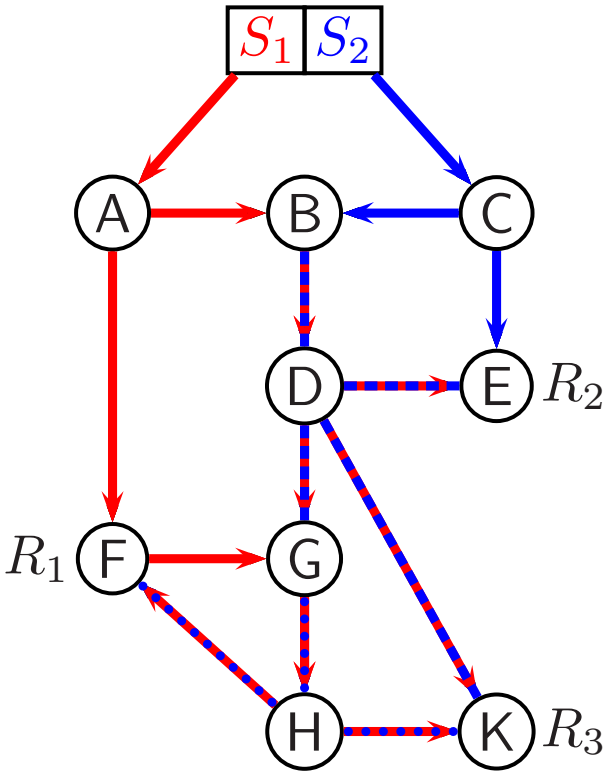
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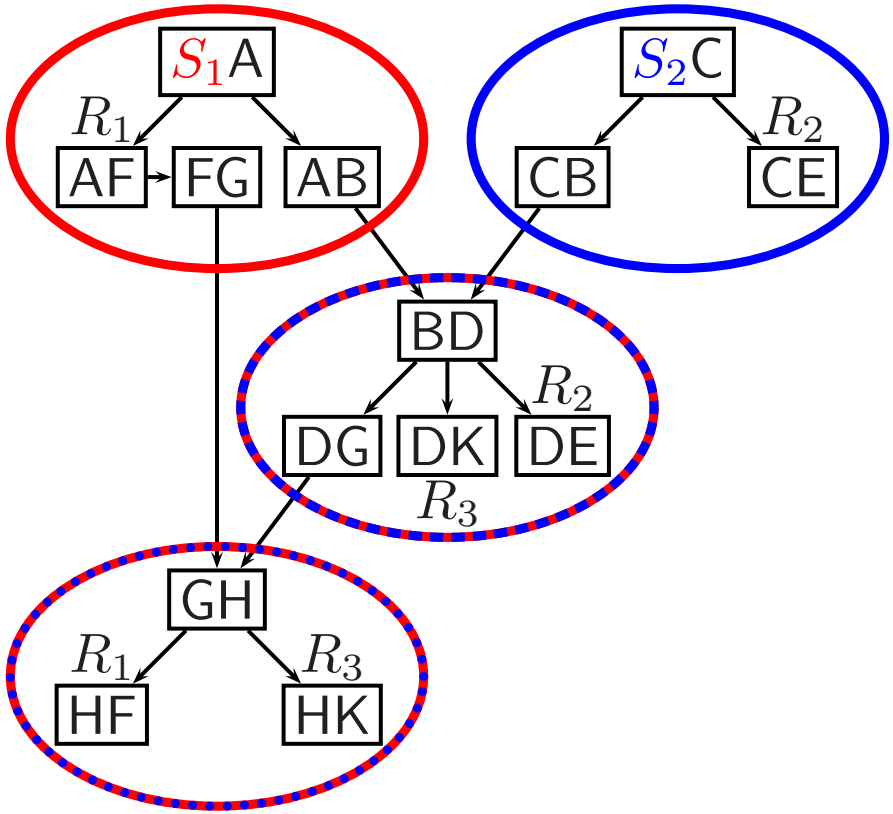




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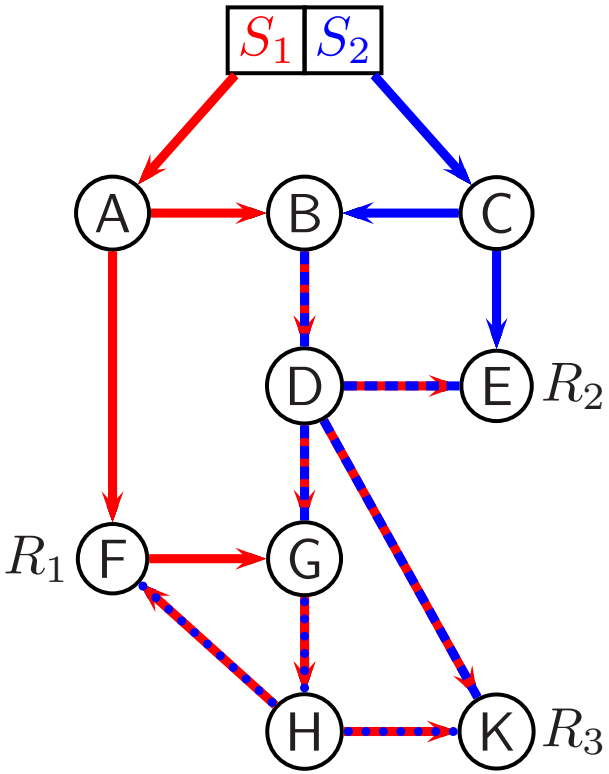
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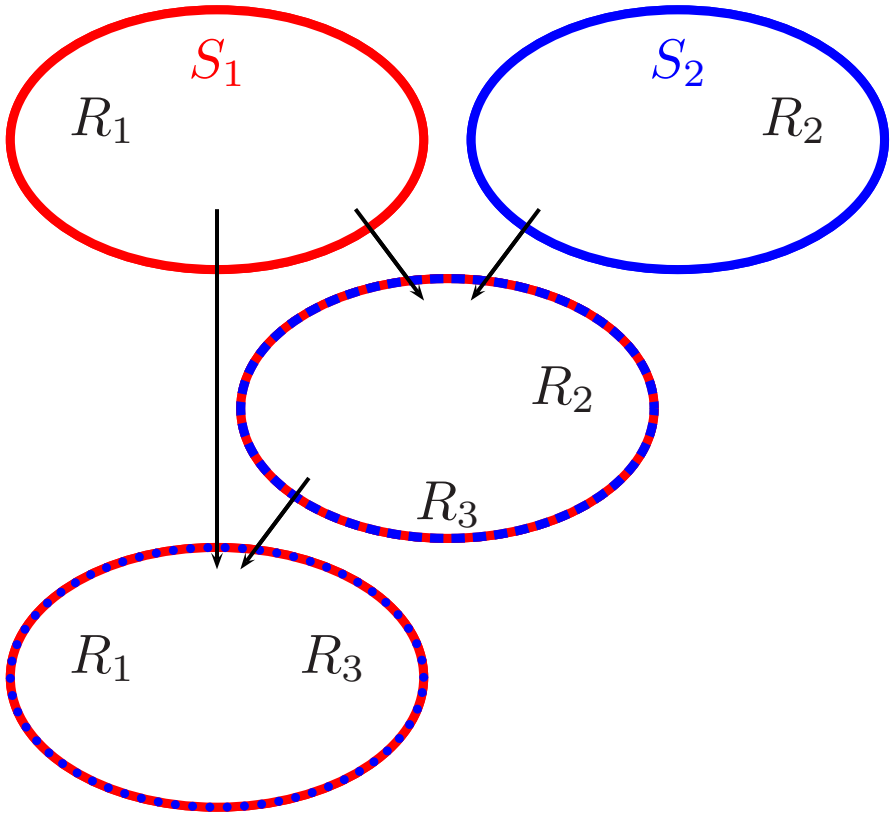




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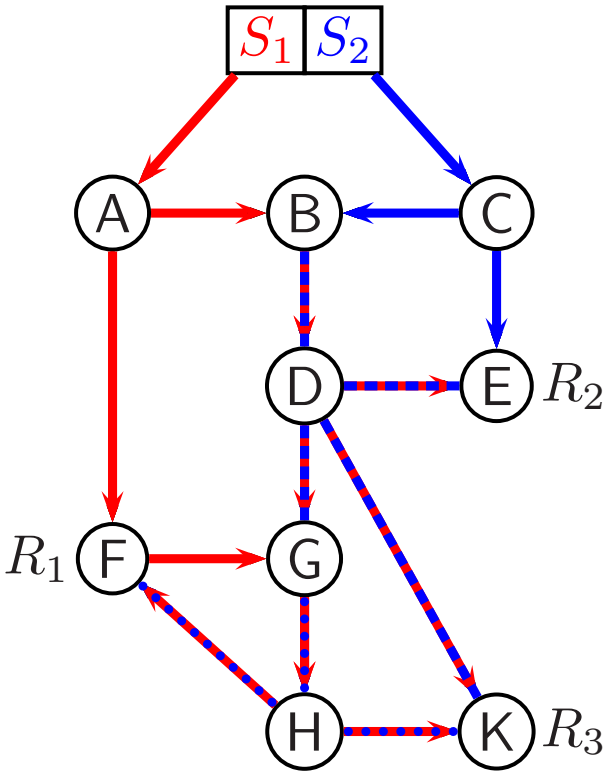
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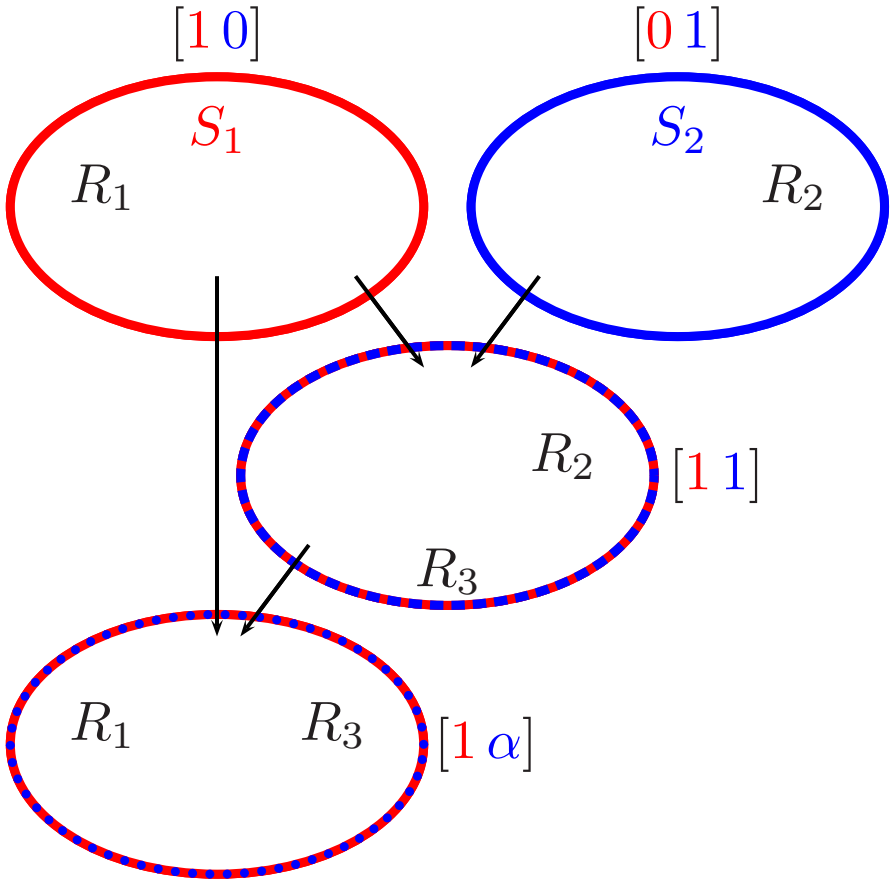




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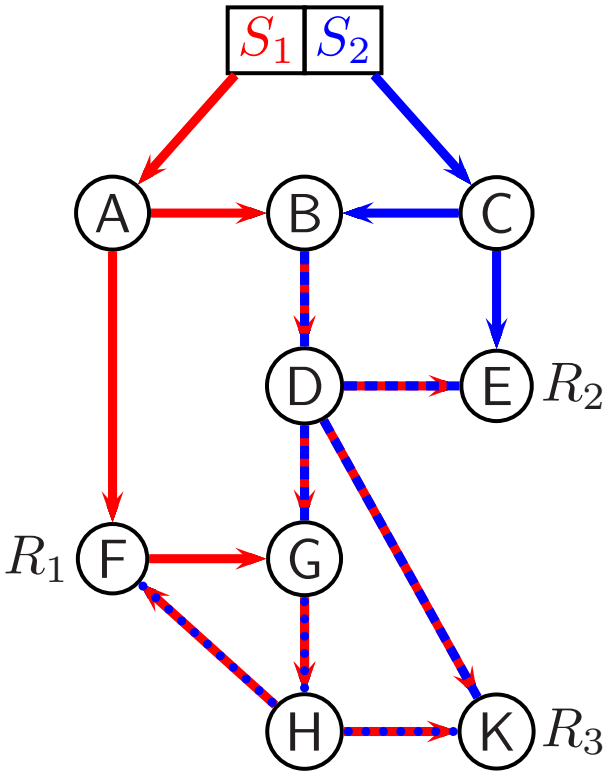
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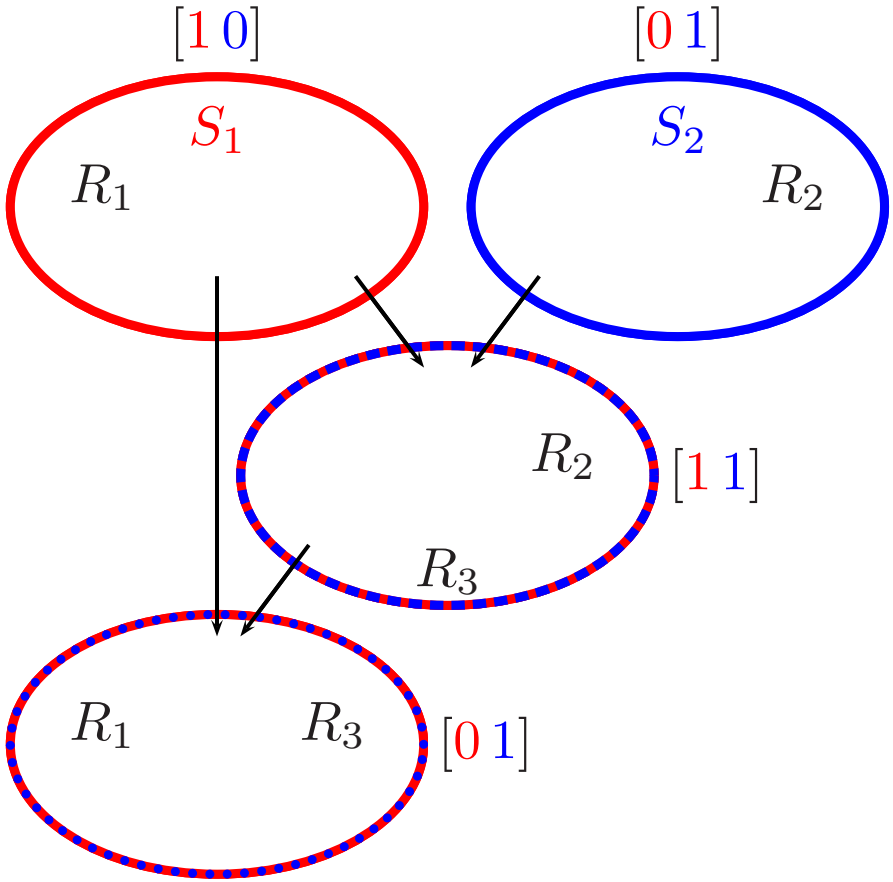




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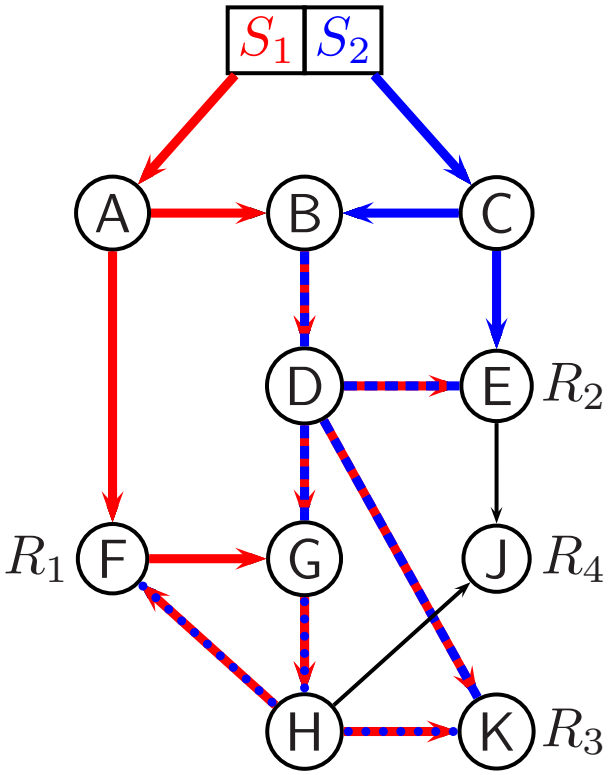
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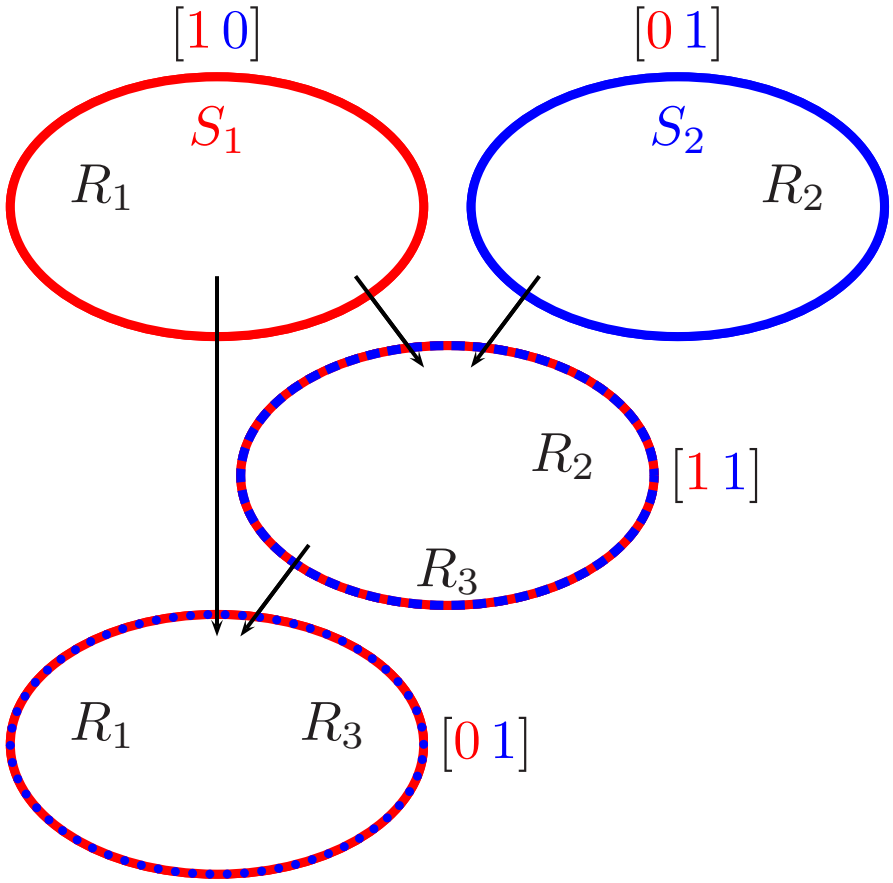




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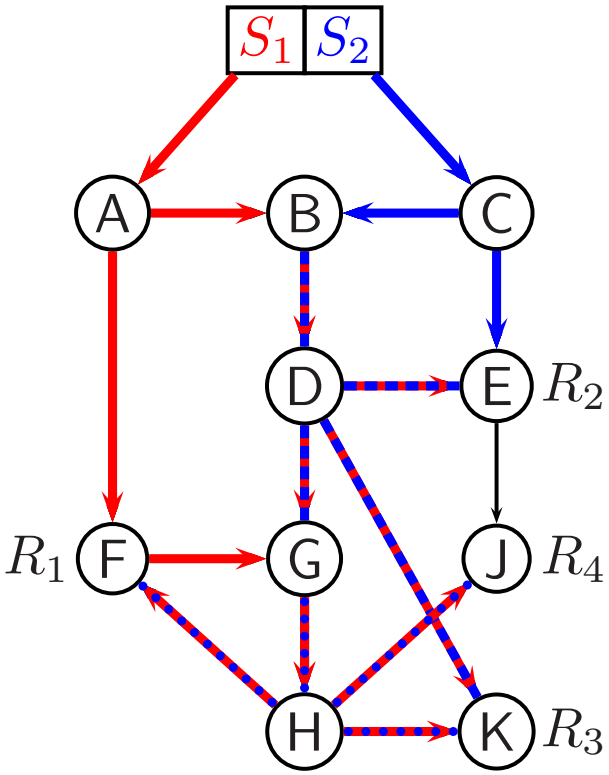
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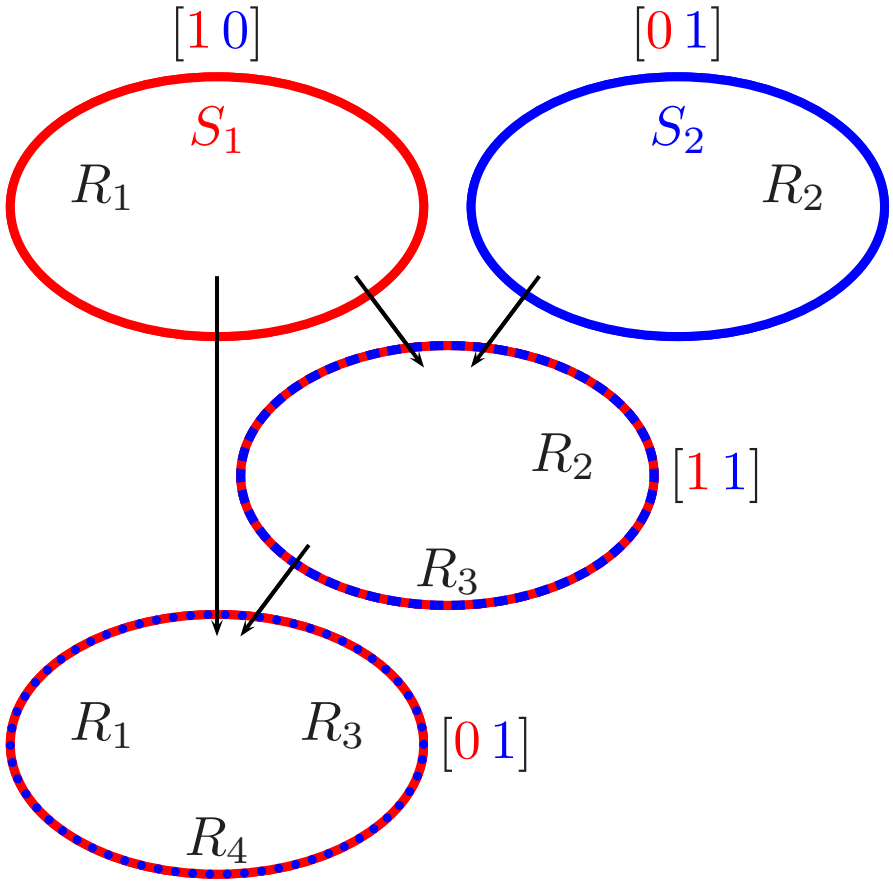




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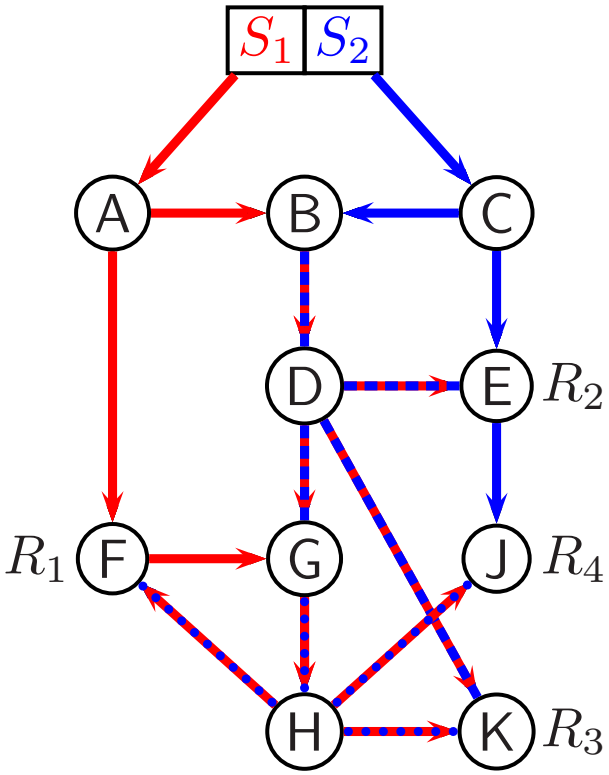
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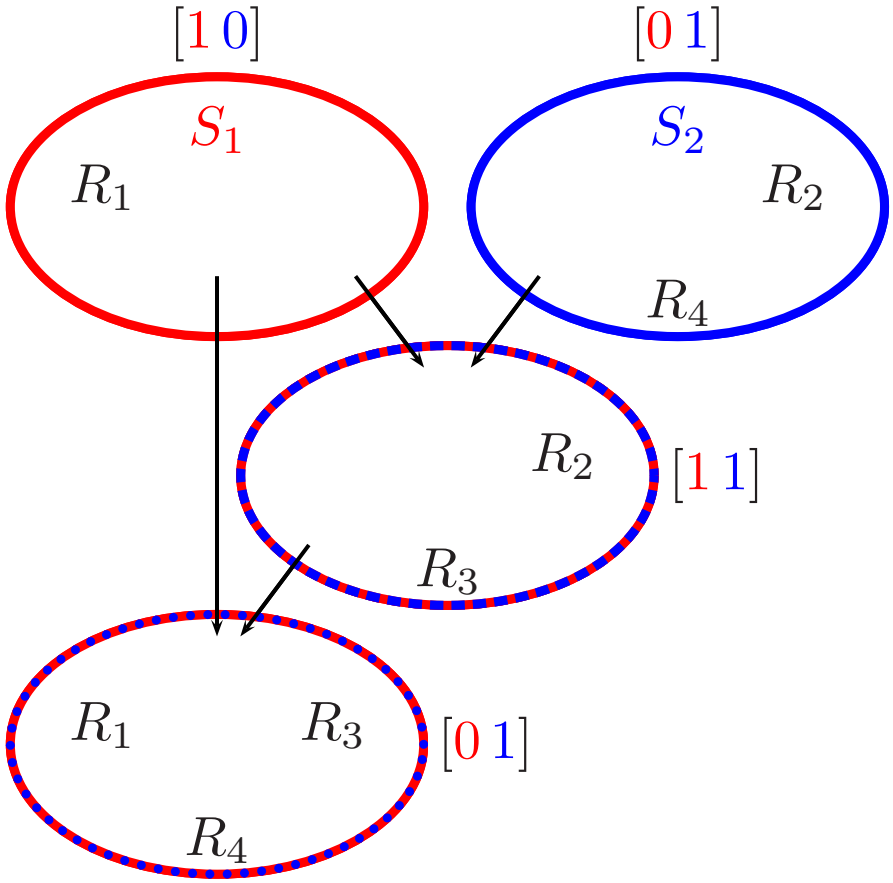




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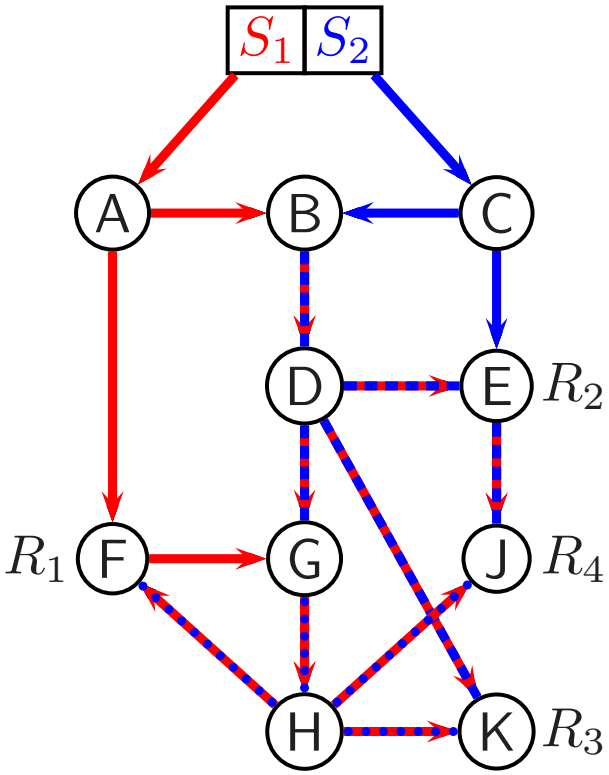
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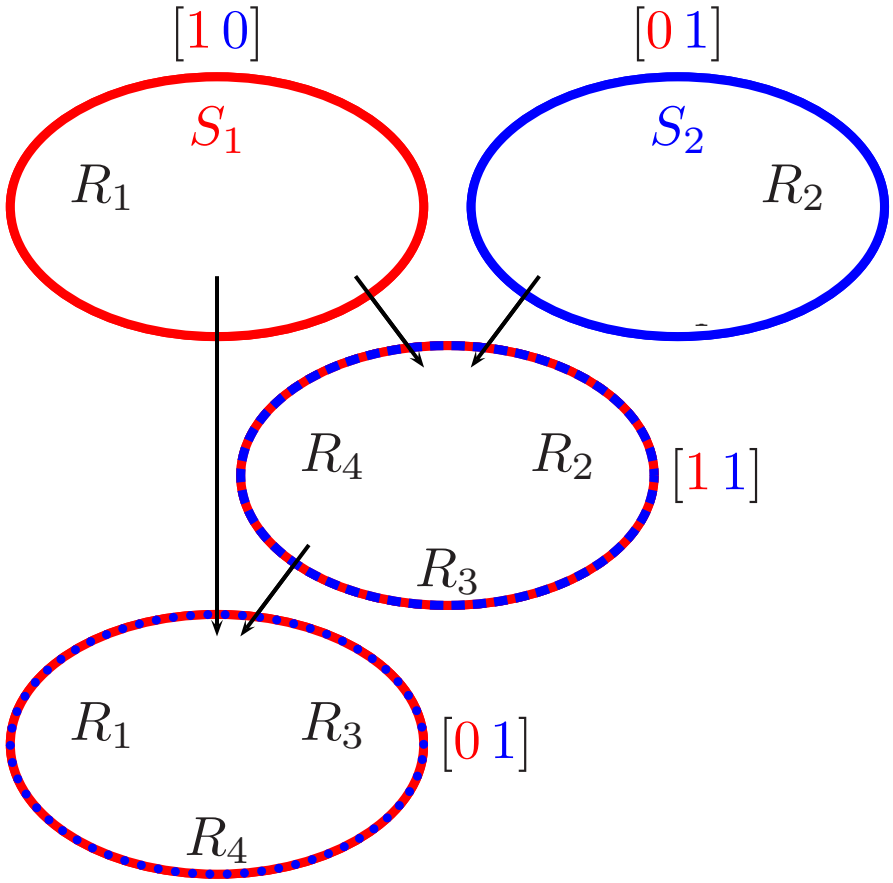




NETWORK MULTICAST

Example





DECOMPOSITION INTO SUBTREES

- Let G'_{ij} be the line graph of (S_i, R_j) and $G' = \bigcup G'_{ij}$.
- Coding points are the nodes of G' with two or more inputs.

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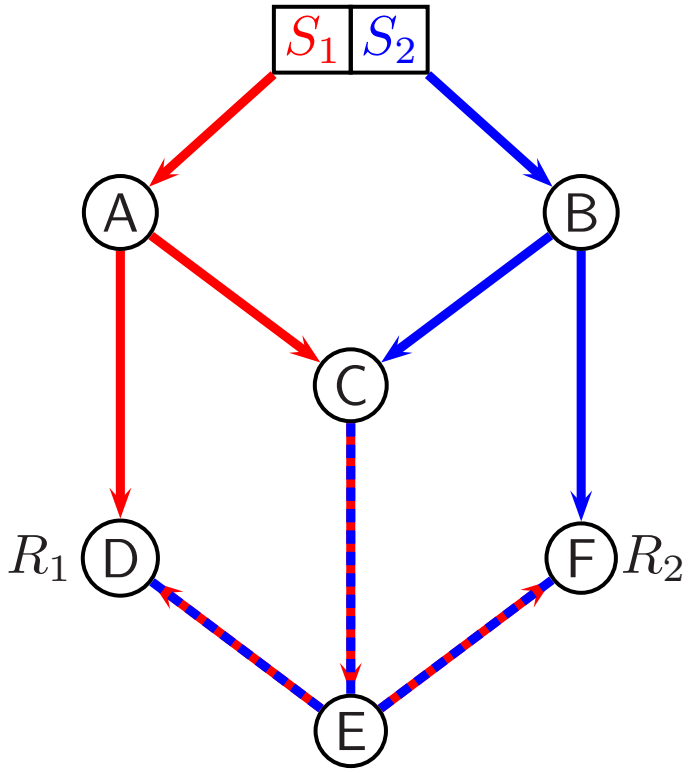
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- For coding, we need to know how T_i's are connected and
  1. which sources are available at the input of each T_i,
  2. which receiver nodes are in each T_i.

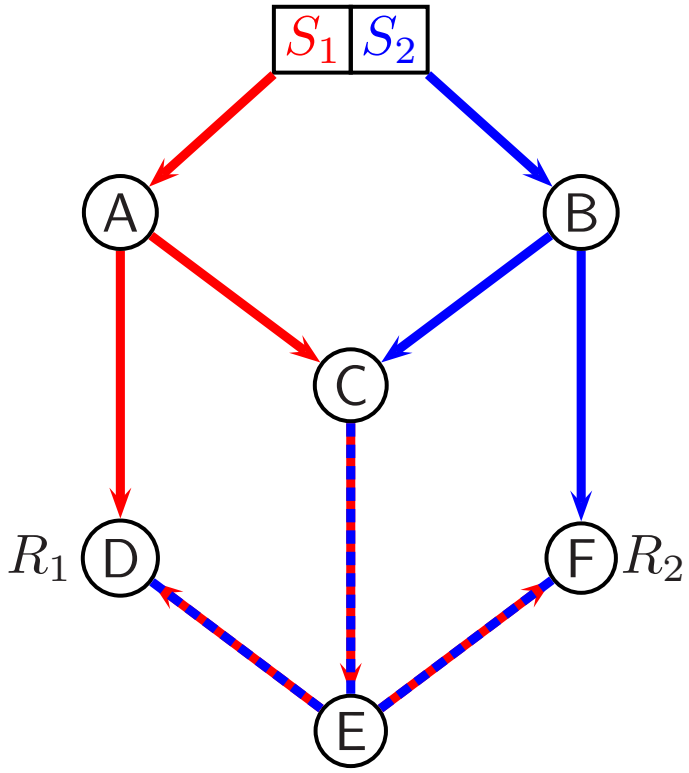
SUBTREE DECOMPOSITION OF NETWORKS

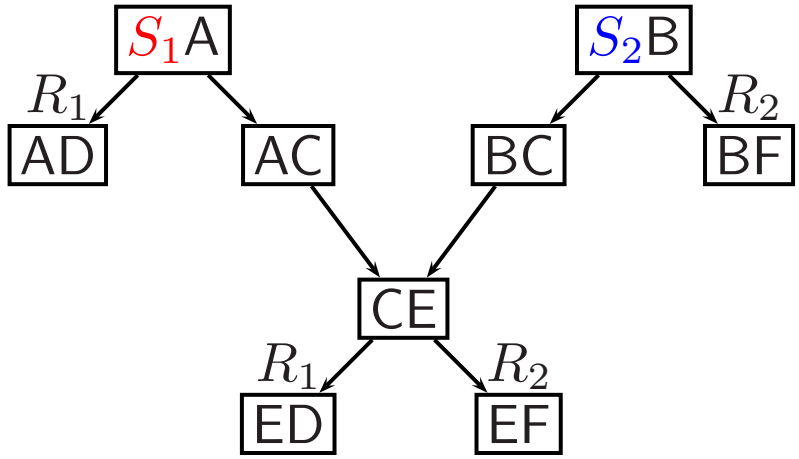
Two Sources and Two Receivers



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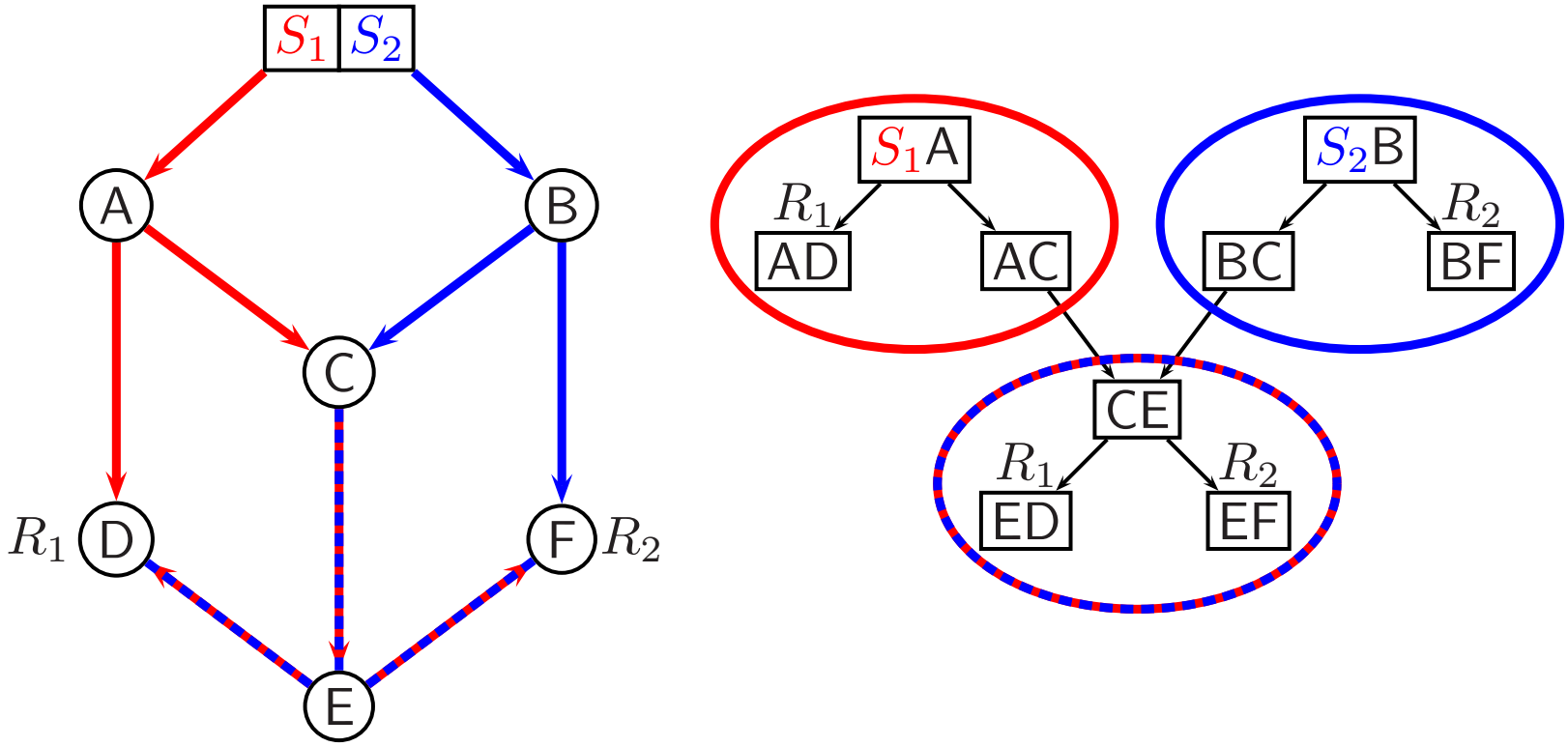
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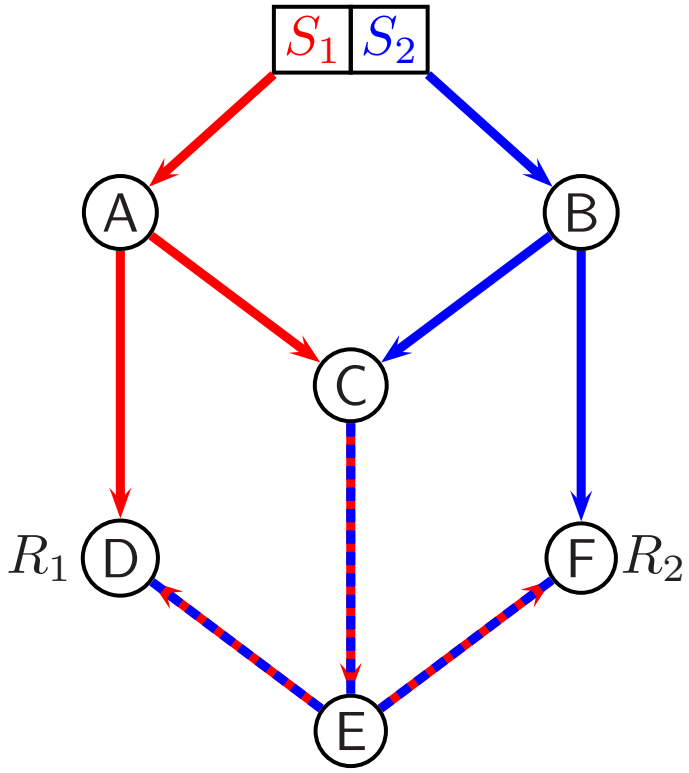
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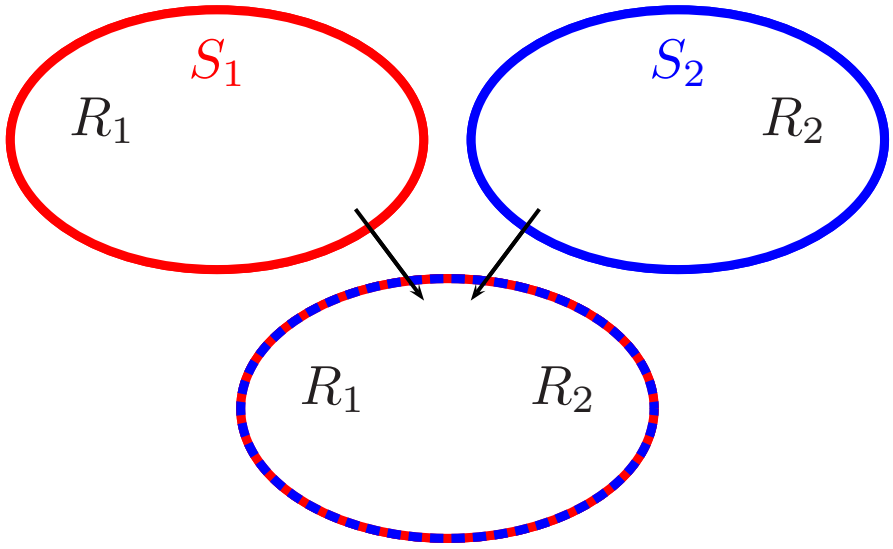
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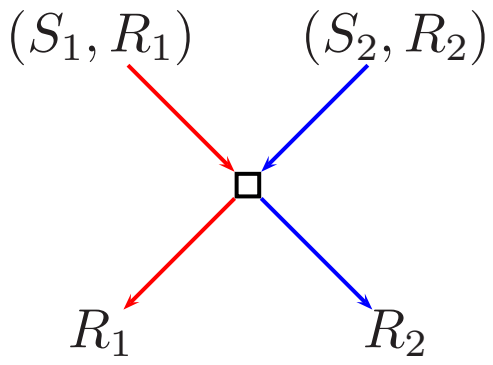


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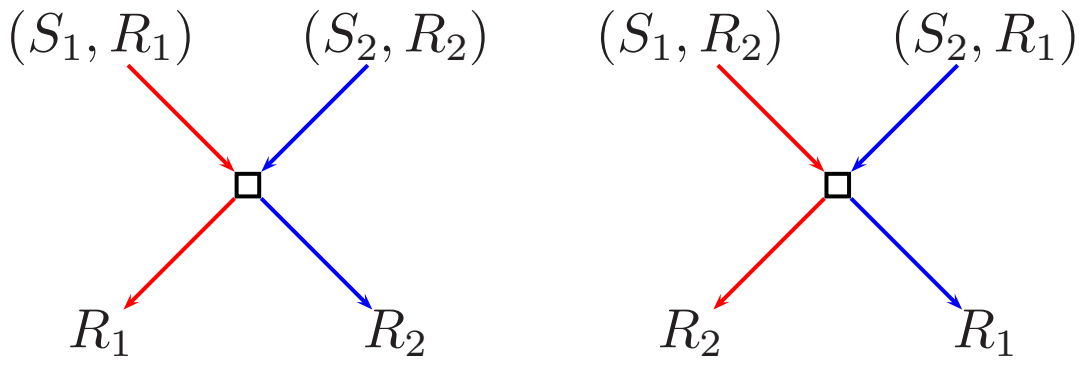
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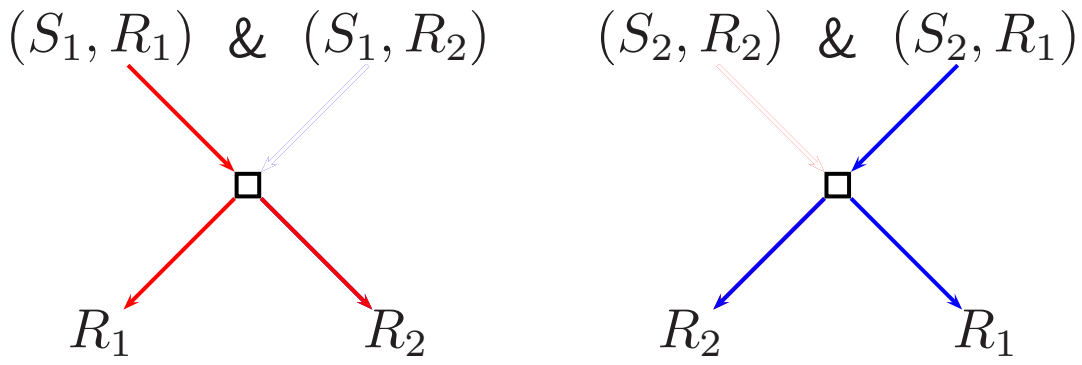
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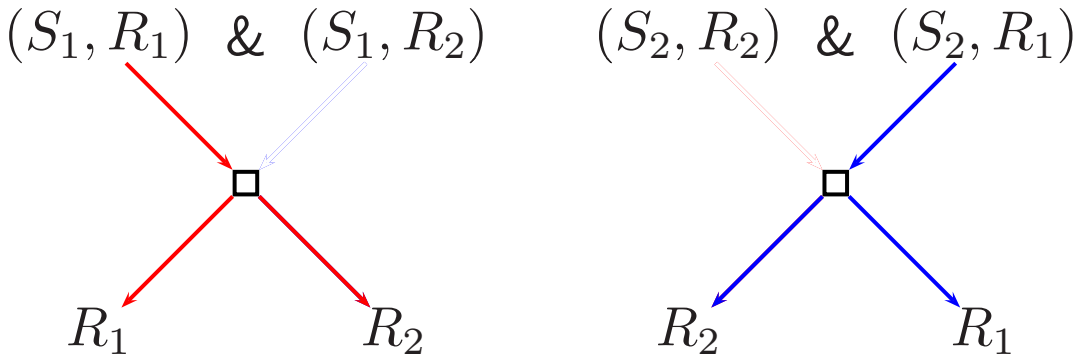
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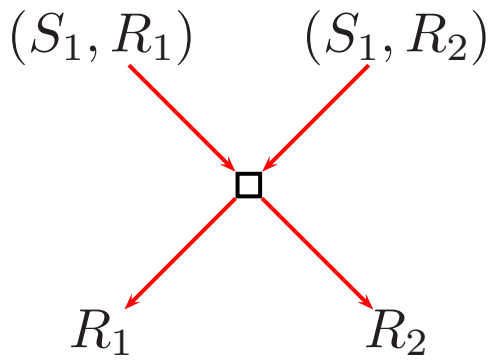
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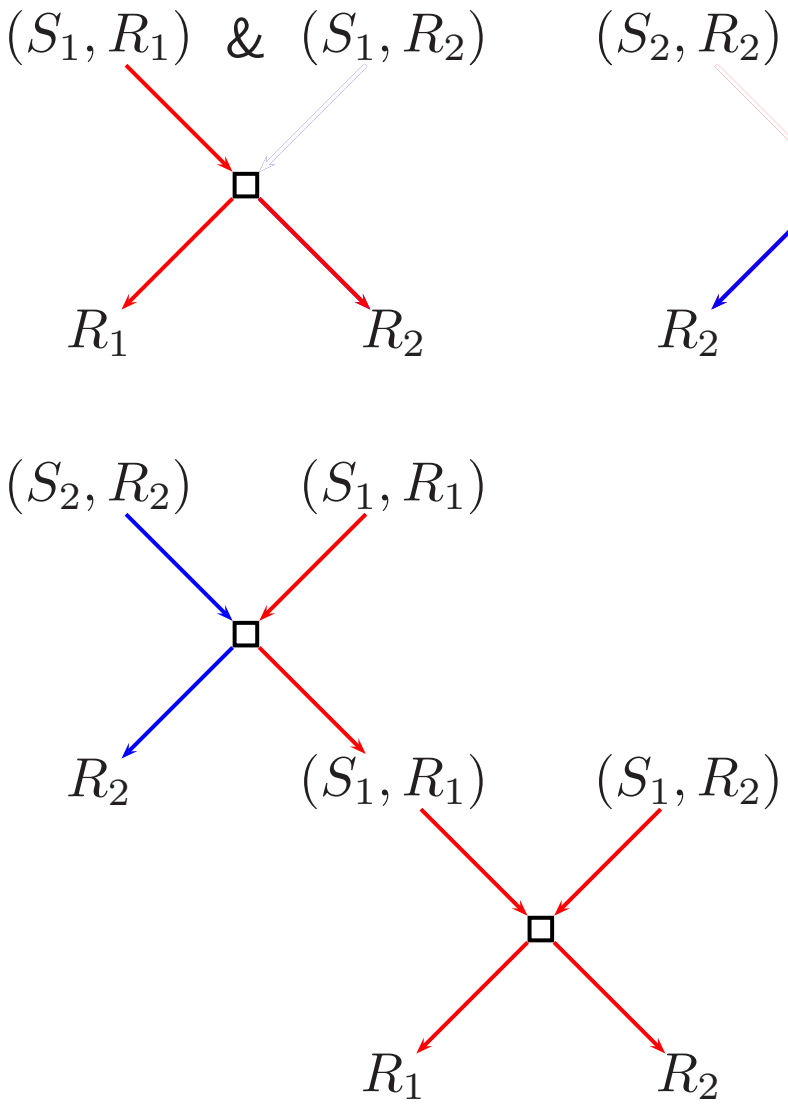
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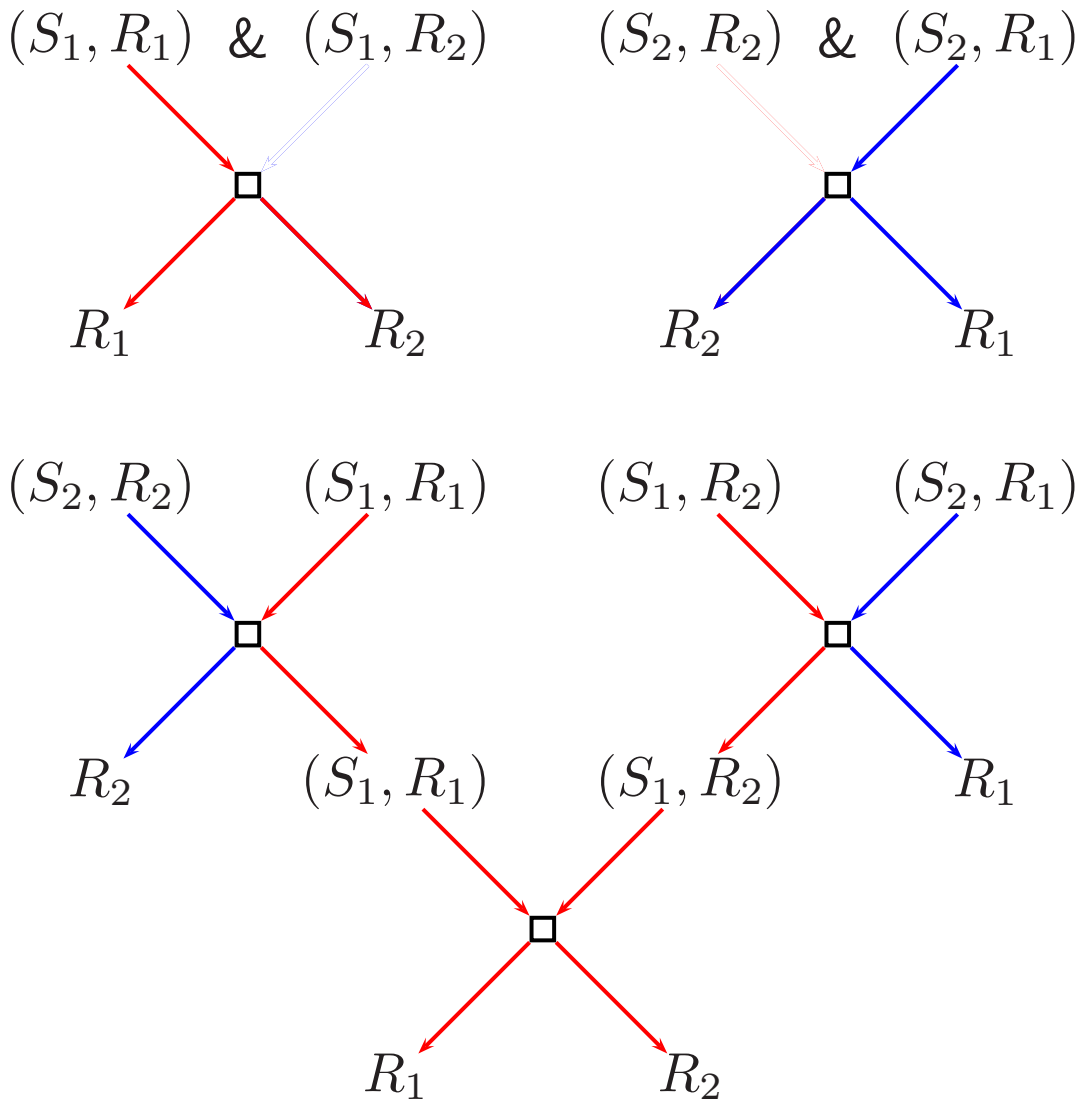
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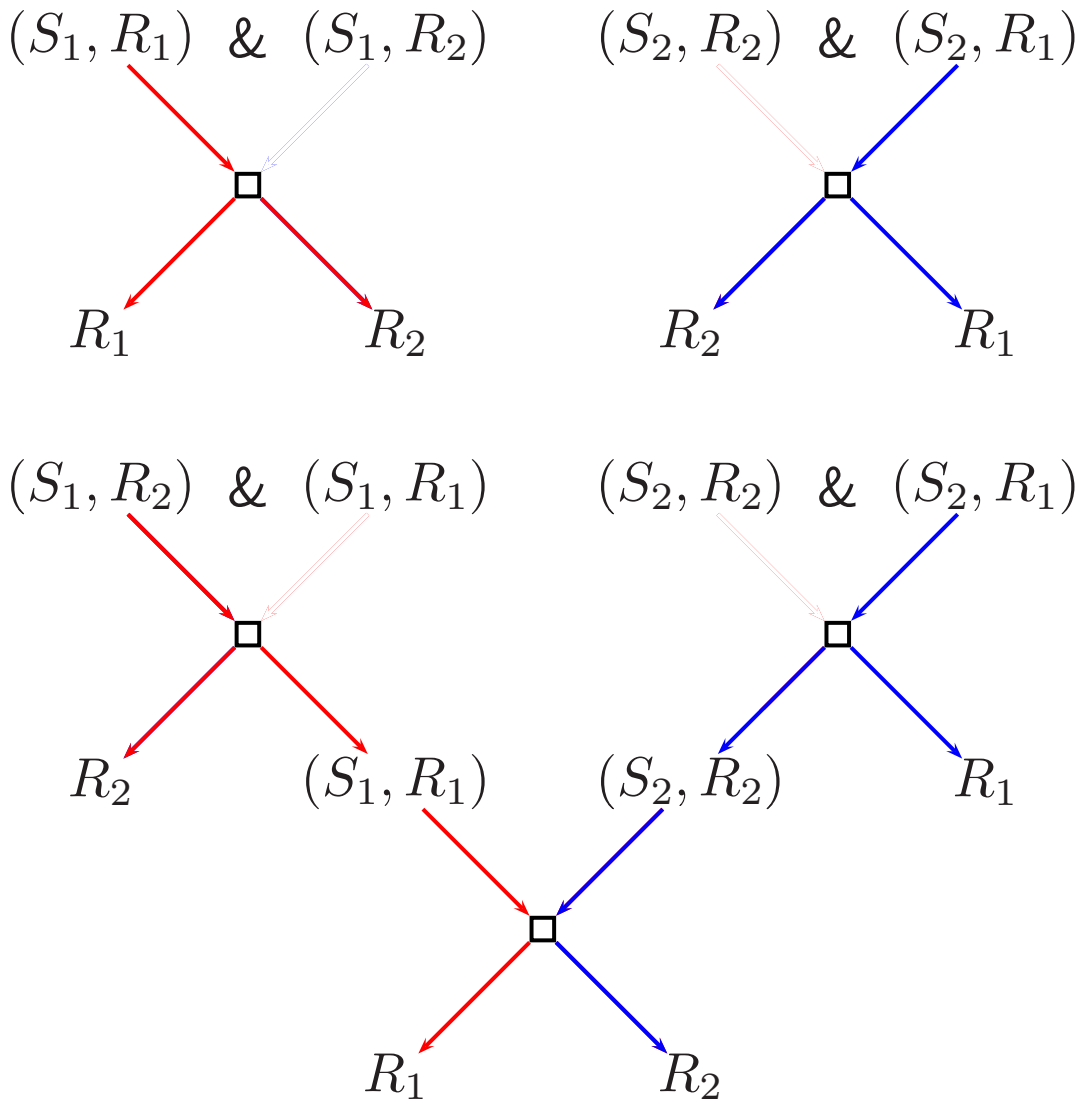
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SUBTREE DECOMPOSITION OF NETWORKS

Some Graph Theory Problems

- Can paths (S_i, R_j), $1 \leq i \leq h$, $1 \leq j \leq N$ be found so that

  1. for each receiver j, paths (S_i, R_j) for $1 \leq i \leq h$ are edge disjoint,
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- Related problems in multicommodity flow:

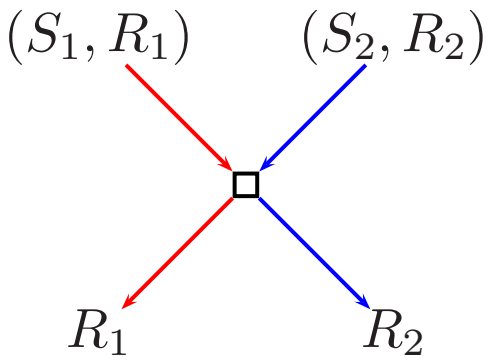
  – packing Steiner trees,
  – Edmonds' theorem on edge connectivity.

SUBTREE DECOMPOSITION OF NETWORKS

Two Sources and N Receivers

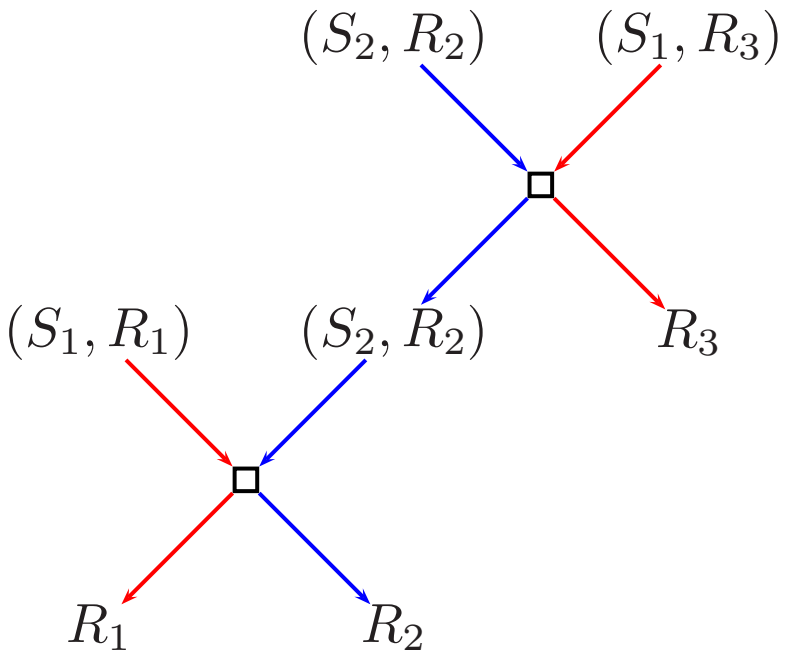
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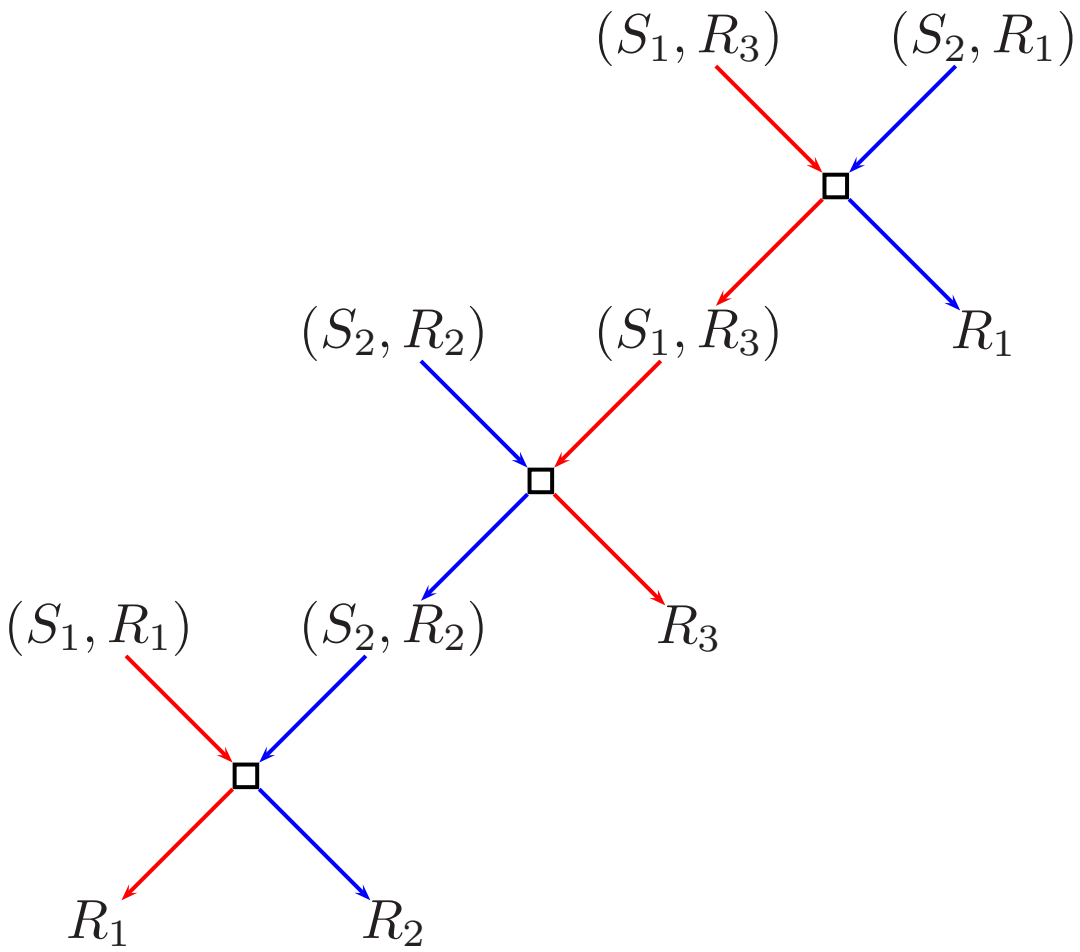
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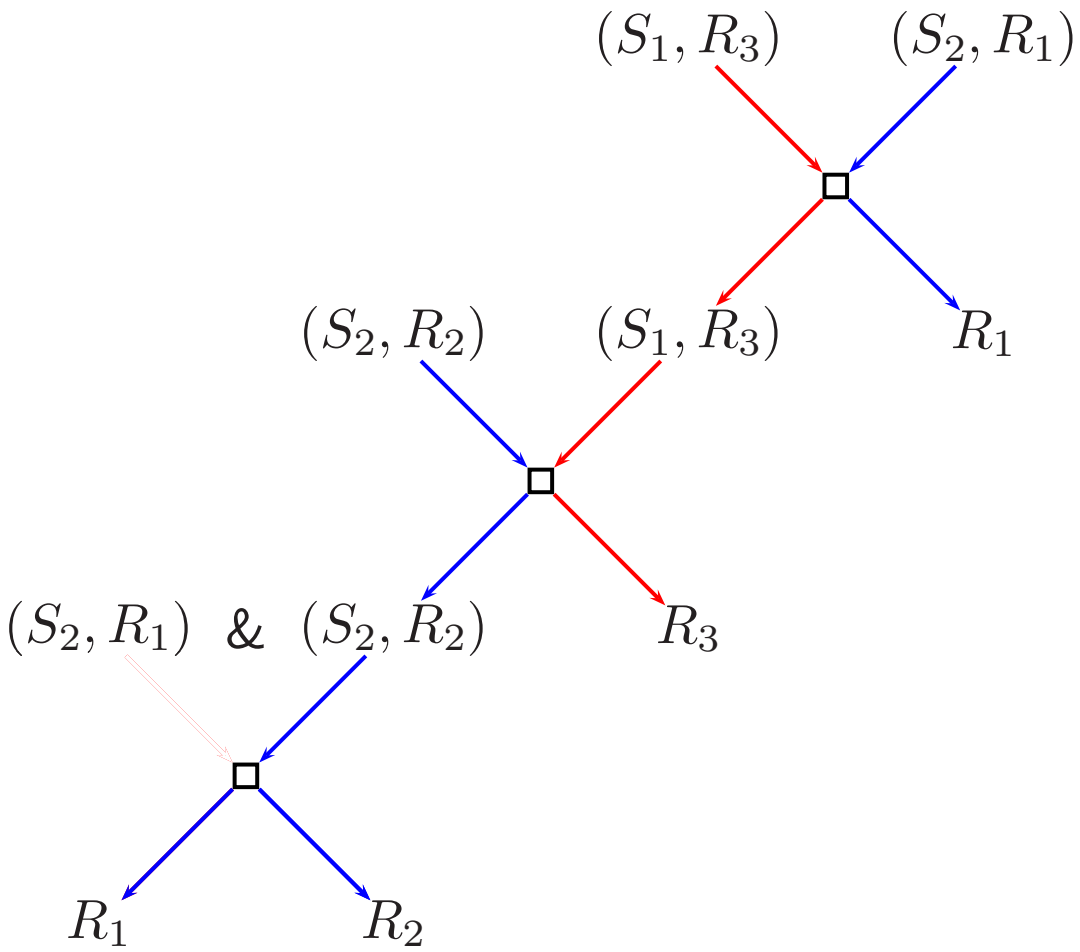
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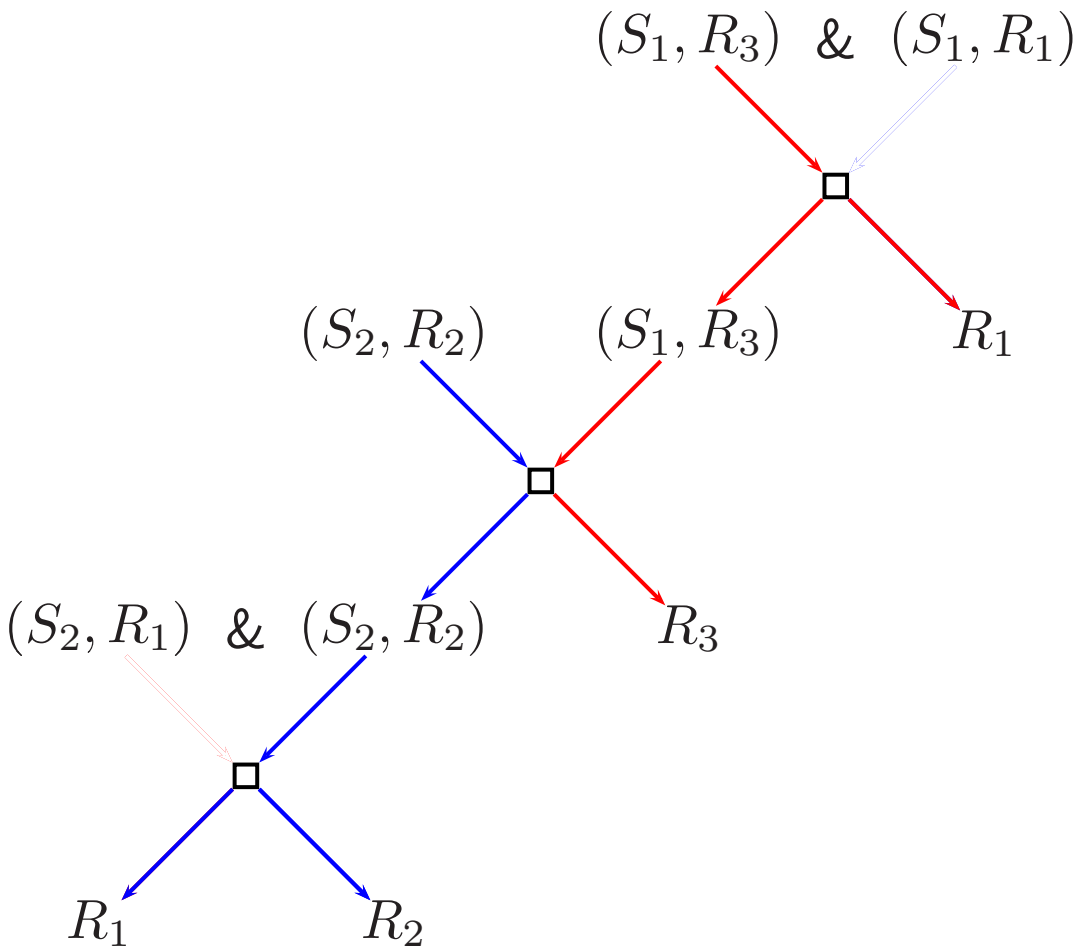
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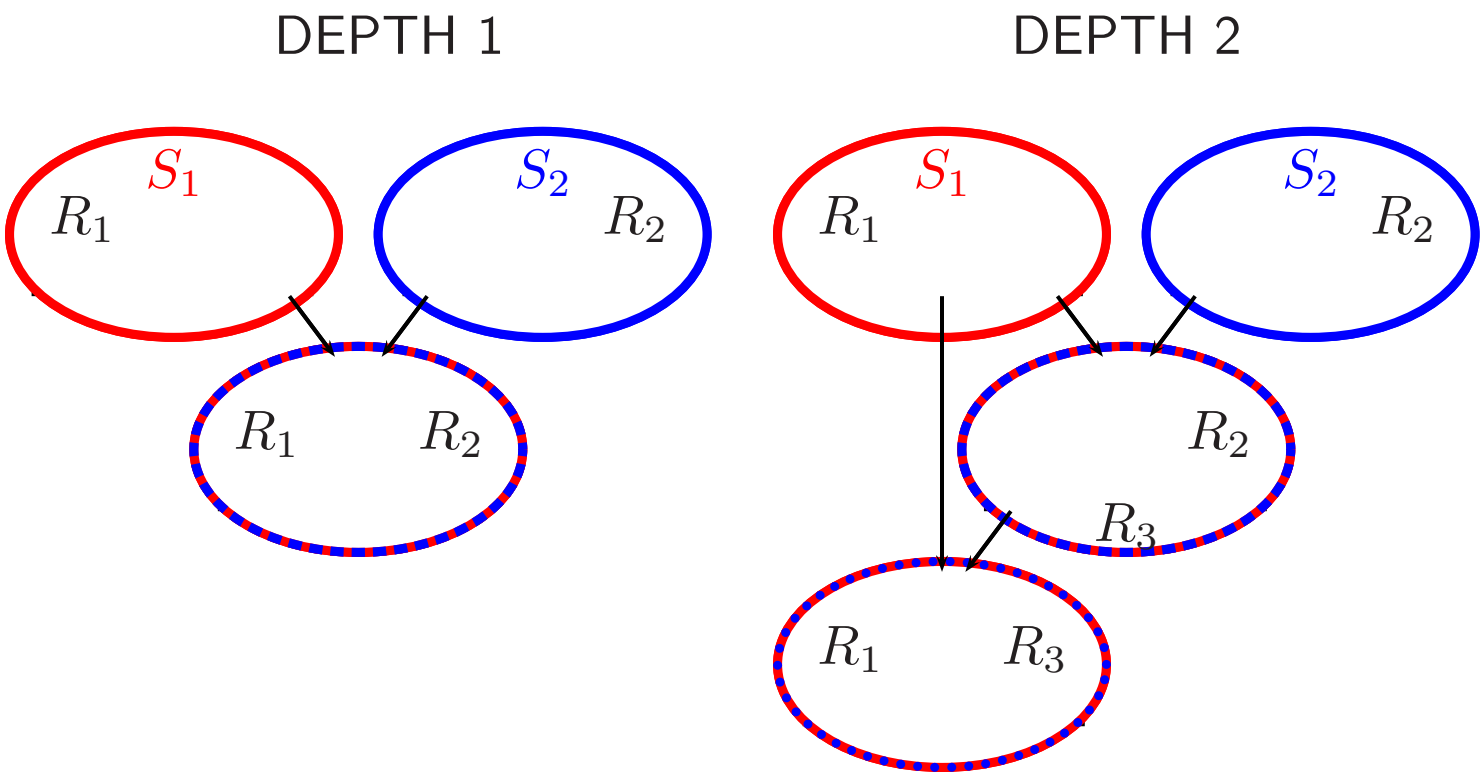
SUBTREE DECOMPOSITION OF NETWORKS

Two Sources and N Receivers



ASSIGNING LABELS AND VERTEX COLORING

Depth of the Subtree Graph



ALPHABET SIZE - 2 Sources, Depth 1
of Coding Vectors v.s. # of Receivers

- Subtrees can have the same label iff they do not share a receiver.
- The field with $n_c - 1$ elements provides exactly n_c coding vectors.

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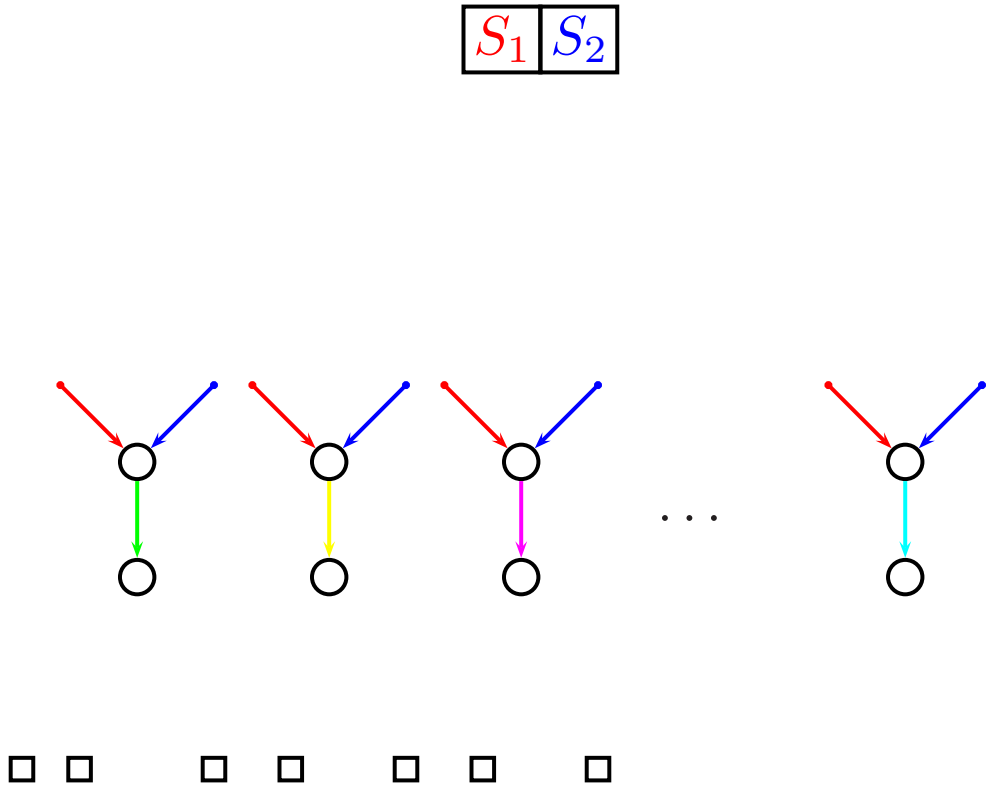
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  2. each pair of coding trees has a receiver in common.
- In such a network, the number of receivers is $n_c(n_c - 1)/2$.
- For fewer receivers, the field with $n_c - 1$ elements is sufficient.

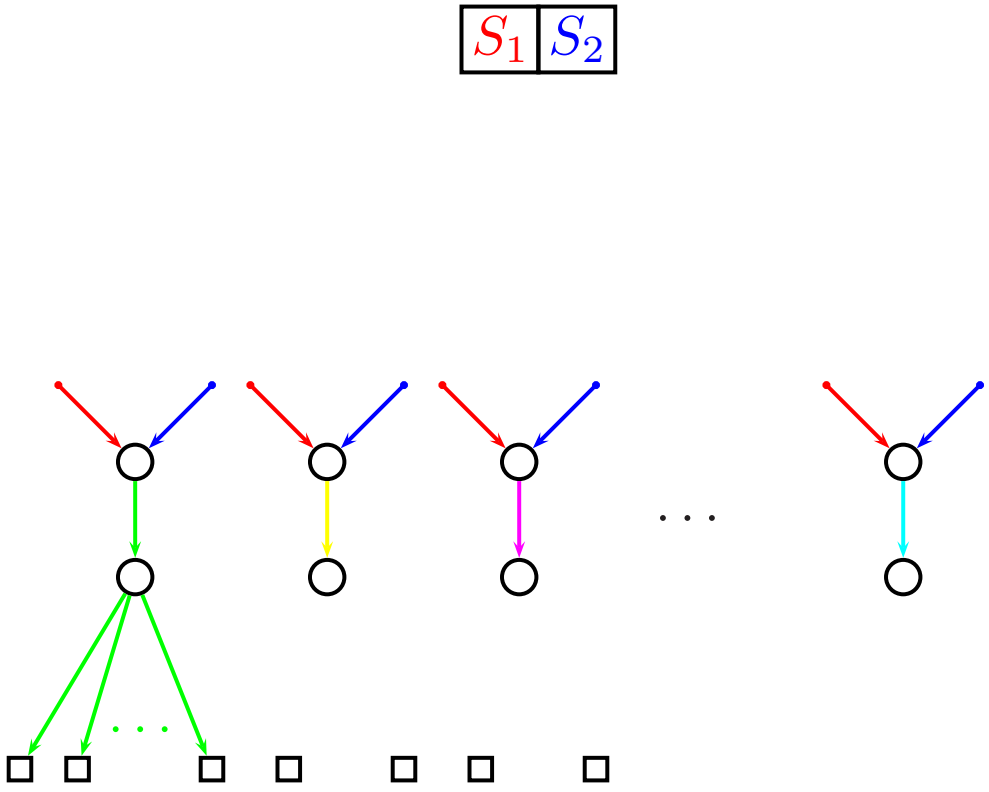
A LOWER BOUND ON THE ALPHABET SIZE

An Example



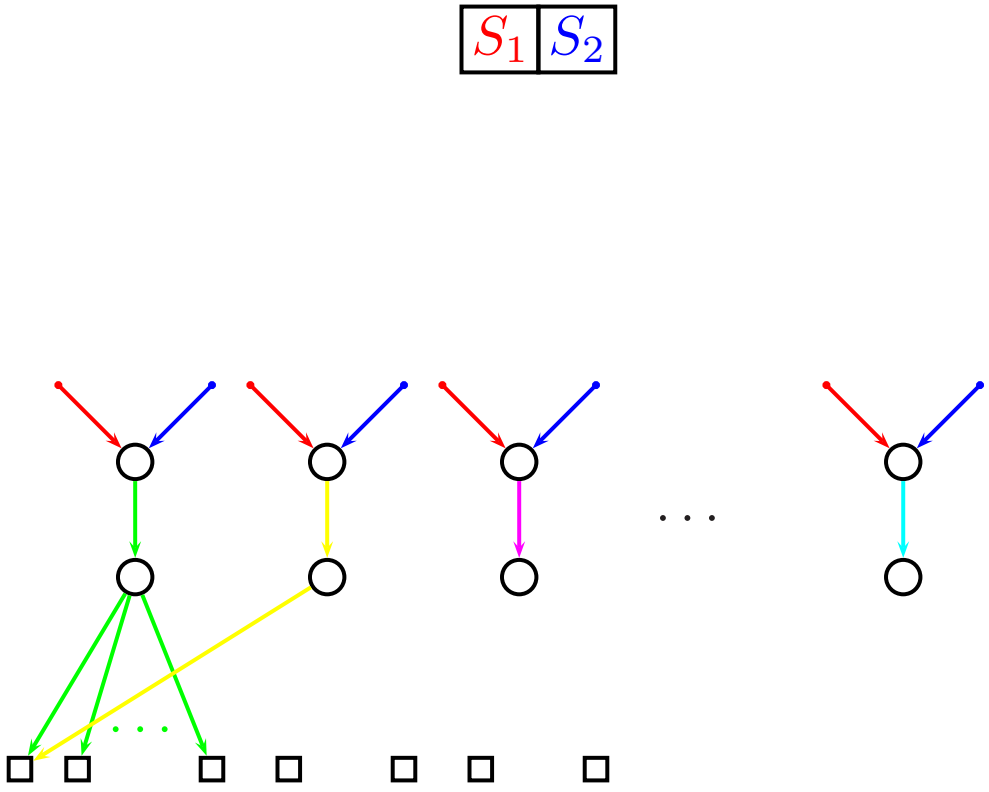
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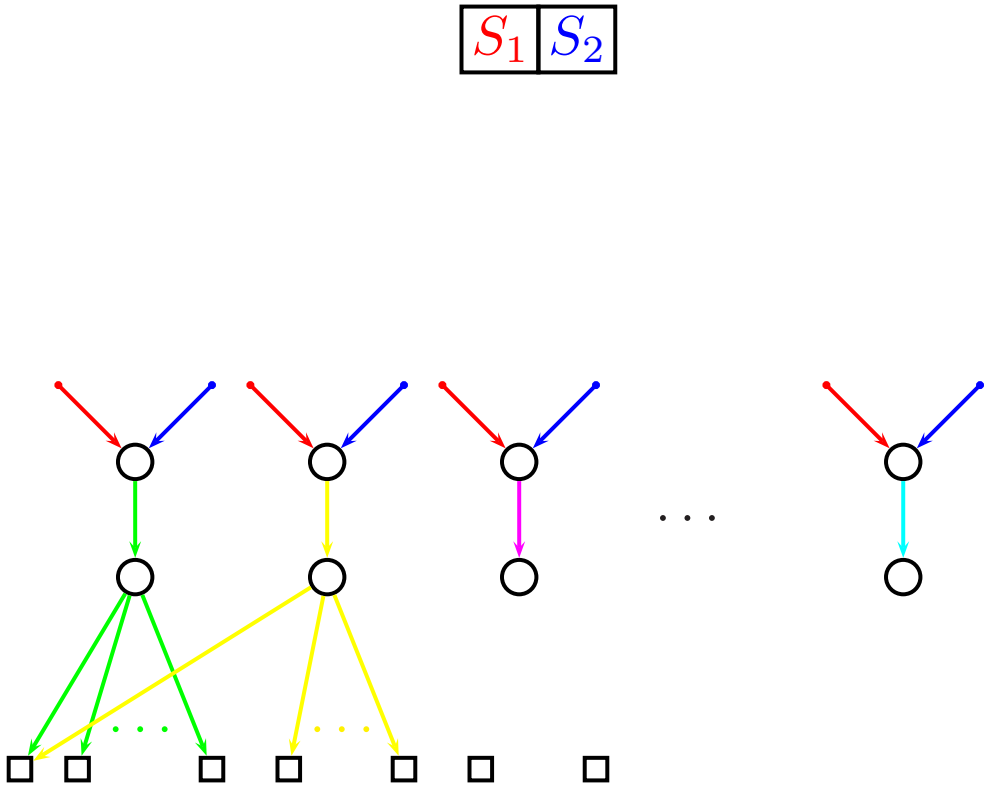
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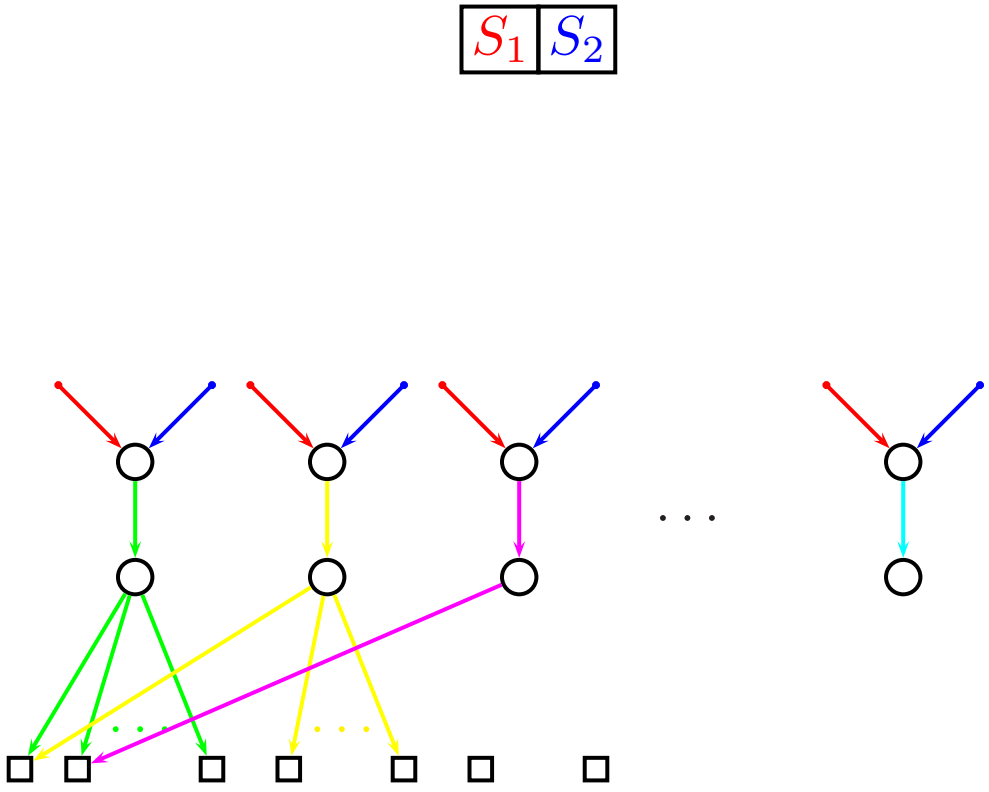
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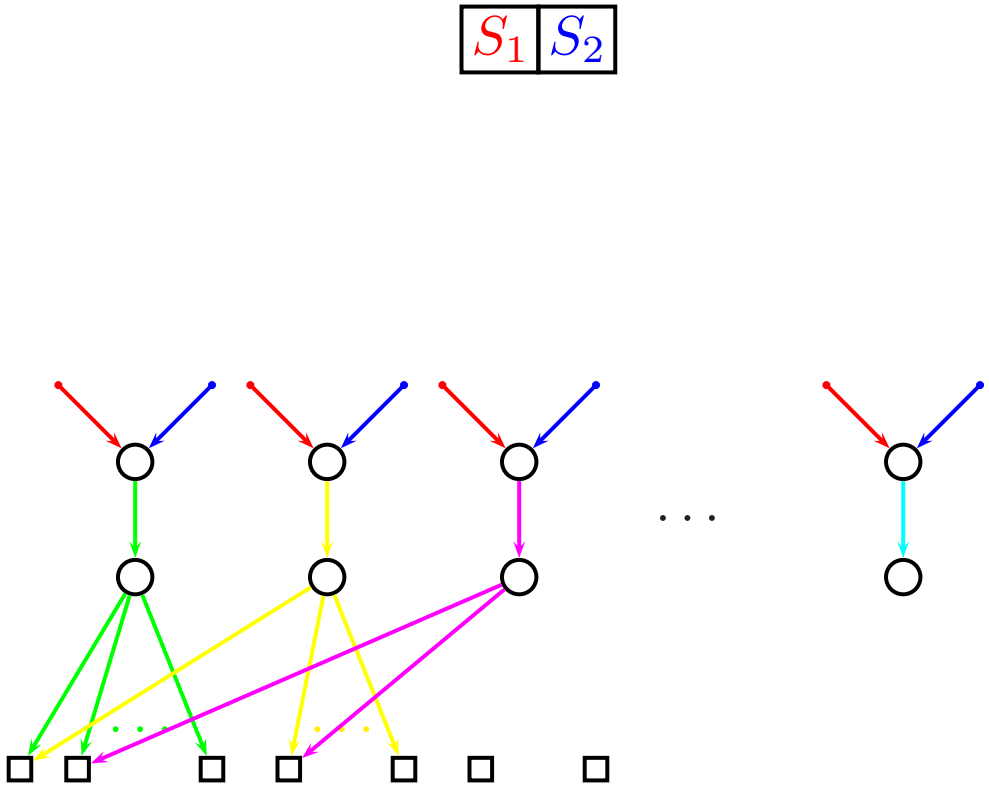
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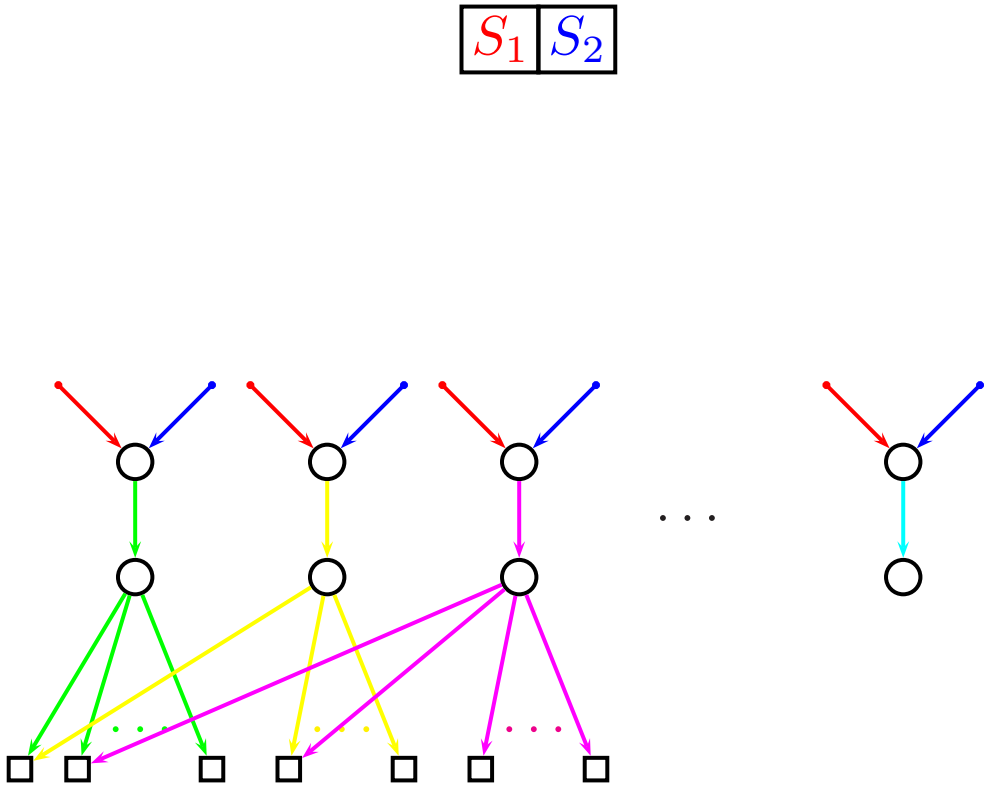
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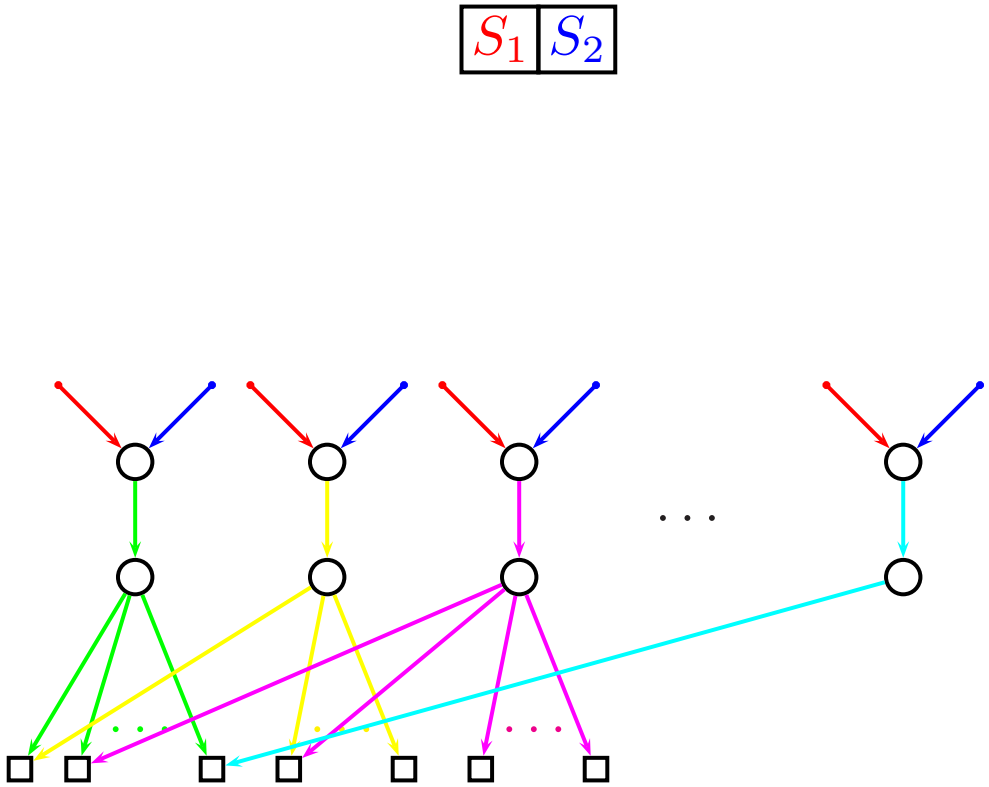
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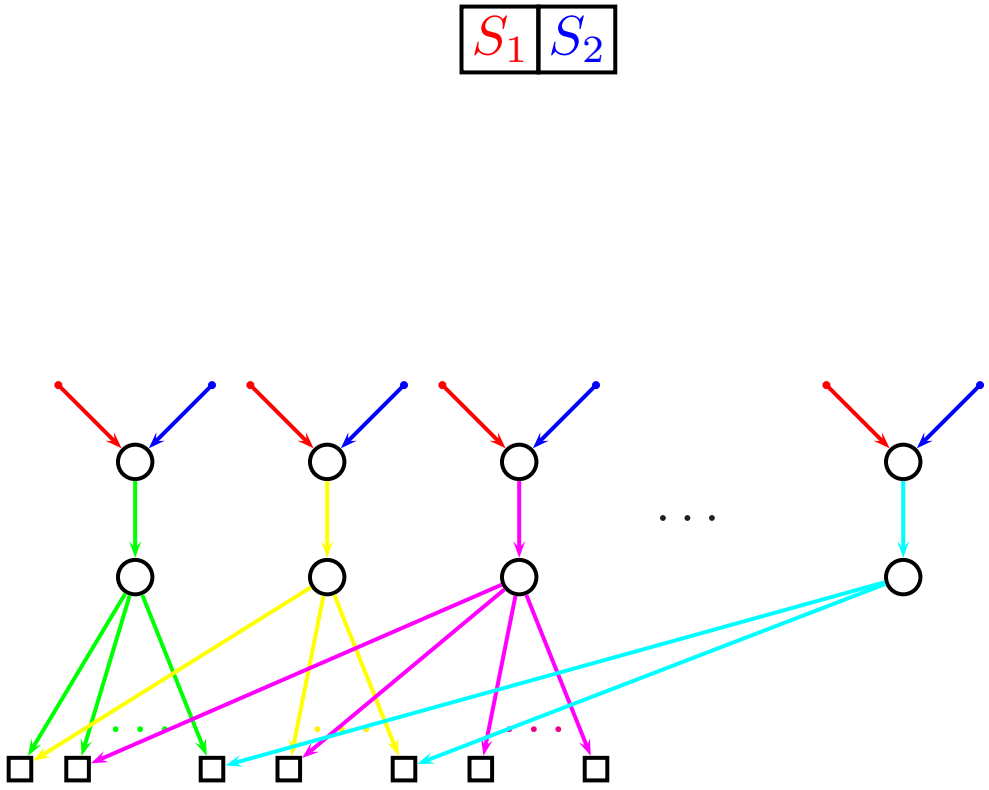
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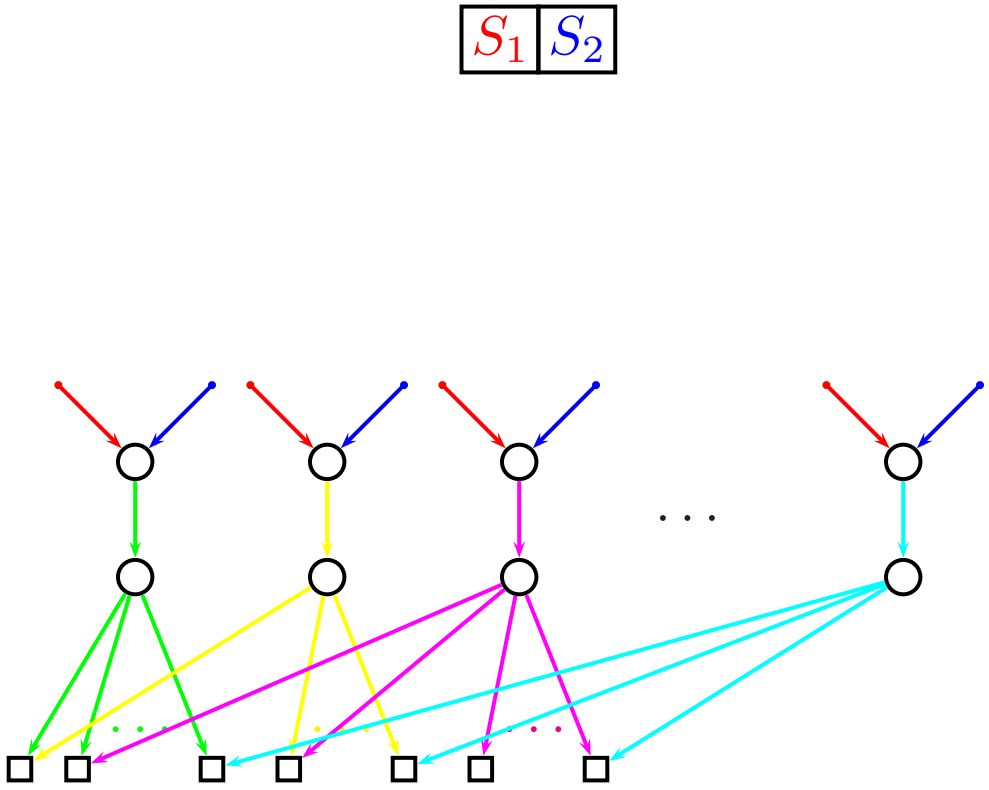
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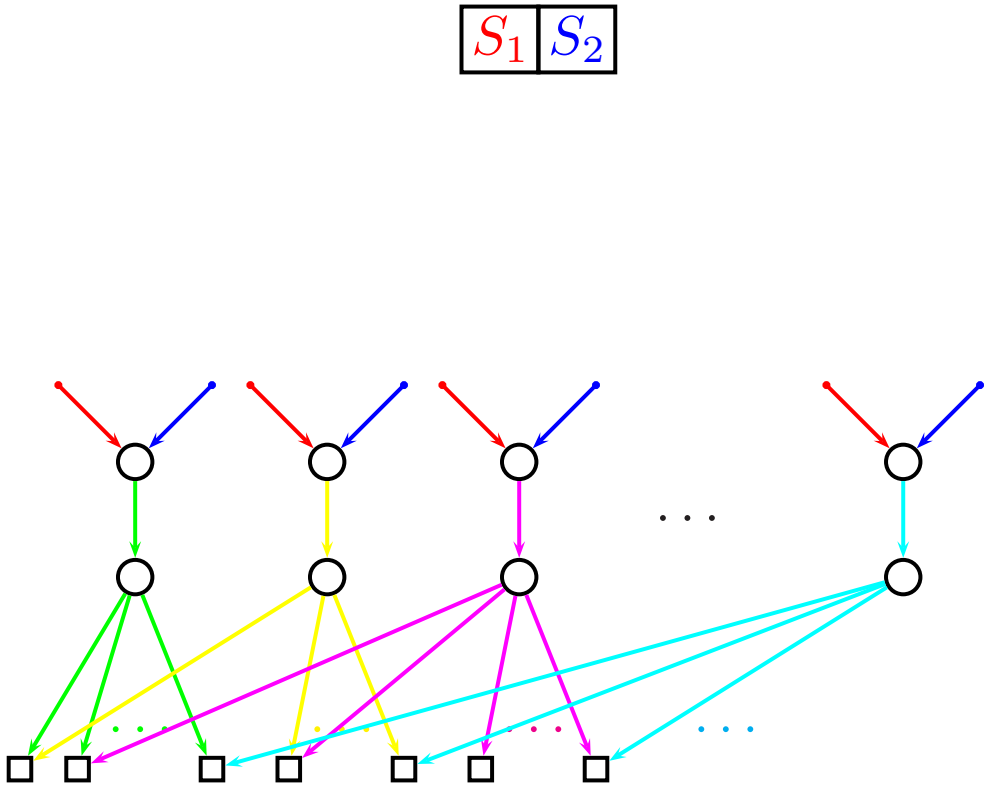
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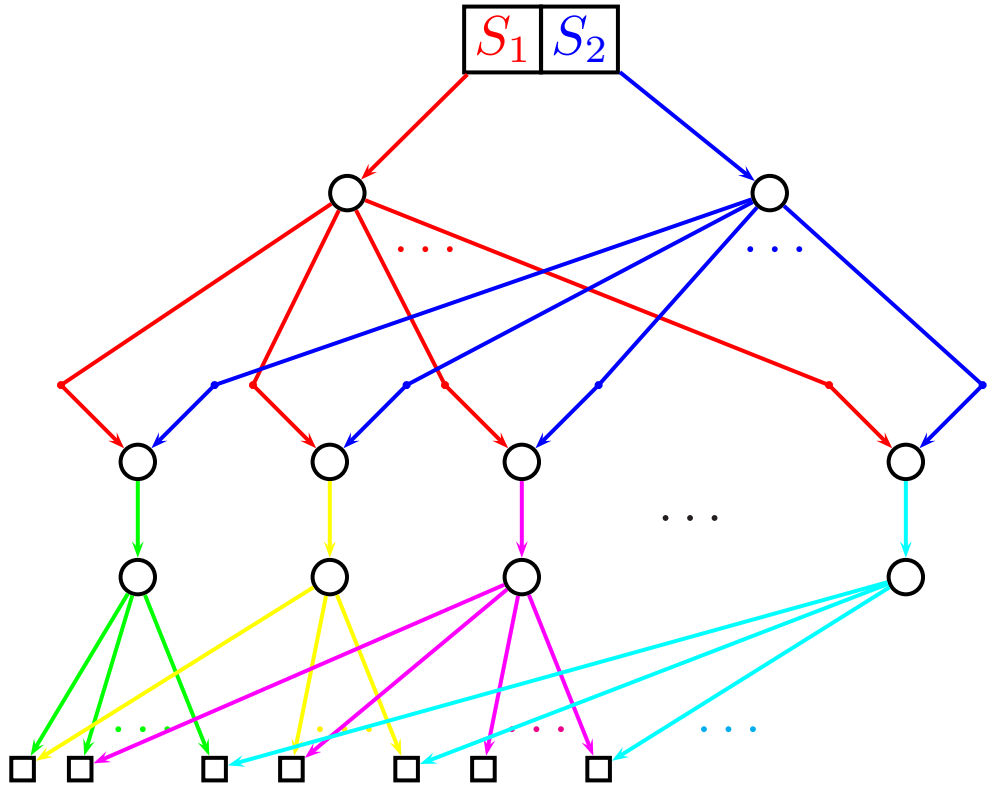
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An Example



A LOWER BOUND ON THE ALPHABET SIZE

An Example



ALPHABET SIZE - 2 Sources
of Coding Vectors v.s. # of Receivers

Claim: For any minimal configuration with N receivers, the code alphabet $\mathbb{F}_q$ of size

$$q = \lfloor \sqrt{2N - 7/4} + 1/2 \rfloor$$

is sufficient. There exist configurations for which it is necessary.

CODING VECTORS

Projective Space $\mathbb{PG}(h-1,q)$

- Coding vectors for networks with two sources:

$$[0\,1],\ [1\,0],\ \text{and}\ [1\,\alpha^i]\ \text{for}\ 0 \le i \le q-2.$$

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DECENTRALIZED CODING

Coding Vectors

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- For decentralized coding, we need arcs in $\mathbb{PG}(h-1, q)$.
- An arc has to have size and structure appropriate for the network.

DECENTRALIZED CODING

Vectors in General Position in $\mathbb{F}_q^h$

- Let $\mathcal{A}$ be a set of vectors in $\mathbb{F}_q^h$.
- We require that any h vectors in $\mathcal{A}$ be linearly independent.
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- Two bounds:

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ & & & \end{bmatrix} \Rightarrow \; g(h,q) \geq h$$

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VECTORS IN GENERAL POSITION

MDS Codes over $\mathbb{F}_q$

- Consider an MDS code of dimension h and generator matrix G.
- The columns of G are vectors in general position in $\mathbb{F}_q^h$.
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$g(4,q) = g(5,q) = q+1$. [Thas]

$g(6,q) = g(7,q) = q+1$ if q is even. [Maruta, Kaneta]

$g(6,q) = q+1$ if $q = 11$ or 13. [Ali, Hirschfeld, Kaneta]

VECTORS IN GENERAL POSITION

Set $\mathcal{A}$ and Codes for Networks

- Size of $\mathcal{A}$:
  - For $h \geq q$, we know that $g(h, q) = h + 1$.
  - For $h \leq q$, it is widely believed that

$$g(h,q) = \begin{cases} q+2 & \text{if } q \text{ is even and either } h = 3 \text{ or } h = q-1, \\ q+1 & \text{otherwise.} \end{cases}$$

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- Structure of $\mathcal{A}$:

  To handle possible constraints on coding vectors, we can

  - develop algorithms to generate $\mathcal{A}$'s, or
  - use theorems about geometry of complete arcs to derive $\mathcal{A}$'s.

VECTORS IN GENERAL POSITION IN $\mathbb{F}_q^h$

Arcs in $\mathbb{PG}(h-1,q)$

- An example:
  - Let $\mathcal{A}$ be a set of points in general position over $\mathbb{F}_q^3$, and $\mathcal{A}'$ the corresponding set in $\mathbb{PG}(2,q)$.

VECTORS IN GENERAL POSITION IN $\mathbb{F}_q^h$

Arcs in $\mathbb{PG}(h-1,q)$

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- For an arc in an $(h-1)$-space, no h points lie in an $h-1$-hyperplane.
- What is the length of the maximal arc in $\mathbb{PG}(h-1,q)$?

THE STRUCTURE AND SIZE OF ARCS

Three Problems of Segre

1. What is $g(h,q)$ and which arcs have $g(h,q)$ points?

2. For which q and $h < q$ are all arcs with $q+1$ points projectively equivalent?

3. What are the sizes of the complete arcs?
   What is the size of the second largest complete arc?

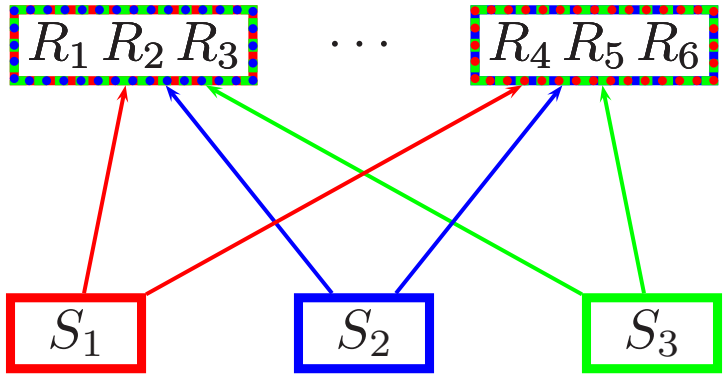
NETWORKS WITH THREE SOURCES

Depth 1

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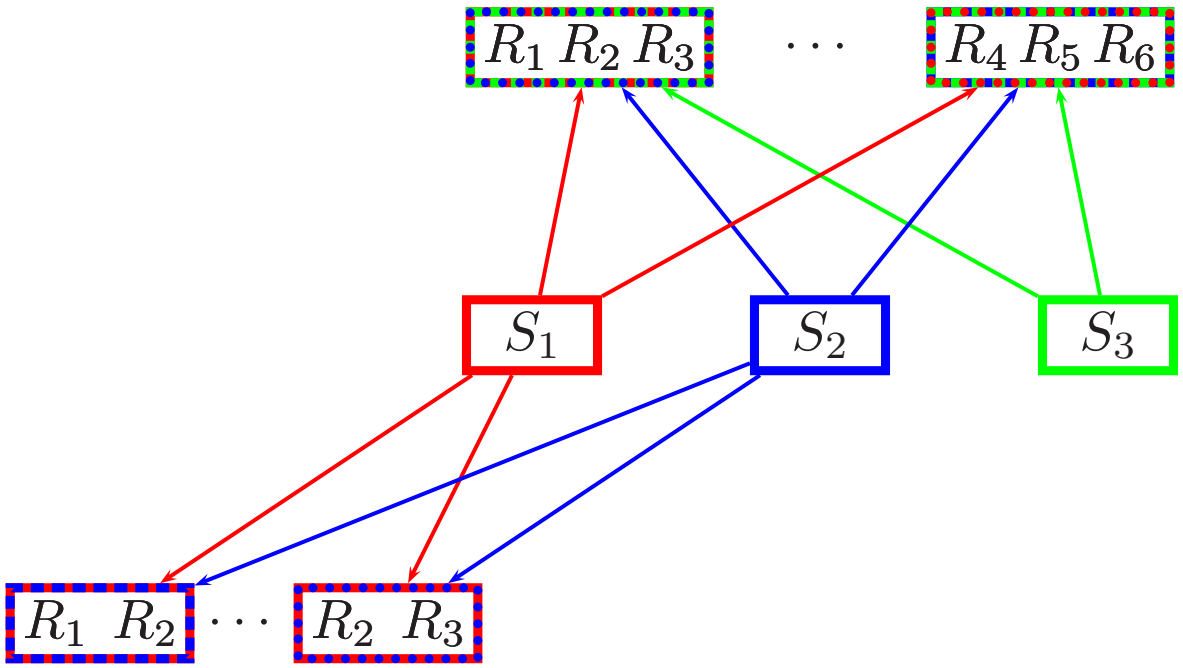
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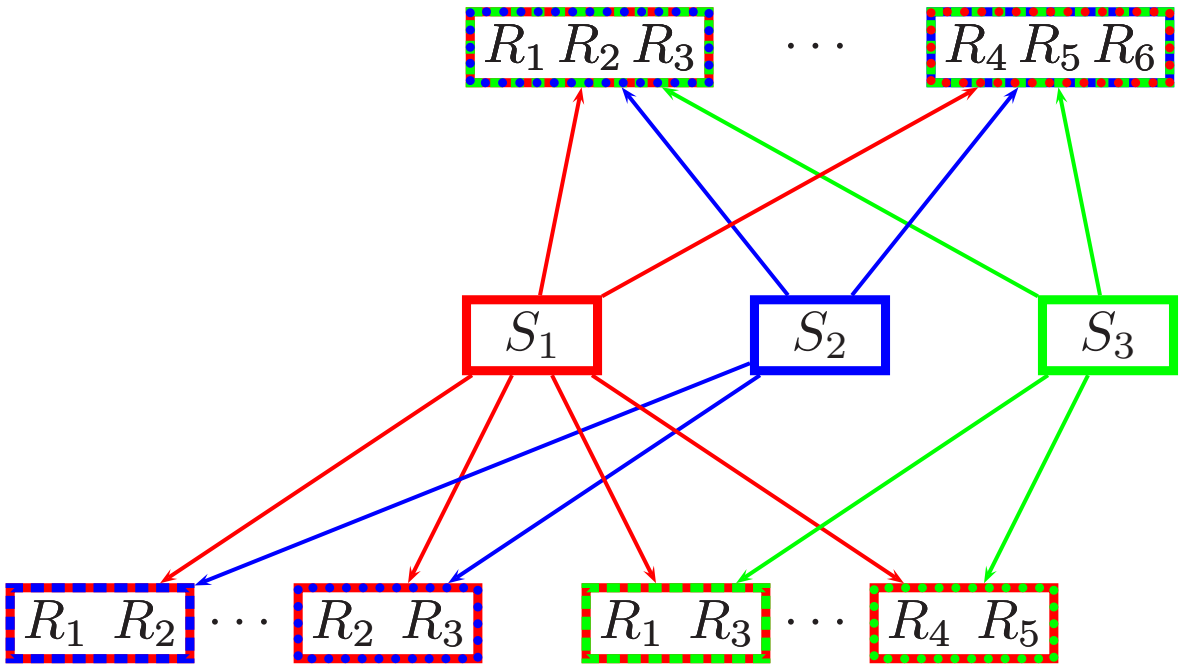


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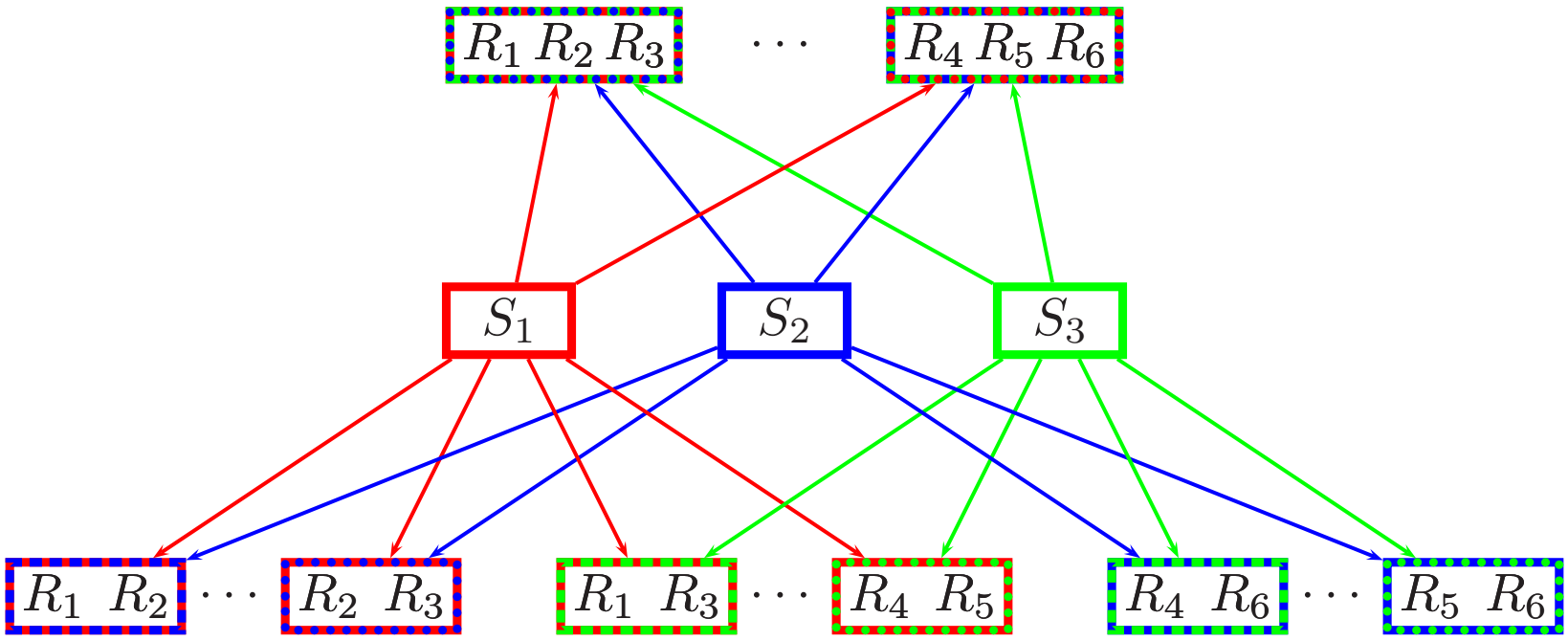


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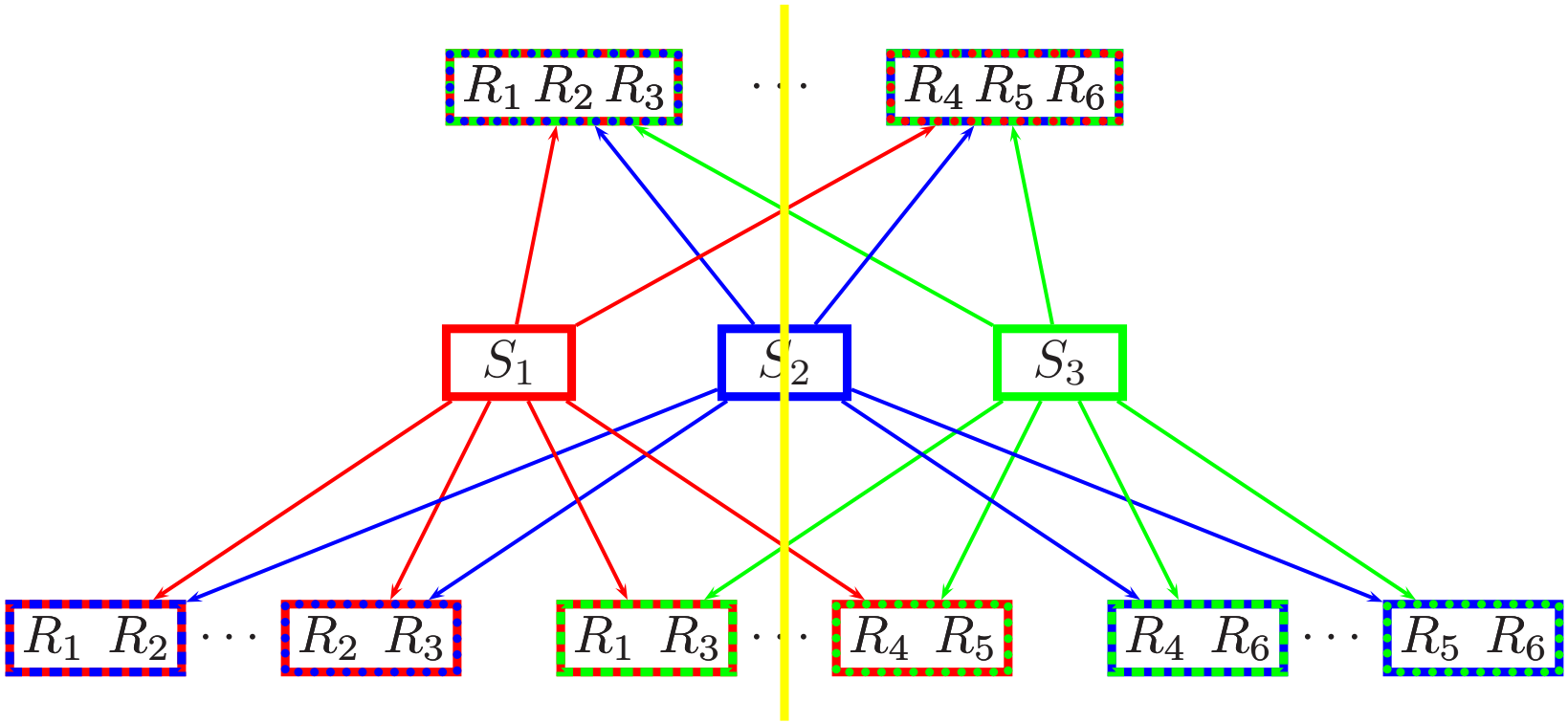


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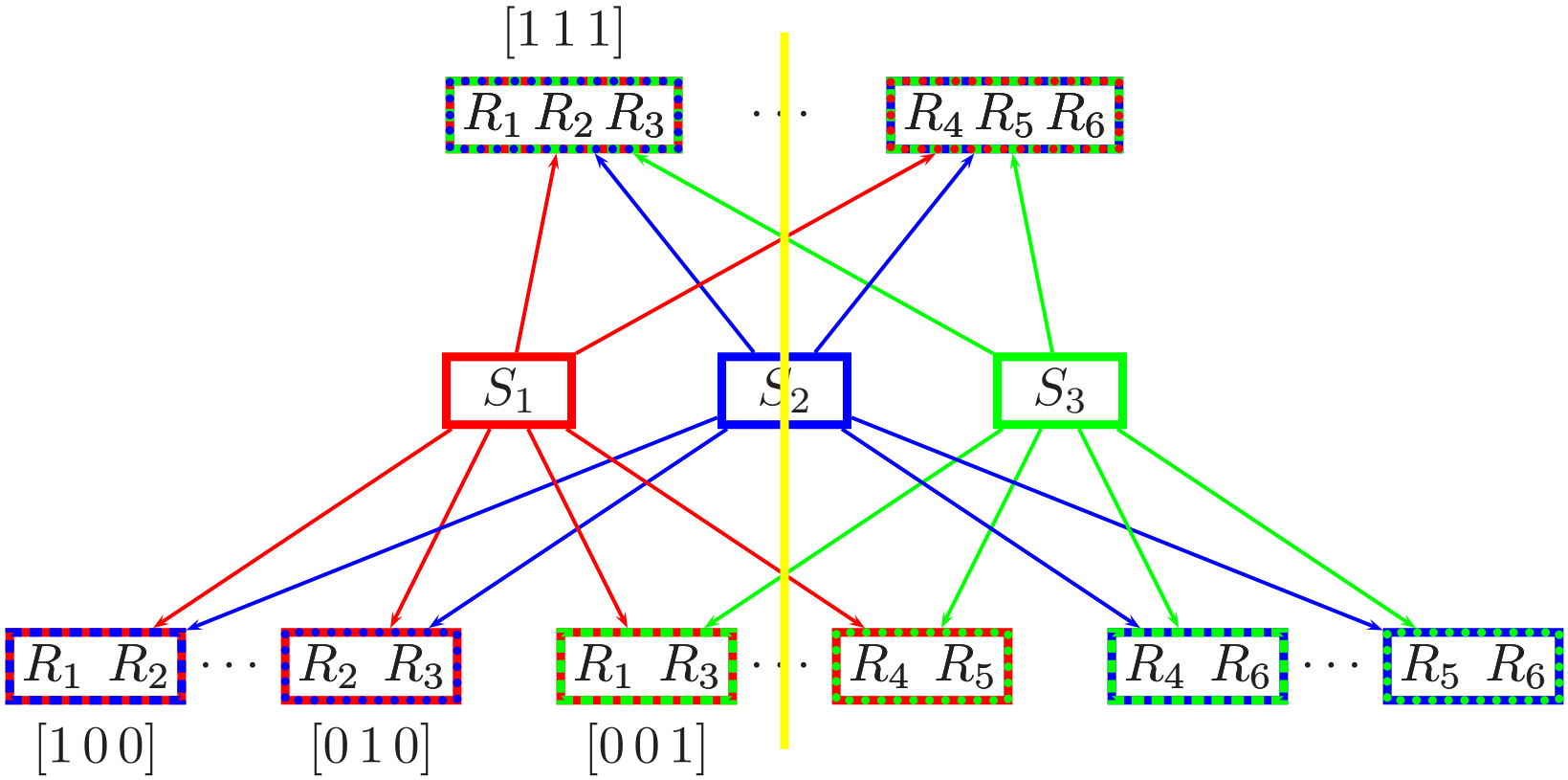


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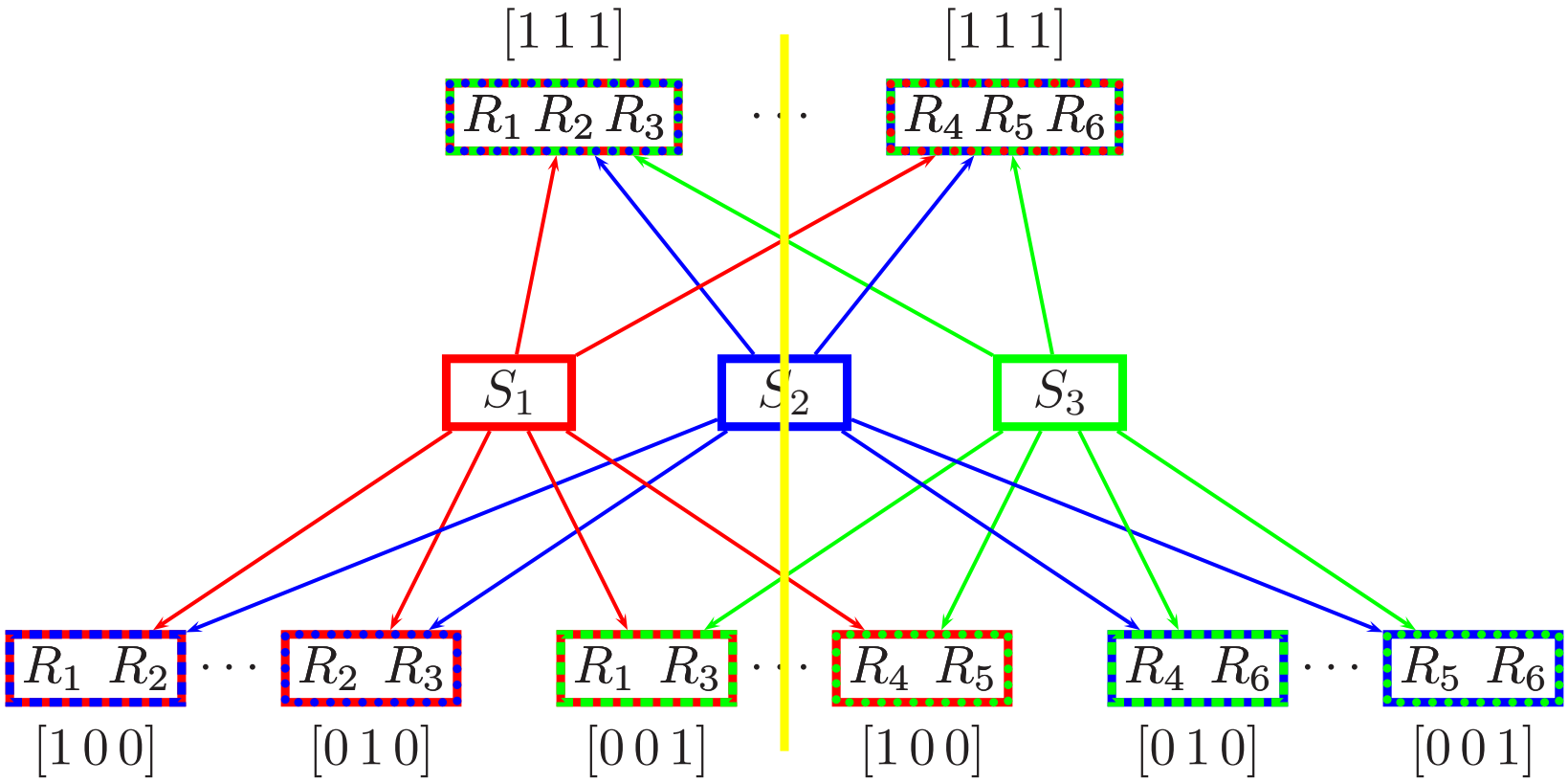


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