ON MULTICAST IN QUANTUM NETWORKS

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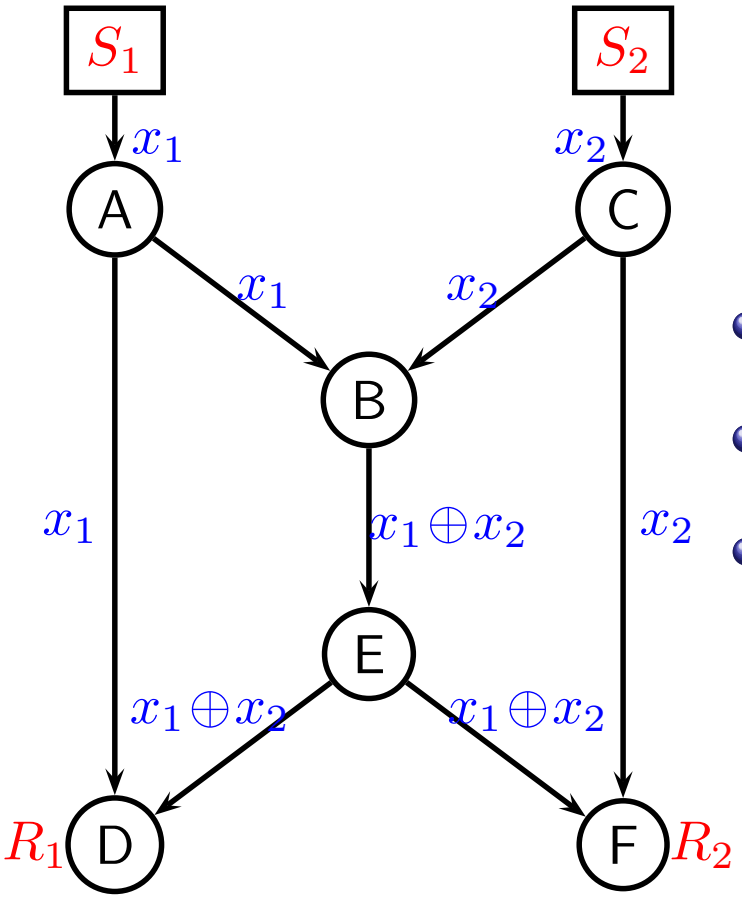
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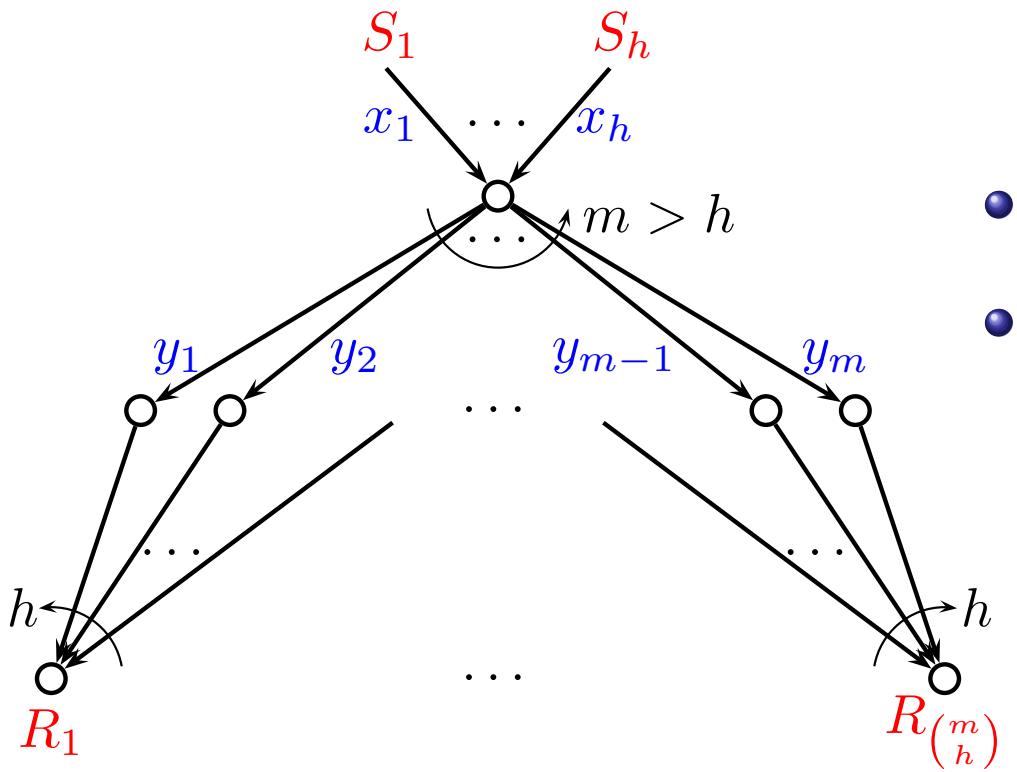
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NETWORK MULTICAST – The Butterfly



- Sources S_1 and S_2 produce bits x_1 and x_2.
- Each receiver needs bits from both sources.
- The edges have unit capacity.

NETWORK MULTICAST – Combination Network



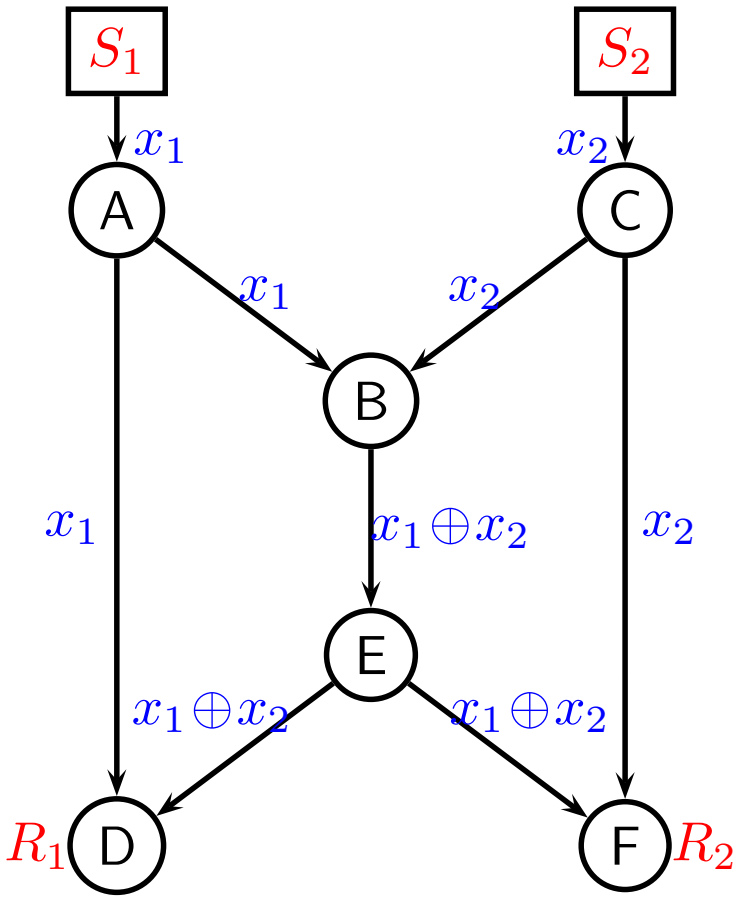
- Assume $x_j \in \mathbb{C}$, $j = 1, \ldots, h$.
- Apply an overcomplete FT to $\{x_i\}$:

$$y_k = \frac{1}{\sqrt{h}} \sum_{j=1}^{h} x_j W_m^{(j-1)(k-1)}$$

where $k = 1, \ldots, m$ and $W_m = e^{\imath \frac{2\pi}{m}}$.

Each receiver has access to h of the m numbers $\{y_k\}$ and can recover $\{x_j\}$.

QUANTUM BUTTERFLY



- What if x_1 and x_2 are qubits?
- Problems:
  - "No information without representation!"
  - The no-cloning principle.

Is quantum multicast just a multicommodity flow problem?

CLASSICAL MEMORYLESS SOURCES

- Discrete: produces sequences of letters.
- Letters belong to a finite alphabet $\mathcal{X}$.
- Memoryless: each letter is produced independently.
- Probability of letter a is P_a.
- Example: coin tossing with $\mathcal{X} = \{H, T\}$.

QUANTUM MEMORYLESS SOURCES

- Letters are transmitted as d-dimensional unit-length vectors in $\mathcal{H}_d$. When $d = 2$ we have qubits.
- $|e_0\rangle$ and $|e_1\rangle$ are the basis vectors of $\mathcal{H}_2$:

$$|e_0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |e_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Example: Alphabet $\mathcal{X} = \{0, 1, 2, 3\}$ mapped onto 4 qubits

$$|\psi_0\rangle = \alpha_0|e_0\rangle + \beta_0|e_1\rangle \quad |\psi_1\rangle = \alpha_1|e_0\rangle + \beta_1|e_1\rangle$$
$$|\psi_2\rangle = \alpha_2|e_0\rangle + \beta_2|e_1\rangle \quad |\psi_3\rangle = \alpha_3|e_0\rangle + \beta_3|e_1\rangle.$$

THE DENSITY MATRIX AND ENTROPY

- Source density matrix:

$$\rho = \sum_{a \in \mathcal{X}} P_a \underbrace{|\psi_a\rangle\langle\psi_a|}_{\rho_a}.$$

- Von Neumann entropy of the source:

$$\begin{aligned} S(\rho) =& -\operatorname{Tr} \rho \log \rho \\ =& -\sum_i \lambda_i \log \lambda_i, \end{aligned}$$

where λ_i are the eigenvalues of ρ.

THE MB EXAMPLE

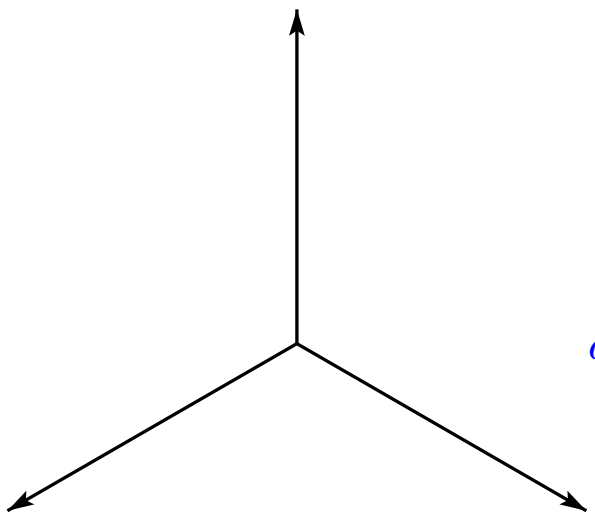
$$\mathcal{X} = \{1,2,3\} \quad P_1 = P_2 = P_3 = 1/3$$

$$\rho = \tfrac{1}{3}|\psi_1\rangle\langle\psi_1| + \tfrac{1}{3}|\psi_2\rangle\langle\psi_2| + \tfrac{1}{3}|\psi_3\rangle\langle\psi_3|$$

$$= \tfrac{1}{2}I$$

$$S(\rho) = 1$$

$$|\psi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$d = 2$

$$|\psi_3\rangle = \begin{bmatrix} -1/2 \\ -\sqrt{3}/2 \end{bmatrix} \qquad |\psi_2\rangle = \begin{bmatrix} -1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

VECTOR SEQUENCES

- Source vector-sequence (state)

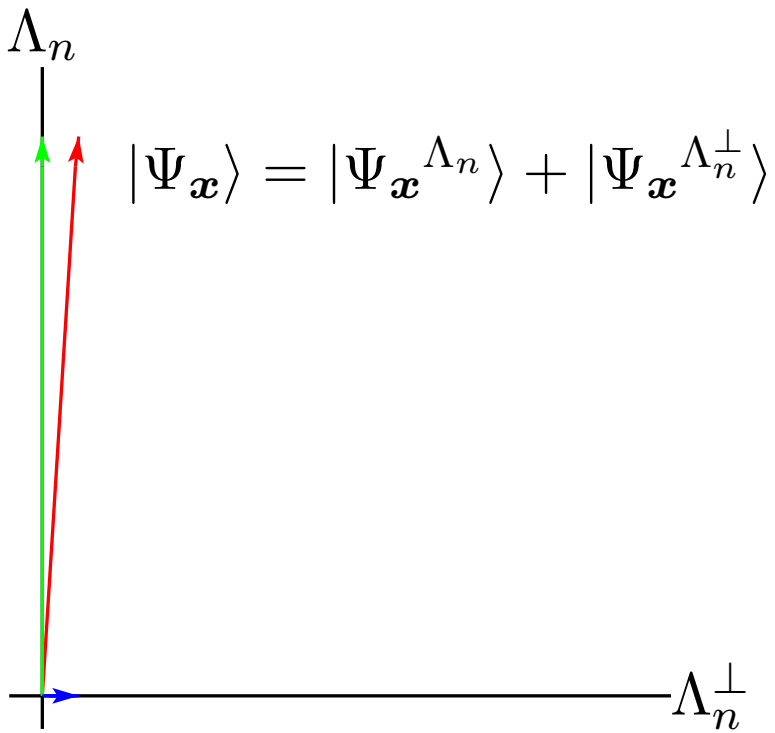
$$|\Psi_{\boldsymbol{x}}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \cdots \otimes |\psi_{x_n}\rangle, \qquad x_i \in \mathcal{X},$$

  appears with probability $P_{\boldsymbol{x}} = P_{x_1} \cdot P_{x_2} \cdot \ldots \cdot P_{x_n}$.

- Typical states $|\Psi_{\boldsymbol{x}}\rangle \in \mathcal{H}^{2^n}$ correspond to typical sequences $\boldsymbol{x}$.
- There are approximately $2^{nH(P)}$ typical states.

TYPICAL SUBSPACE

- Typical states $|\Psi_{\boldsymbol{x}}\rangle \in \mathcal{H}^{2^n}$ "live" in the typical subspace Λ_n



- The dimension of Λ_n is approximately $2^{nS(\rho)}$.

QUANTUM OPERATIONS – Evolution

- State ρ can be transformed to another state $\mathcal{E}(\rho)$ only by a physical process consistent with the lows of quantum theory:
  - unitary evolution:

$$\mathcal{E}(\rho) = U\rho U^\dagger \quad \text{where} \quad UU^\dagger = I,$$

  - completely positive, trace-preserving map:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad \text{where} \quad \sum_k E_k^\dagger E_k = I.$$

- An example of a two-qubit quantum gate is the XOR:

$$\texttt{XOR} : |x, y\rangle \rightarrow |x, x \oplus y\rangle \quad U_{\texttt{XOR}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CLONING, BROADCASTING, AND DELETING

- The No-Cloning Principle:

  There is no physical process that leads to an evolution

$$|\phi\rangle \otimes |s\rangle \rightarrow |\phi\rangle \otimes |\phi\rangle$$

  where $|\phi\rangle$ is an arbitrary state and $|s\rangle$ is a fixed state.

- The No-Broadcasting Principle – generalization of no-cloning.

- The No-Deleting Principle:

  There is no physical process that leads to an evolution

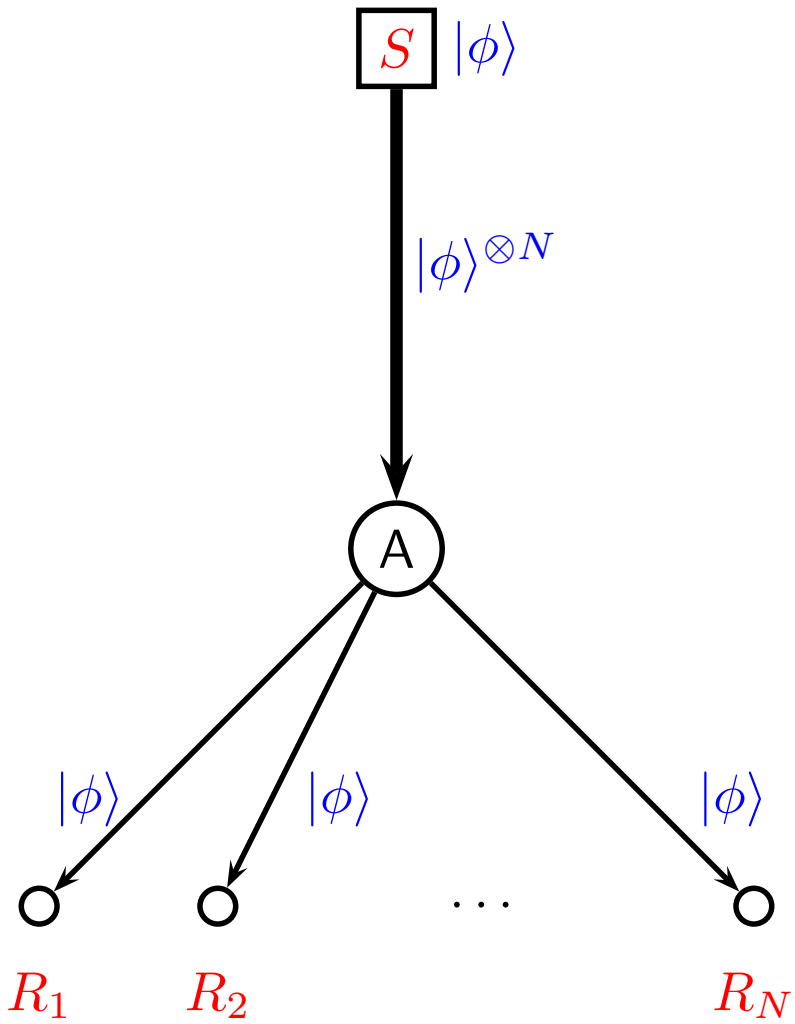
$$|\phi\rangle \otimes |\phi\rangle \rightarrow |\phi\rangle \otimes |s\rangle$$

  where $|\phi\rangle$ is an arbitrary state and $|s\rangle$ is a fixed state.

QUANTUM OPERATIONS – Measurement

- Von Neumann:
  - A set of pairwise orthogonal projection operators $\{\Pi_i\}$.
  - They form a complete resolution of the identity: $\sum_i \Pi_i = I$.
  - For input $|\psi_j\rangle$, output $\Pi_i|\psi_j\rangle$ happens with probability $\langle\psi_j|\Pi_i|\psi_j\rangle$.
- Positive Operator-Valued Measure (POVM):
  - Any set of positive-semidefinite operators $\{E_i\}$.
  - They form a complete resolution of the identity: $\sum_i E_i = I$.
  - For input $|\psi_j\rangle$, output $E_i|\psi_j\rangle$ happens with probability $\langle\psi_j|E_i|\psi_j\rangle$.

SINGLE-SOURCE QUANTUM MULTICAST



$|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$

- $\mathcal{H}$ is a d-dimensional vector space with a basis $\{|e_i\rangle,\ i = 1, \ldots, d\}$.
- $|\phi\rangle \in \mathcal{H}$ can be written as

$$|\phi\rangle = \sum_{i=1}^{d} \alpha_i |e_i\rangle.$$

- Let $\mathcal{X} = \{1, \ldots, d\}$. Then $|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N}$ can be written as

$$\begin{aligned} |\phi\rangle^{\otimes N} &= \Big(\sum_{i=1}^{d} \alpha_i |e_i\rangle\Big)^{\otimes N} \\ &= \sum_{\boldsymbol{x} \in \mathcal{X}^N} \alpha_{\boldsymbol{x}} |e_{\boldsymbol{x}}\rangle, \end{aligned}$$

where $\alpha_{\boldsymbol{x}} = \prod_{j=1}^{N} \alpha_{x_j}$ and $|e_{\boldsymbol{x}}\rangle = |e_{x_1}\rangle \otimes \cdots \otimes |e_{x_N}\rangle$.

THE SYMMETRIC SUBSPACE

- $|\phi\rangle^{\otimes N}$ has fewer degrees of freedom then d^N.
- $|\phi\rangle^{\otimes N}$ lives in $\mathcal{SYM}(\mathcal{H}^{\otimes N})$, the symmetric subspace of $\mathcal{H}^{\otimes N}$.
- The symmetric subspace is (equivalently)
  - the smallest subspace of $\mathcal{H}^{\otimes N}$ that contains all vectors of the form

$$|\phi\rangle^{\otimes N} \in \mathcal{H}^{\otimes N},$$

  - or, the subspace of $\mathcal{H}^{\otimes N}$ invariant under permutations of the qubits.
- What is the dimension of $\mathcal{SYM}(\mathcal{H}^{\otimes N})$?

TYPES AND TYPE CLASSES

- Let $\mathcal{X}$ be a finite set, $\boldsymbol{x} \in \mathcal{X}^N$ and $a \in \mathcal{X}$, and $\eta(a|\boldsymbol{x})$ denote the number occurrences of a in $\boldsymbol{x}$.
- Type of a sequence $\boldsymbol{x} \in \mathcal{X}^N$ is the distribution $P_{\boldsymbol{x}}$ given by

$$P_{\boldsymbol{x}}(a) = \frac{1}{N}\eta(a|\boldsymbol{x}) \quad \text{for every} \quad a \in \mathcal{X}.$$

- Type class of a distribution P is the set T_P^N:

$$\mathsf{T}_P^N = \{\boldsymbol{x} : \boldsymbol{x} \in \mathcal{X}^N \text{ and } P_{\boldsymbol{x}} = P\}.$$

- $\mathcal{P}_n(\mathcal{X})$ is the set consisting of all possible types of sequences in $\mathcal{X}^N$.
- The number of types is polynomial in N:

$$|\mathcal{P}_N(\mathcal{X})| = \binom{N + |\mathcal{X}| - 1}{|\mathcal{X}| - 1}.$$

A BASIS FOR THE SYMMETRIC SUBSPACE

- $|\phi\rangle^{\otimes N}$ has fewer degrees of freedom then d^N:

$$|\phi\rangle^{\otimes N} = \sum_{\boldsymbol{x}\in\mathcal{X}^N} \alpha_{\boldsymbol{x}} |e_{\boldsymbol{x}}\rangle$$

$$= \sum_{P\in\mathcal{P}_N(\mathcal{X})} \prod_{i=1}^{d} \alpha_i^{NP(i)} \sum_{\boldsymbol{x}\in\mathsf{T}_P^N} |e_{\boldsymbol{x}}\rangle$$

- An orthonormal basis for $\mathcal{SYM}(\mathcal{H}^{\otimes N})$:

$$|E_P\rangle = \frac{1}{\sqrt{|\mathsf{T}_P^N|}} \sum_{\boldsymbol{x}\in\mathsf{T}_P^N} |e_{\boldsymbol{x}}\rangle, \quad P \in \mathcal{P}_N(\mathcal{X}).$$

LOSSLESS COMPRESSION INTO $\mathcal{SYM}(\mathcal{H}^{\otimes N})$

- We have to map $|\phi\rangle^{\otimes N}$ into a $O(\log N)$-qubit state.
- Consider the $d = 2$ case (qubits):

$$|E_i\rangle = \frac{1}{\sqrt{\binom{N}{i}}} \sum_{x\in\{0,1\}^N, |x|=i} |x\rangle, \qquad 0 \le i \le N.$$

- A quantum circuit W that maps $|i\rangle \mapsto |E_i\rangle$ can be constructed.
- W^{-1} does the compression.

QUANTUM NETWORK MULTICAST

- Network represented as a directed graph $G = (V, E)$.
- There are h information sources $S_1, \ldots, S_h$,
  each producing a qubit per unit time.
- There are N receivers $\mathcal{R} = \{R_1, \ldots, R_N\}$.
- For each receiver,
  there are h edge disjoint paths to it, one from each of the h sources.
- The h sources need to multicast h qubits to all N receivers.

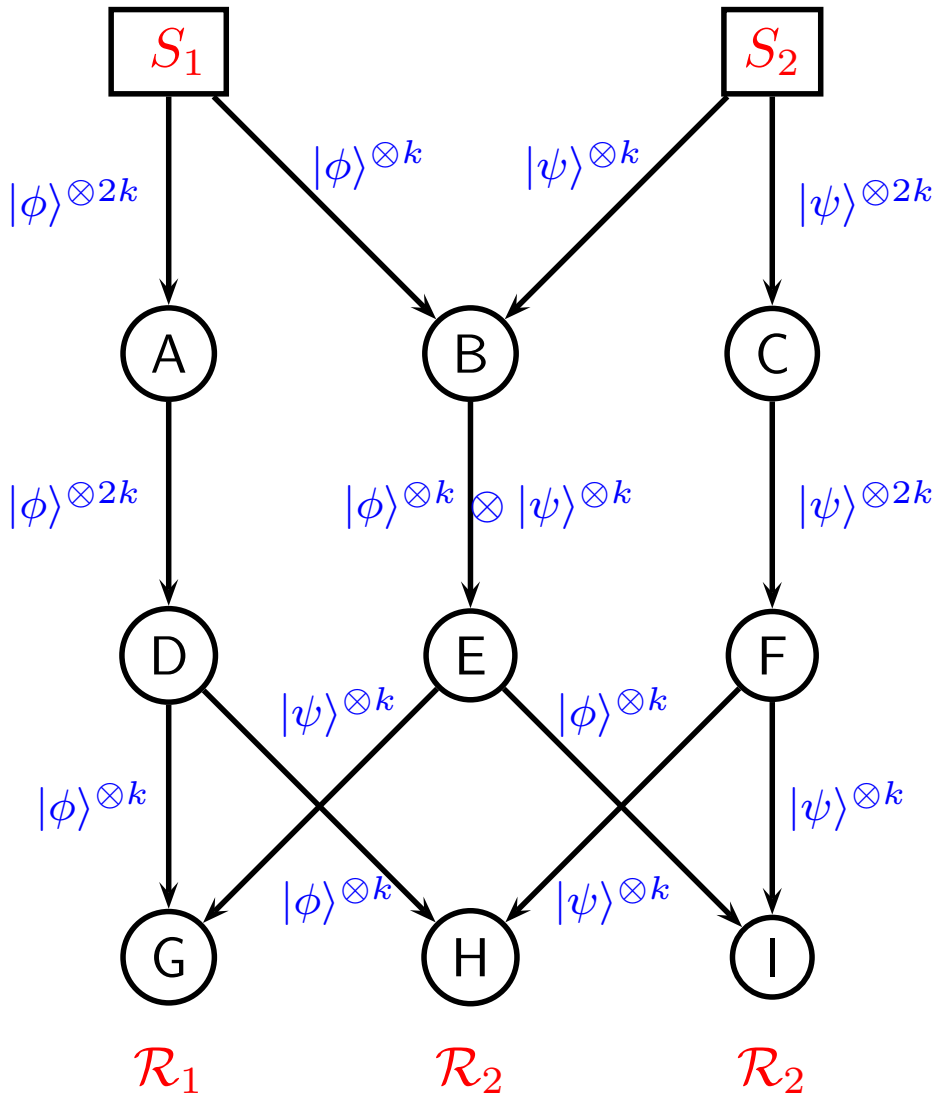
QUANTUM NETWORK MULTICAST

- $\mathcal{I}(e) \subseteq \mathcal{I}$ comprises the labels of sources present at edge $e \in E$.
- $r(i;e)$ is the number of all receivers $R_j \in \mathcal{R}$ for which the paths (S_i, R_j) form source S_i pass through edge e.
- The quantum state at edge e lives in the space of dimension

$$\prod_{i \in \mathcal{I}(e)} (r(i;e)+1) \leq \left[\frac{\sum_{i \in \mathcal{I}(e)} (r(i;e)+1)}{|\mathcal{I}(e)|} \right]^{|\mathcal{I}(e)|}$$

- Further upper bounds on the sufficient edge capacity:
  $|\mathcal{I}(e)| \log(N+1)$, $N \log_2 e$ qubits.

QUANTUM NETWORK MULTICAST



ROUTING FOR QUANTUM MULTICAST

