Punctured vs Rateless Codes for Hybrid ARQ

Emina Soljanin

Mathematical and Algorithmic Sciences Research, Bell Labs

Collaborations with R. Liu, P. Spasojevic, N. Varnica and P. Whiting

Tsinghua University

10/30/2006

AUTOMATIC REPEAT REQUEST

- The receiving end detects frame errors and requests retransmissions.
- If P_e is the frame error rate, the average number of transmissions is

$$1 \cdot (1 - P_e) + \cdots + n \cdot P_e^{n-1}(1 - P_e) + \cdots = \frac{1}{1 - P_e}$$

AUTOMATIC REPEAT REQUEST

- The receiving end detects frame errors and requests retransmissions.
- If P_e is the frame error rate, the average number of transmissions is

$$1 \cdot (1 - P_e) + \cdots + n \cdot P_e^{n-1}(1 - P_e) + \cdots = \frac{1}{1 - P_e}$$

- Hybrid ARQ uses a code that can correct some frame errors.
- In (type I and II) HARQ schemes
  - the average number of transmissions is reduced, but
  - each transmission carries redundant information.
- Incremental redundancy (IR) HARQ is
  - appropriate for time-varying channels (mobile wireless),
  - used in 3G wireless standards based on Turbo codes.

IR-HARQ BASED ON PUNCTURING

- Information bits are encoded by a (low rate) mother code.
- Information and a selected number of parity bits are transmitted.
- If a retransmission is not successful:
  - transmitter sends additional selected parity bits
  - receiver puts together the new bits and those previously received.
- Each retransmission produces a codeword of a stronger code.

IR-HARQ BASED ON PUNCTURING

- Information bits are encoded by a (low rate) mother code.
- Information and a selected number of parity bits are transmitted.
- If a retransmission is not successful:
  - transmitter sends additional selected parity bits
  - receiver puts together the new bits and those previously received.
- Each retransmission produces a codeword of a stronger code.

TSINGHUA

IR-HARQ BASED ON PUNCTURING

- Information bits are encoded by a (low rate) mother code.
- Information and a selected number of parity bits are transmitted.
- If a retransmission is not successful:
  - transmitter sends additional selected parity bits
  - receiver puts together the new bits and those previously received.
- Each retransmission produces a codeword of a stronger code.

TSINGHUA UNIVERSITY

IR-HARQ BASED ON PUNCTURING

- Information bits are encoded by a (low rate) mother code.
- Information and a selected number of parity bits are transmitted.
- If a retransmission is not successful:
  - transmitter sends additional selected parity bits
  - receiver puts together the new bits and those previously received.
- Each retransmission produces a codeword of a stronger code.

TSINGHUA UNIVERSITY NORTH WEST

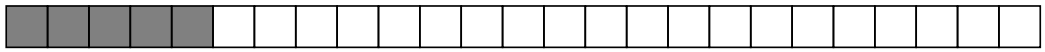
IR-HARQ BASED ON PUNCTURING

- Information bits are encoded by a (low rate) mother code.
- Information and a selected number of parity bits are transmitted.
- If a retransmission is not successful:
  - transmitter sends additional selected parity bits
  - receiver puts together the new bits and those previously received.
- Each retransmission produces a codeword of a stronger code.

TSINGHUA UNIVERSITY NORTH WEST BEIJING

IR-HARQ EXAMPLE – Punctured Rate $1/5$ Mother Code

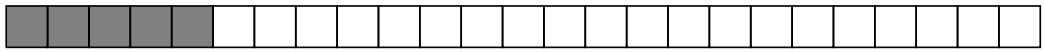
TSINGHUA UNIVERSITY NORTH WEST BEIJING



at the transmitter

IR-HARQ EXAMPLE – Punctured Rate $1/5$ Mother Code

TSINGHUA UNIVERSITY NORTH WEST BEIJING



at the transmitter



transmission # 1

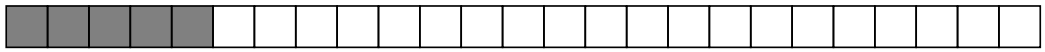


at the receiver

TSINGHUA

IR-HARQ EXAMPLE – Punctured Rate $1/5$ Mother Code

TSINGHUA UNIVERSITY NORTH WEST BEIJING



at the transmitter



transmission # 1



transmission # 2

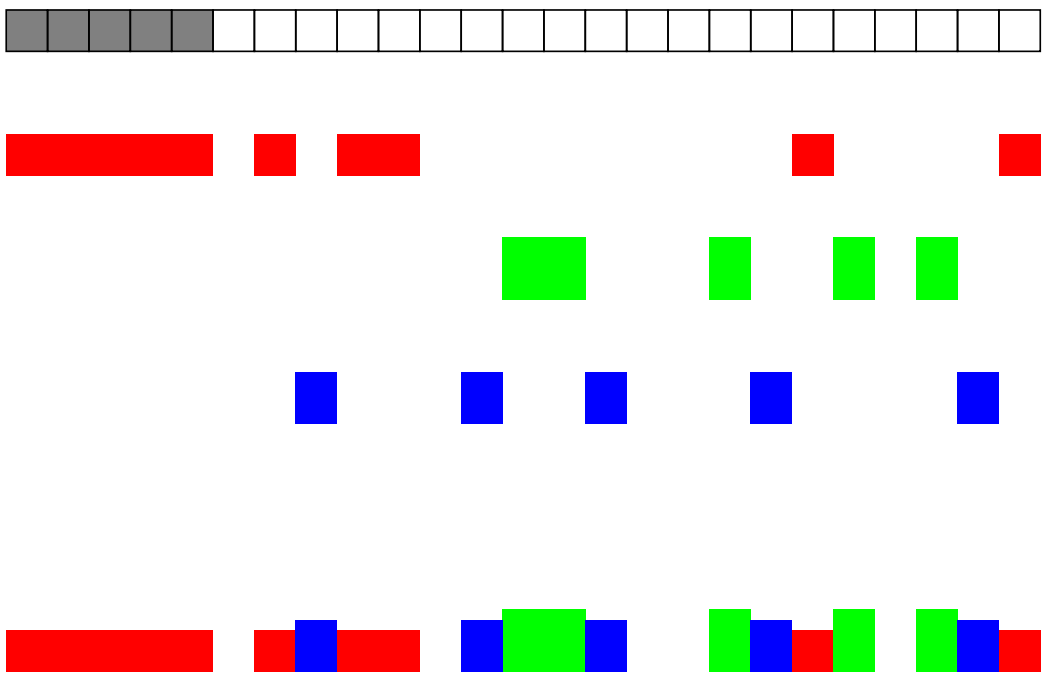


at the receiver

TSINGHUA UNIVERSTY

IR-HARQ EXAMPLE – Punctured Rate $1/5$ Mother Code

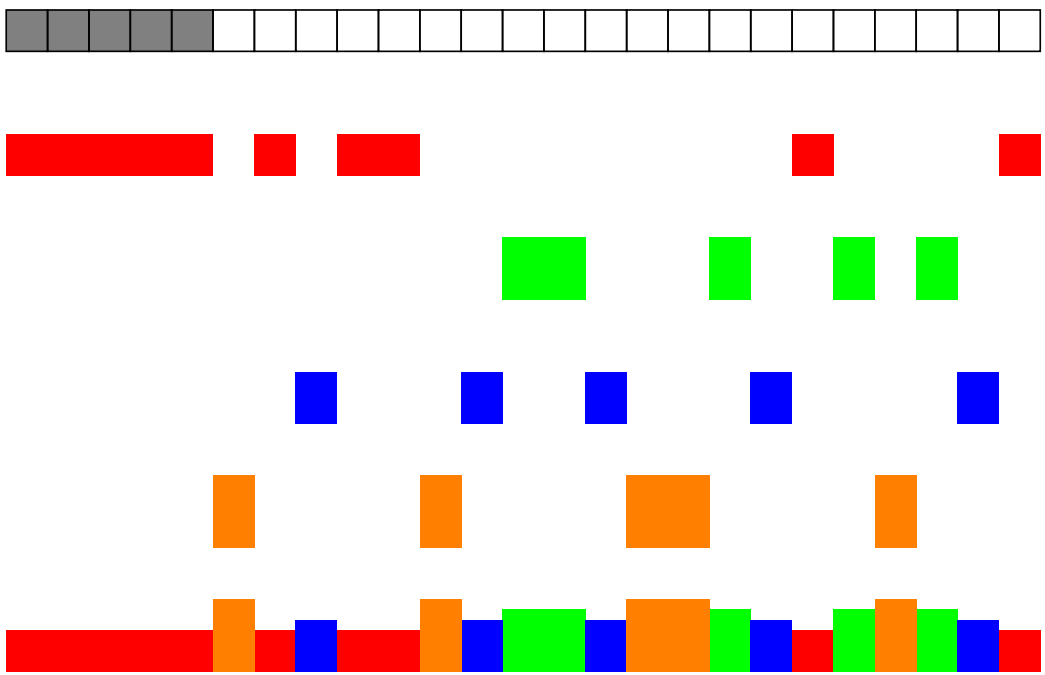
TSINGHUA UNIVERSITY NORTH WEST BEIJING



TSINGHUA UNIVERSTY NORTH WEST

IR-HARQ EXAMPLE – Punctured Rate $1/5$ Mother Code

TSINGHUA UNIVERSITY NORTH WEST BEIJING



TSINGHUA UNIVERSTY NORTH WEST BEIJING

ABOUT THIS TALK

- If transmission $m-1$ fails, one has to decide
  - how many bits to send in transmission m,
  - which bits to send in transmission m,
  - at which power to send the bits in transmission m,

  and we will look into these questions for two HARQ schemes.

ABOUT THIS TALK

- If transmission $m-1$ fails, one has to decide
  - how many bits to send in transmission m,
  - which bits to send in transmission m,
  - at which power to send the bits in transmission m,

  and we will look into these questions for two HARQ schemes.
- Talk outline:
  - introduce linear codes
  - define memoryless communications channels
  - explain how Fountain codes work
  - derive transmission rules for two schemes
  - compare the schemes

BINARY LINEAR CODES

- Suppose we want to communicate k bits $\omega_1, \ldots, \omega_k$.
- If we encode $\omega_1, \ldots, \omega_k$ into $n = k + 1$ bits $x_1, \ldots, x_k, x_{k+1}$ s.t.

$$x_i = \omega_i \text{ for } 1 \leq i \leq n-1 \text{ and } x_n = \omega_1 + \cdots + \omega_{n-1},$$

  we can decode all $\omega_1, \ldots, \omega_k$ even if one of $x_1, \ldots, x_n$ is erased.
- The generator matrix of this (n, k) code is

$$G = \begin{bmatrix} 1 & 0 & \ldots & 0 & 1 \\ 0 & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 1 \end{bmatrix}_{k \times n}$$

BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?

BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?
  - performance

BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?
  - performance
  - encoding/decoding complexity

BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?
  - performance
  - encoding/decoding complexity
- Which code choices are good?

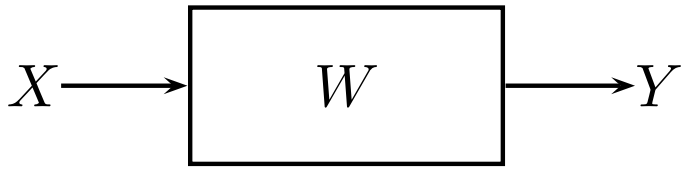
BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?
  - performance
  - encoding/decoding complexity
- Which code choices are good?
  - random-like codes

BINARY LINEAR CODES

- G defines n equations featuring k information symbols.
- Which code features are important?
  - performance
  - encoding/decoding complexity
- Which code choices are good?
  - random-like codes
  - 3G wireless implements a turbo code based HARQ
  - we will consider LDPC and Fountain codes

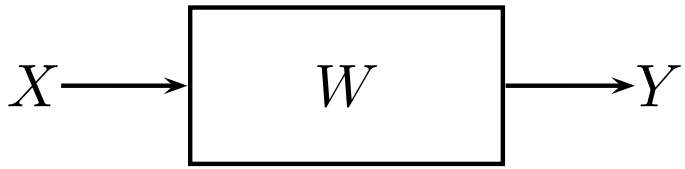
THE MEMORYLESS CHANNEL MODEL



- X and Y are random variables.
- We assume binary input alphabet $\{0, 1\}$ and output alphabet $\mathcal{Y}$.
- Constant in time with

  transition probabilities $W(b|0)$ and $W(b|1)$, $b \in \mathcal{Y}$.

THE MEMORYLESS CHANNEL MODEL

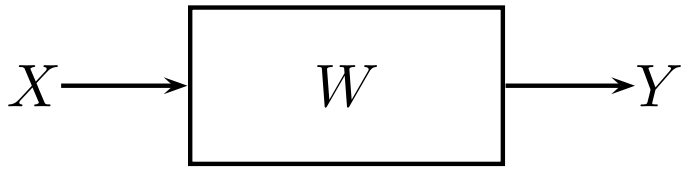


- X and Y are random variables.
- We assume binary input alphabet $\{0, 1\}$ and output alphabet $\mathcal{Y}$.
- Constant in time with

  transition probabilities $W(b|0)$ and $W(b|1)$, $b \in \mathcal{Y}$.

  Bhattacharyya noise parameter $\gamma = \sum_{b \in \mathcal{Y}} \sqrt{W(b|0)W(b|1)}$.

THE MEMORYLESS CHANNEL MODEL



- X and Y are random variables.
- We assume binary input alphabet $\{0, 1\}$ and output alphabet $\mathcal{Y}$.
- Constant in time with

  transition probabilities $W(b|0)$ and $W(b|1)$, $b \in \mathcal{Y}$.

  Bhattacharyya noise parameter $\gamma = \sum_{b \in \mathcal{Y}} \sqrt{W(b|0)W(b|1)}$.
- Time varying with

  transition probabilities at time i $W_i(b|0)$ and $W_i(b|1)$, $b \in \mathcal{Y}$.
- $W_i(\cdot|0)$ and $W_i(\cdot|1)$ are known at the receiver.

PERFORMANCE MEASURE – Time-Invariant Channel

- An (n, k) binary linear code $\mathcal{C}$ with A_d codewords of weight d.
- The union-Bhattacharyya bound on word error probability:

$$P_W^{\mathcal{C}} \leq \sum_{d=1}^{n} A_d \gamma^d,$$

γ is the Bhattacharyya noise parameter $\gamma = \sum_{b \in \mathcal{Y}} \sqrt{W(b|0)W(b|1)}$.

PERFORMANCE MEASURE – Time-Invariant Channel

- An (n, k) binary linear code $\mathcal{C}$ with A_d codewords of weight d.
- The union-Bhattacharyya bound on word error probability:

$$P_W^{\mathcal{C}} \leq \sum_{d=1}^{n} A_d \gamma^d,$$

  γ is the Bhattacharyya noise parameter $\gamma = \sum_{b \in \mathcal{Y}} \sqrt{W(b|0)W(b|1)}$.

- Weight distribution A_d for a given LDPC code?
- Consider the set of all codes $[\mathcal{C}]$ with a given degree distribution.
- Use the average $\overline{A}_d^{[\mathcal{C}](n)}$ instead of A_d for large n.

THE THRESHOLD BEHAVIOR

$$\overline{P}_W^{[\mathcal{C}](n)} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \gamma^d$$

THE THRESHOLD BEHAVIOR

$$\begin{aligned}\overline{P}_W^{[\mathcal{C}](n)} &\leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \gamma^d \\ &\leq \sum_{d=1}^{\lceil n\theta_0 \rceil} \overline{A}_d^{[\mathcal{C}](n)} + \sum_{d=\lceil n\theta_0 \rceil+1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \gamma^d\end{aligned}$$

THE THRESHOLD BEHAVIOR

$$\begin{aligned}
\overline{P}_W^{[\mathcal{C}](n)} &\leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \gamma^d \\
&\leq \sum_{d=1}^{\lceil n\theta_0 \rceil} \overline{A}_d^{[\mathcal{C}](n)} + \sum_{d=\lceil n\theta_0 \rceil+1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \gamma^d \qquad \exists \theta_0, 0 < \theta_0 < 1 \\
&\leq O(n^{-\frac{1}{2}}) + \sum_{d=\lceil n\theta_0 \rceil+1}^{n} \exp[d(c_0^{[\mathcal{C}]} + \log \gamma)],
\end{aligned}$$

where $c_0^{[\mathcal{C}]}$ is the noise threshold of the code ensemble:

$$c_0^{[\mathcal{C}]} \triangleq \limsup_{n\to\infty} \max_{\lceil n\theta_0 \rceil < d \leq n} \frac{\log \overline{A}_d^{[\mathcal{C}](n)}}{d}.$$

AN HARQ STRATEGY

- We assume that the transmitter
  - wants to maximize the throughput,

AN HARQ STRATEGY

- We assume that the transmitter
  - wants to maximize the throughput,
  - knows the mother code and data rates of the past transmissions,

AN HARQ STRATEGY

- We assume that the transmitter
  - wants to maximize the throughput,
  - knows the mother code and data rates of the past transmissions,
  - is informed of channel information of the past transmissions.

AN HARQ STRATEGY

- We assume that the transmitter
  - wants to maximize the throughput,
  - knows the mother code and data rates of the past transmissions,
  - is informed of channel information of the past transmissions.
- Strategy:

  The transmitter will send only as many codeword symbols as necessary to insure a high probability of successful ML decoding assuming a high SNR channel during the current transmission.

HARQ MODEL

- There are at most M transmissions.
- $\mathcal{I} = \{1, \ldots, n\}$ is the set indexing the bit positions in a codeword.
- $\mathcal{I}$ is partitioned in M subsets $\mathcal{I}(j)$, for $1 \leq j \leq M$.
- Bits at positions in $\mathcal{I}(j)$ are transmitted during j-th transmission.

HARQ MODEL

- There are at most M transmissions.
- $\mathcal{I} = \{1, \ldots, n\}$ is the set indexing the bit positions in a codeword.
- $\mathcal{I}$ is partitioned in M subsets $\mathcal{I}(j)$, for $1 \leq j \leq M$.
- Bits at positions in $\mathcal{I}(j)$ are transmitted during j-th transmission.
- The channel remains constant during a single transmission.
- γ_j is the Bhattacharyya noise parameter for transmission j.

PERFORMANCE MEASURE – Time-Varying Channel

- Assume that all M transmissions have taken place.
- The union bound on the ML decoder word error probability:

$$P_W^{\mathcal{C}} \leq \sum_{d_1=1}^{|\mathcal{I}(1)|} \cdots \sum_{d_M=1}^{|\mathcal{I}(M)|} A_{d_1 \ldots d_M} \prod_{j=1}^{M} \gamma_j^{d_j},$$

$A_{d_1 \ldots d_M}$ is the number of codewords with weight d_j over $\mathcal{I}(j)$.

PERFORMANCE MEASURE – Time-Varying Channel

- Assume that all M transmissions have taken place.
- The union bound on the ML decoder word error probability:

$$P_W^{\mathcal{C}} \leq \sum_{d_1=1}^{|\mathcal{I}(1)|} \cdots \sum_{d_M=1}^{|\mathcal{I}(M)|} A_{d_1 \ldots d_M} \prod_{j=1}^{M} \gamma_j^{d_j},$$

  $A_{d_1 \ldots d_M}$ is the number of codewords with weight d_j over $\mathcal{I}(j)$.
- Problems:
  - How to deal with the split-weight enumerator $A_{d_1 \ldots d_M}$?
  - How to evaluate the error-rate at the m-th transmission for $m < M$?

HARQ PERFORMANCE

- A bit is assigned to transmission j with probability α_j.

HARQ PERFORMANCE

- A bit is assigned to transmission j with probability α_j.
- The bits which are not yet transmitted are transmitted over a really bad channel, that is, a channel with the noise parameter equal to 1.

HARQ PERFORMANCE

- A bit is assigned to transmission j with probability α_j.
- The bits which are not yet transmitted are transmitted over a really bad channel, that is, a channel with the noise parameter equal to 1.
- The performance after m transmissions, $1 \leq m \leq M$, is

$$\overline{P}_W^{[\mathcal{C}](n)} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \overline{\gamma}_m^d, \quad \overline{\gamma}_m = \left(1 - \sum_{j=1}^{m} \alpha_j\right) \cdot 1 + \sum_{j=1}^{m} \gamma_j \alpha_j.$$

HARQ PERFORMANCE

- A bit is assigned to transmission j with probability α_j.
- The bits which are not yet transmitted are transmitted over a really bad channel, that is, a channel with the noise parameter equal to 1.
- The performance after m transmissions, $1 \leq m \leq M$, is

$$\overline{P}_W^{[\mathcal{C}](n)} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \overline{\gamma}_m^d, \quad \overline{\gamma}_m = \left(1 - \sum_{j=1}^{m} \alpha_j\right) \cdot 1 + \sum_{j=1}^{m} \gamma_j \alpha_j.$$

- Require $\overline{\gamma}_m < \exp(-c_0^{[\mathcal{C}]})$.

THE m-th TRANSMISSION

- The average Bhattacharyya parameter after the m-th transmission:

$$\overline{\gamma}_m = \alpha_1 \cdot \gamma_1 + \cdots + \alpha_m \cdot \gamma_m + (1 - \alpha_1 - \cdots - \alpha_m) \cdot 1.$$

- Choose α_m or γ_m or both so that

$$\overline{\gamma}_m < \exp(-c_0^{[\mathcal{C}]}).$$

THE m-th TRANSMISSION

- The average Bhattacharyya parameter after the m-th transmission:

$$\overline{\gamma}_m = \alpha_1 \cdot \gamma_1 + \cdots + \alpha_m \cdot \gamma_m + (1 - \alpha_1 - \cdots - \alpha_m) \cdot 1.$$

- Choose α_m or γ_m or both so that

$$\overline{\gamma}_m < \exp(-c_0^{[\mathcal{C}]}).$$

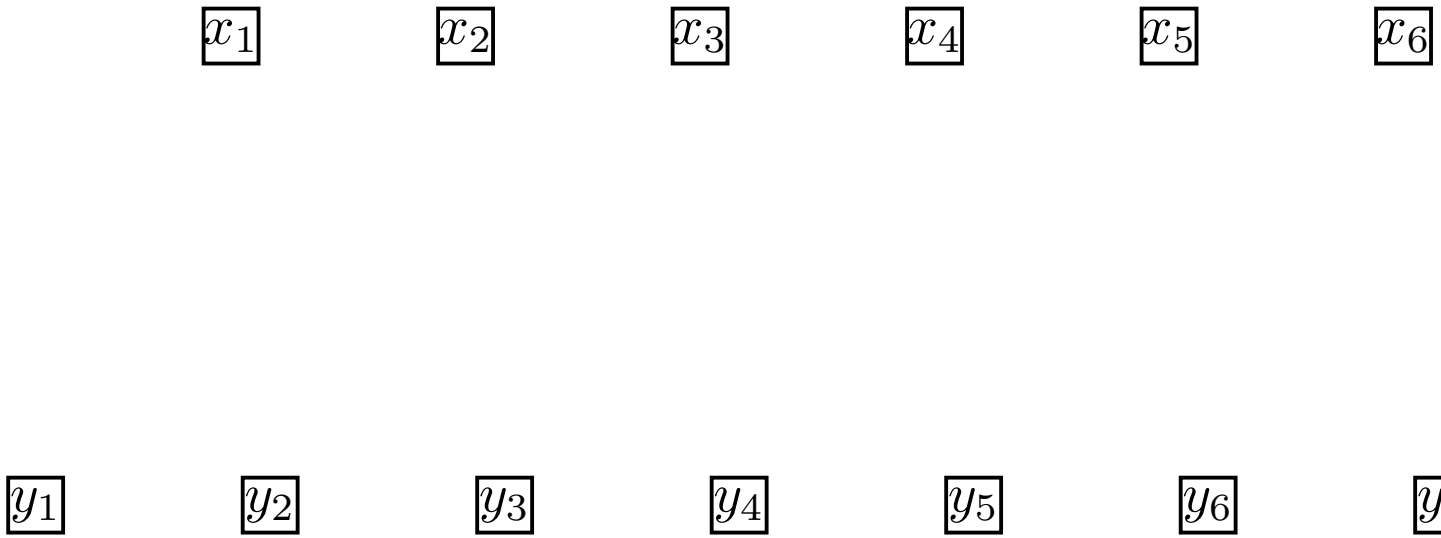
- The tradeoff between the rate and the power:

$$\alpha_m(1 - \gamma_m) > 1 - \exp(-c_0^{[\mathcal{C}]}) - \sum_{j=1}^{m-1} \alpha_j(1 - \gamma_j).$$

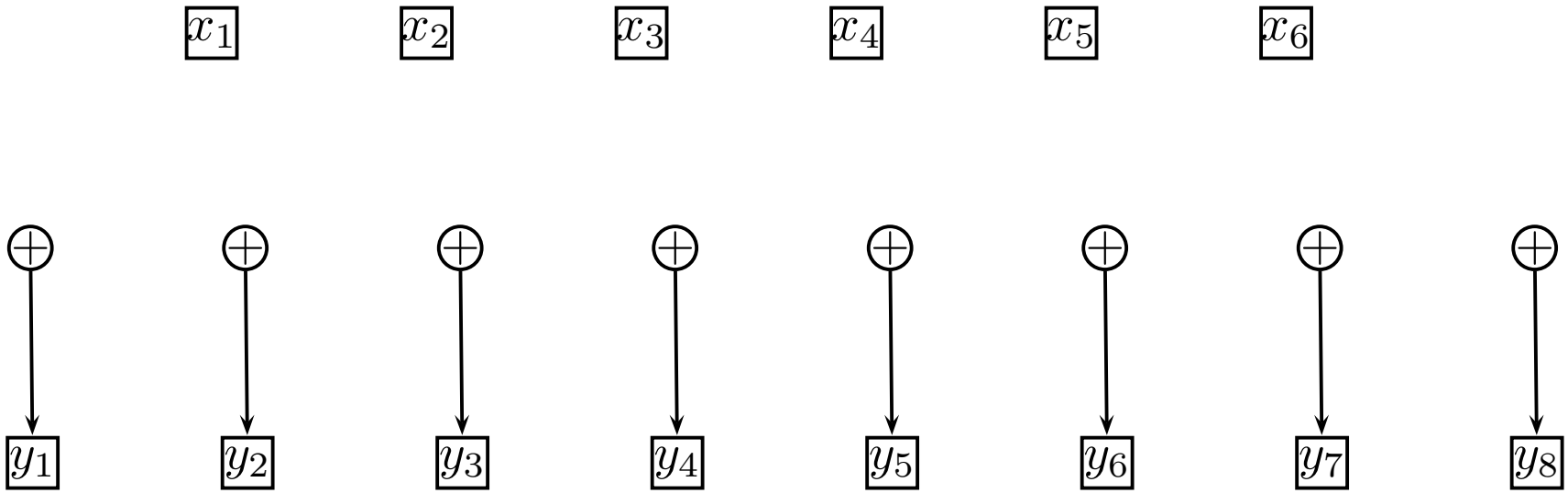
LT CODES – Encoding



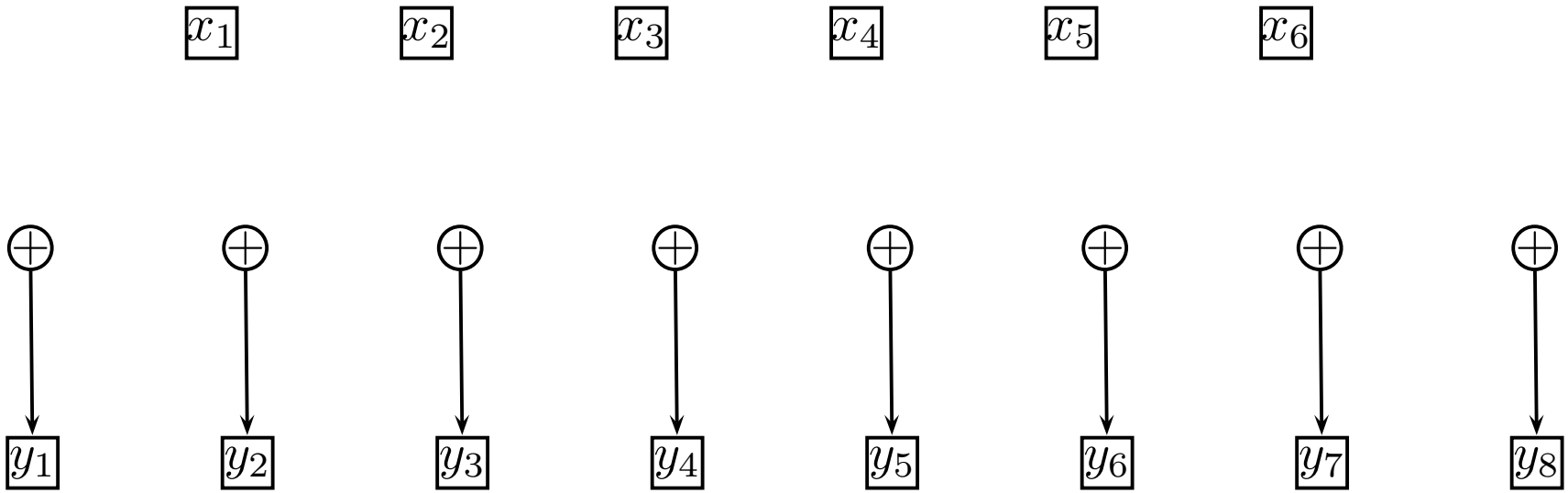
LT CODES – Encoding



LT CODES – Encoding

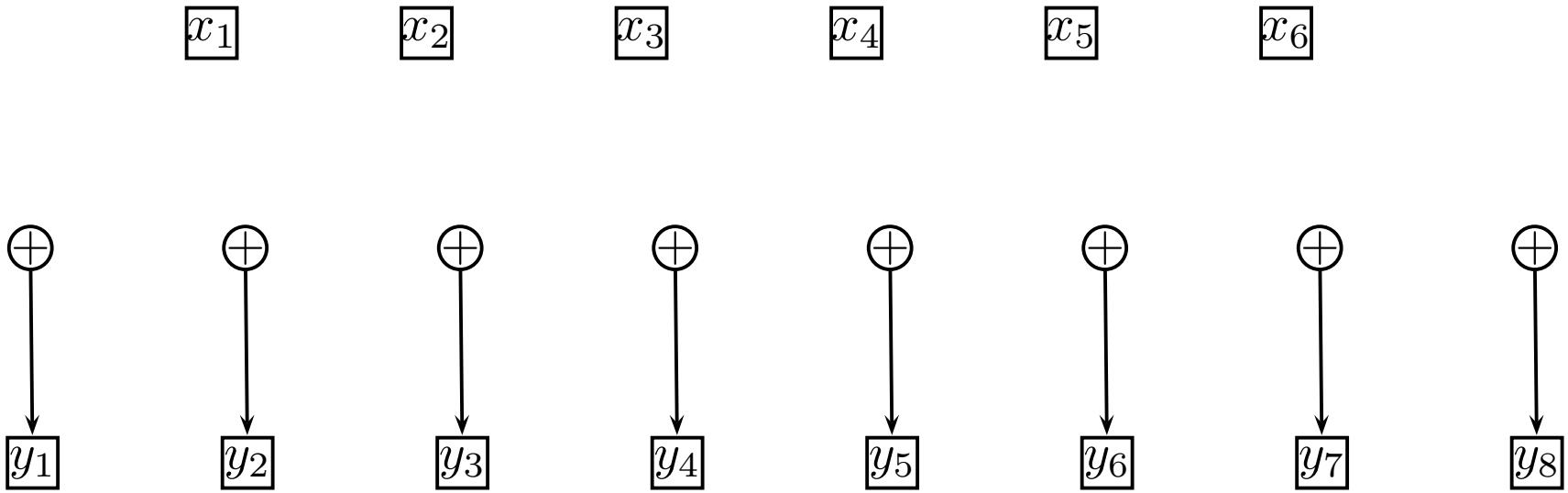


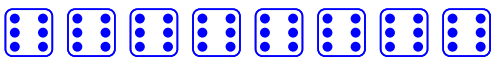
LT CODES – Encoding



face						
Ω	.05	.5	.1	.05	.25	.05

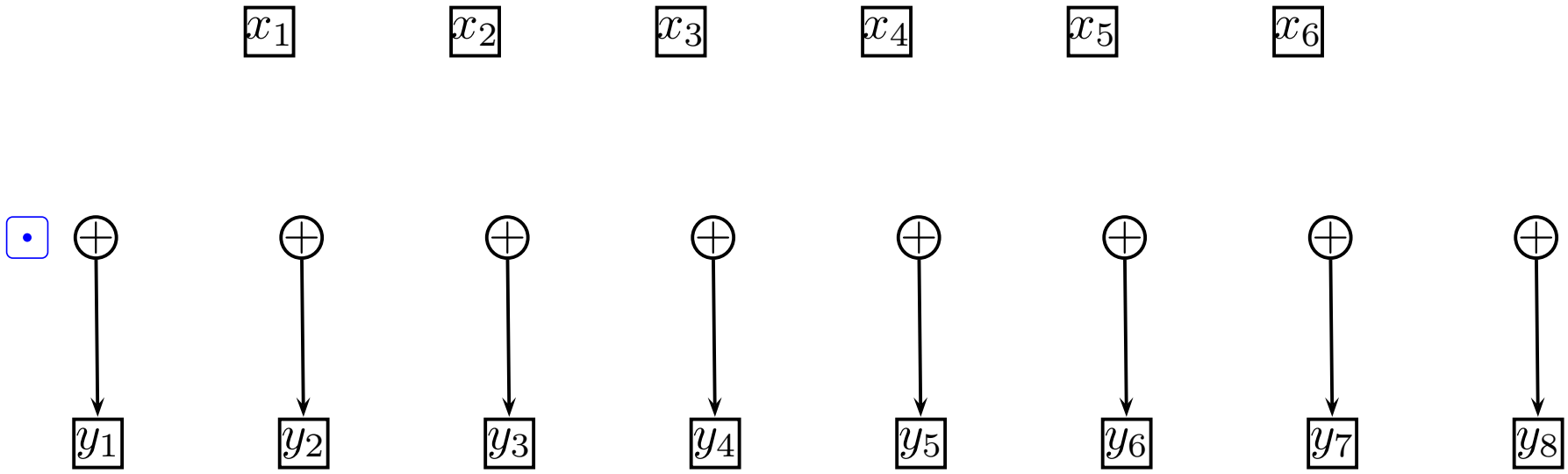
LT CODES – Encoding





face						
Ω	.05	.5	.1	.05	.25	.05

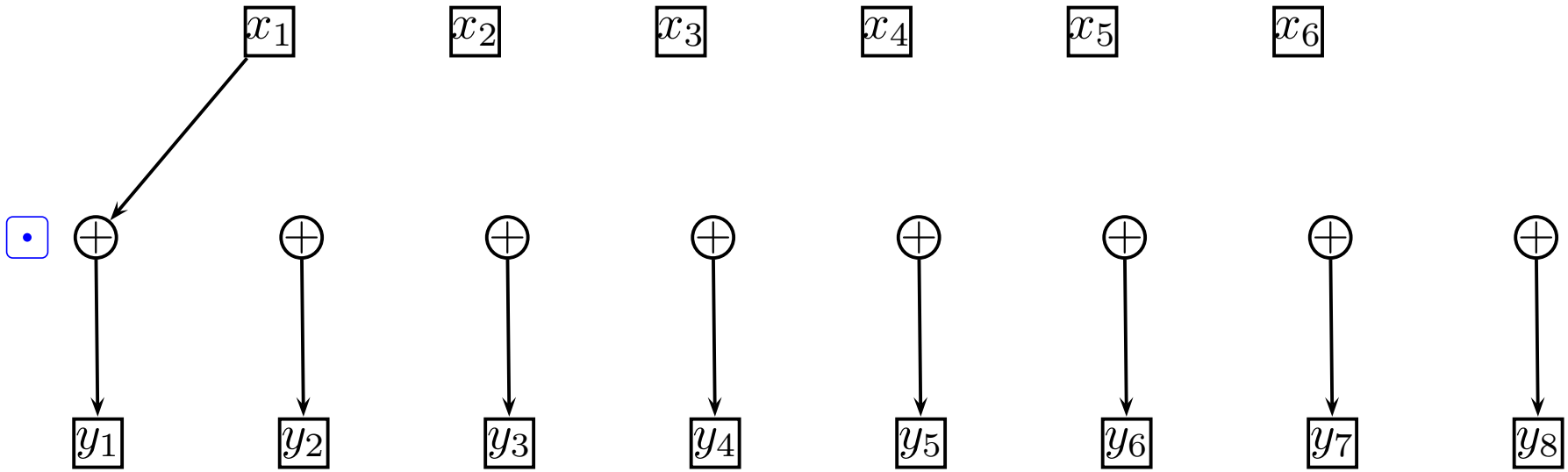
LT CODES – Encoding





face						
Ω	.05	.5	.1	.05	.25	.05

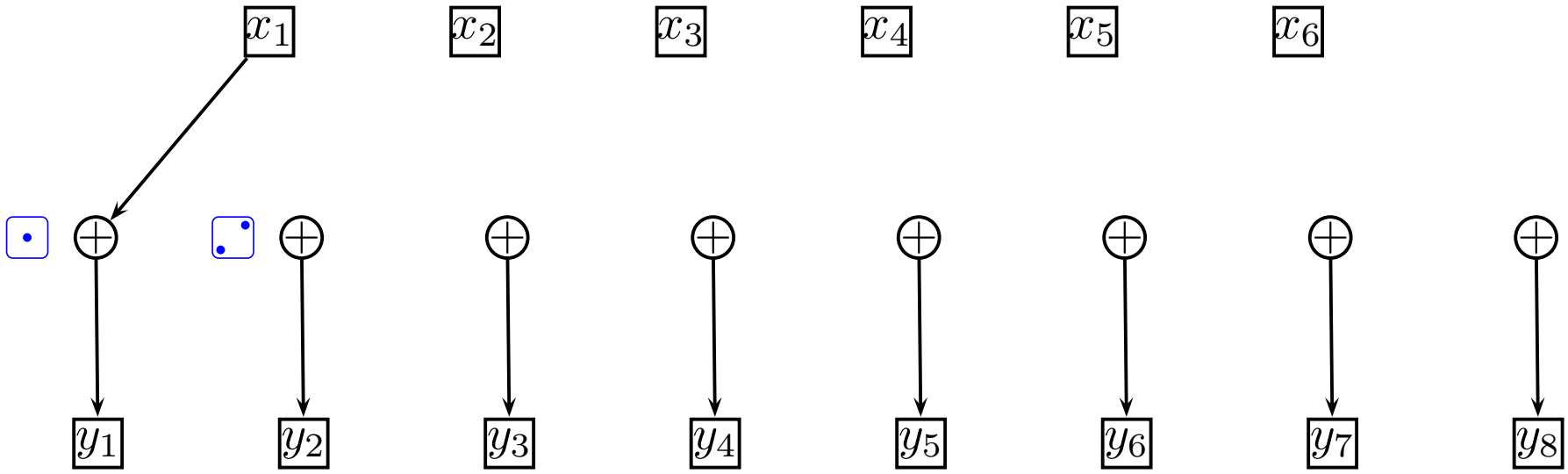
LT CODES – Encoding

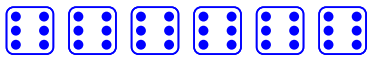




face						
Ω	.05	.5	.1	.05	.25	.05

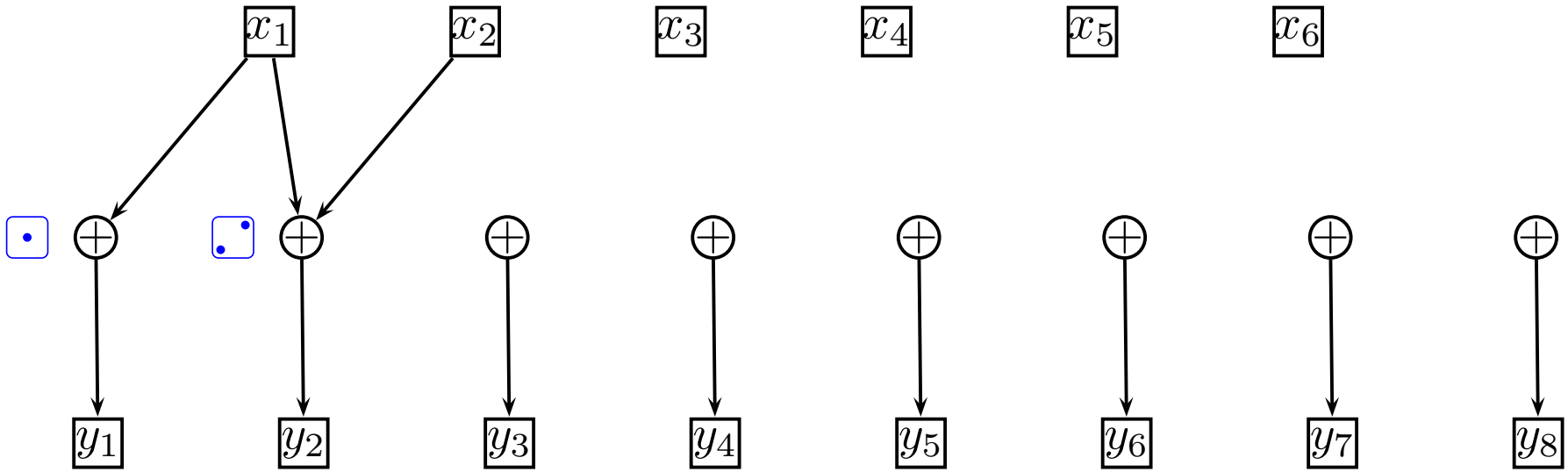
LT CODES – Encoding

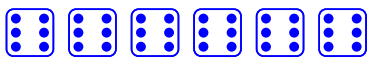




face						
Ω	.05	.5	.1	.05	.25	.05

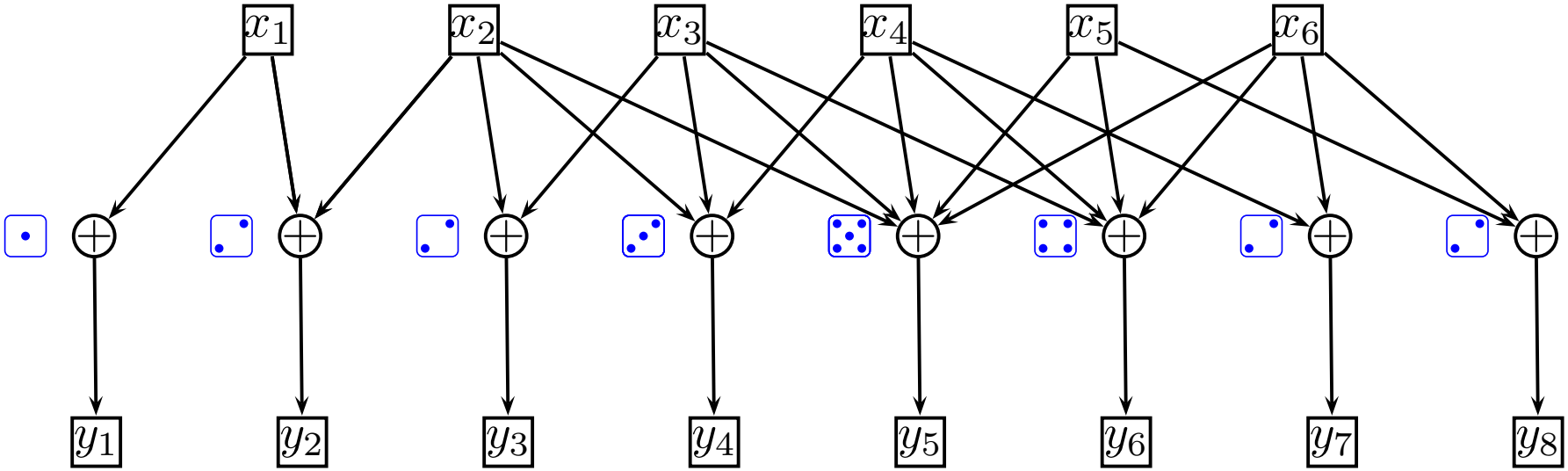
LT CODES – Encoding





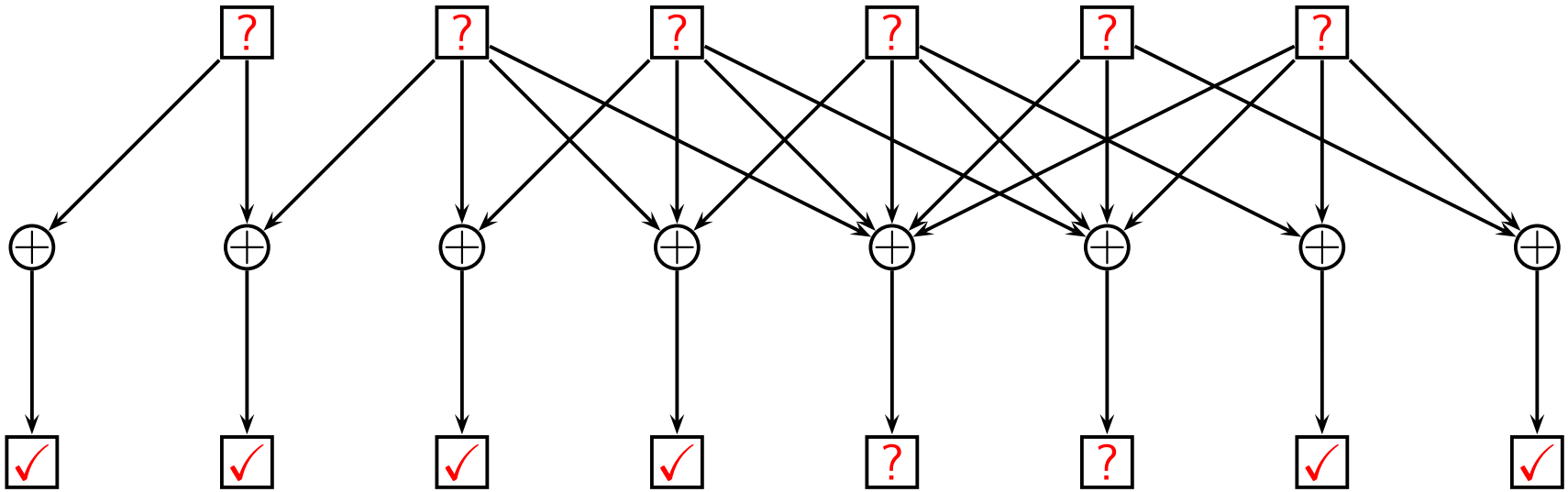
face						
Ω	.05	.5	.1	.05	.25	.05

LT CODES – Encoding

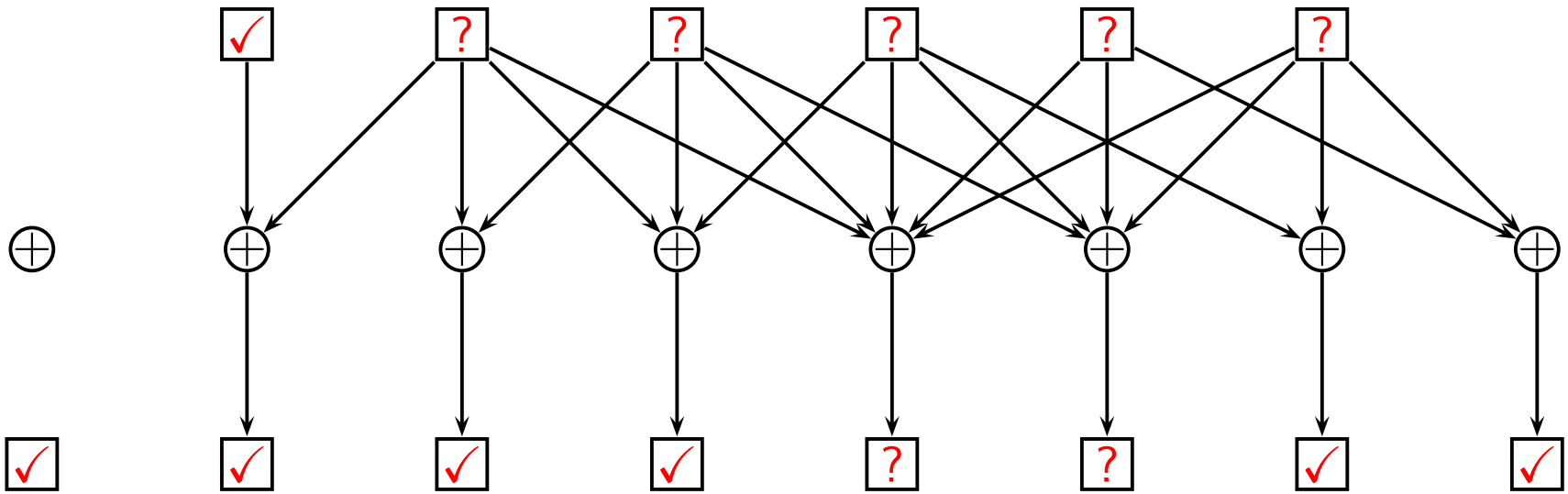


face	⚀	⚁	⚂	⚃	⚄	⚅
Ω	.05	.5	.1	.05	.25	.05

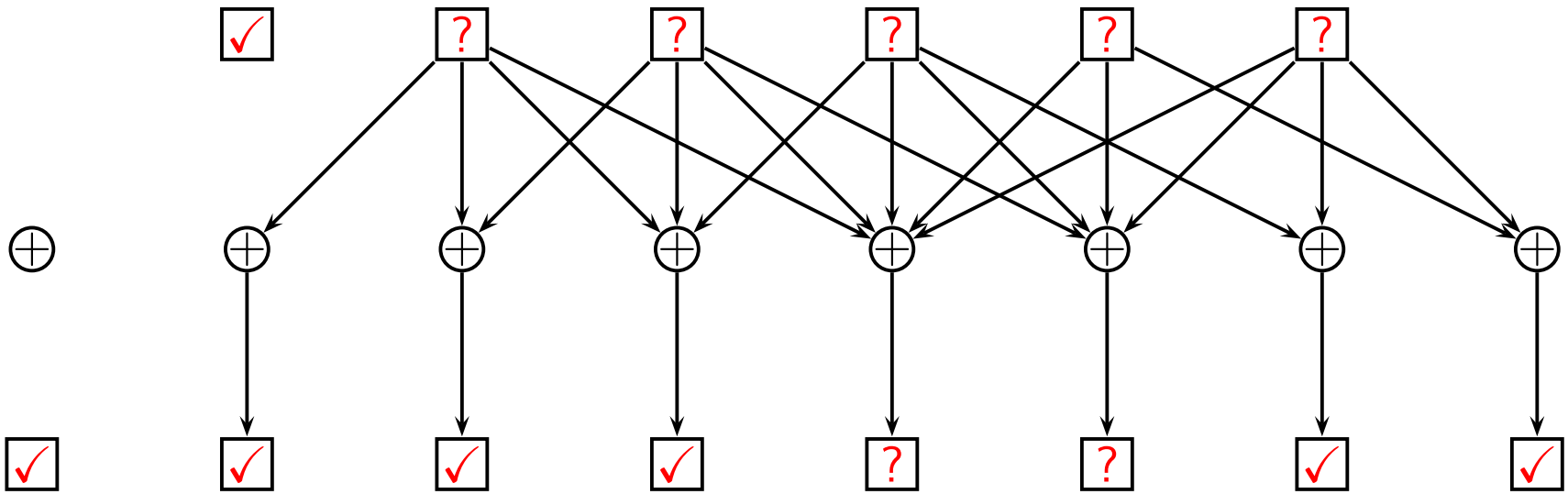
LT CODES – Decoding



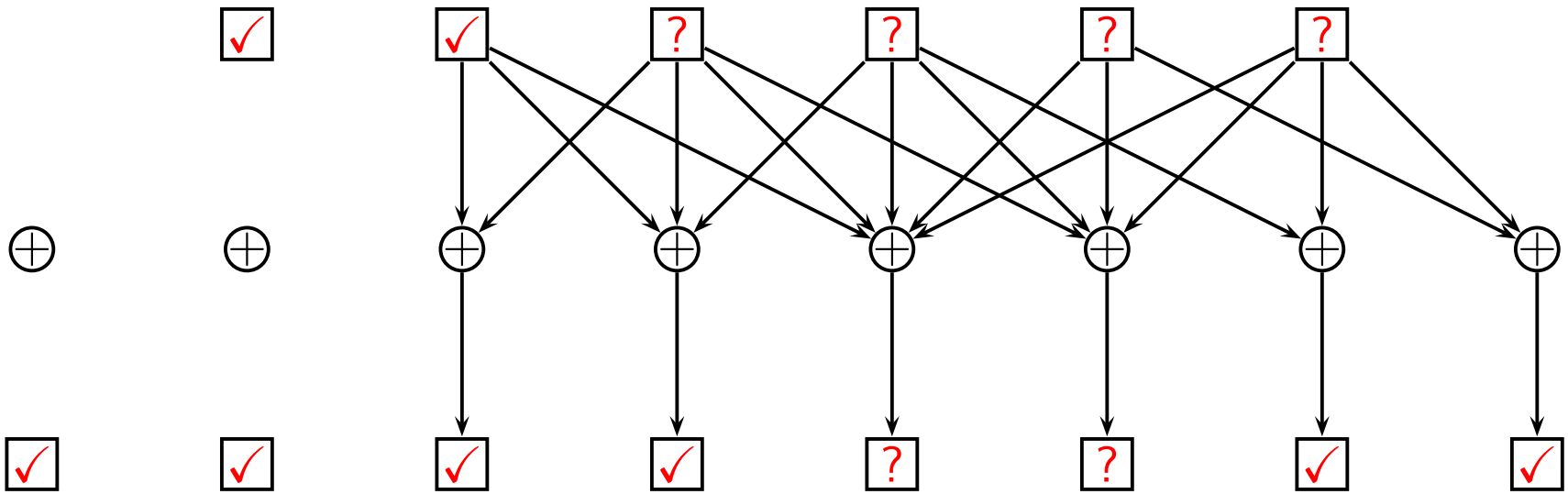
LT CODES – Decoding



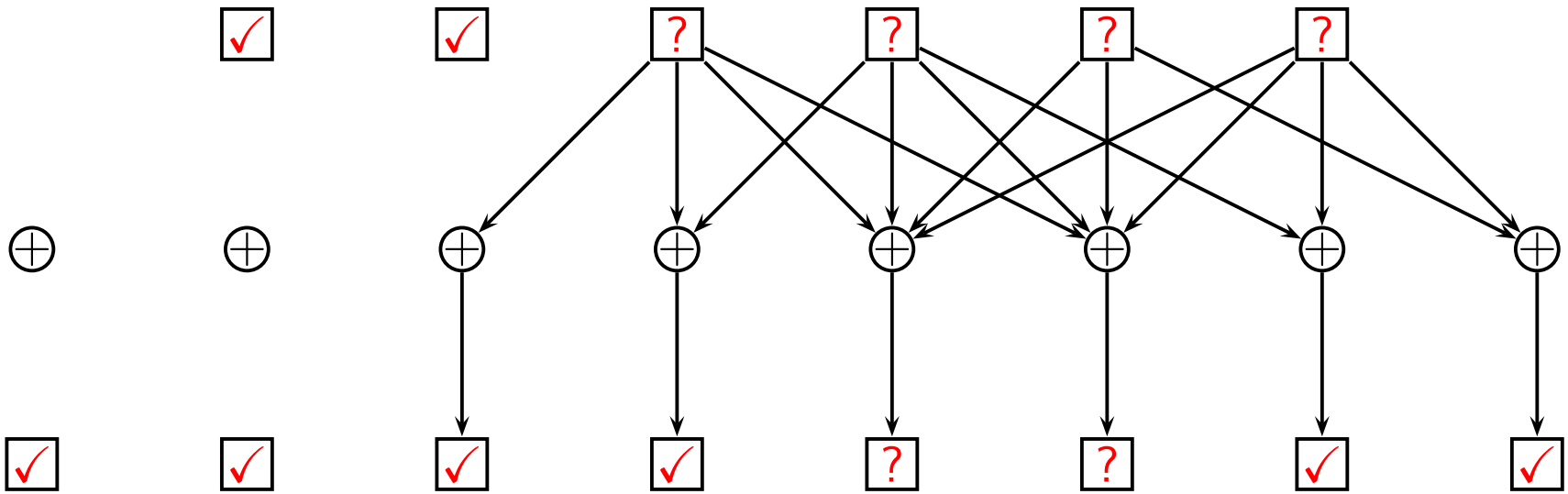
LT CODES – Decoding



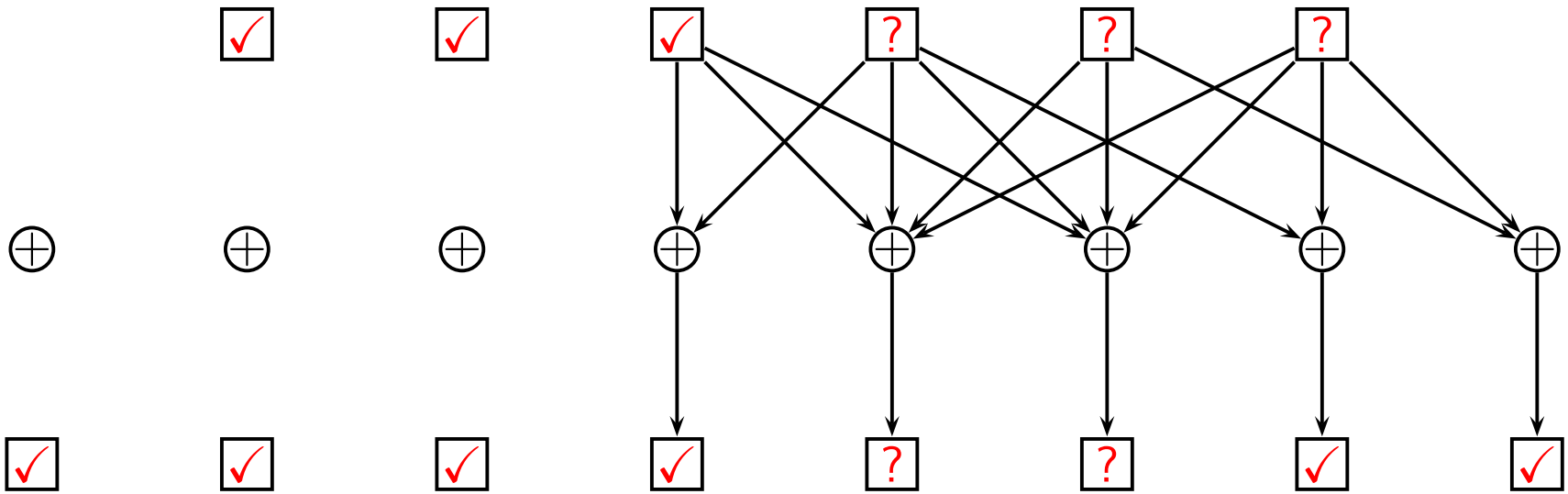
LT CODES – Decoding



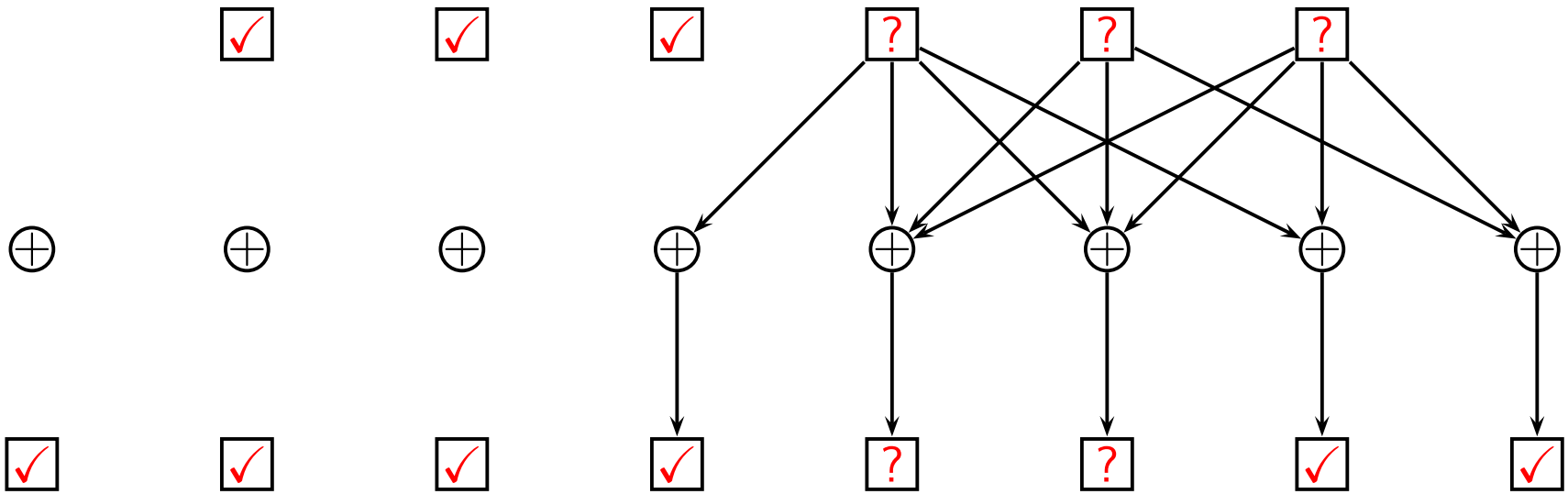
LT CODES – Decoding



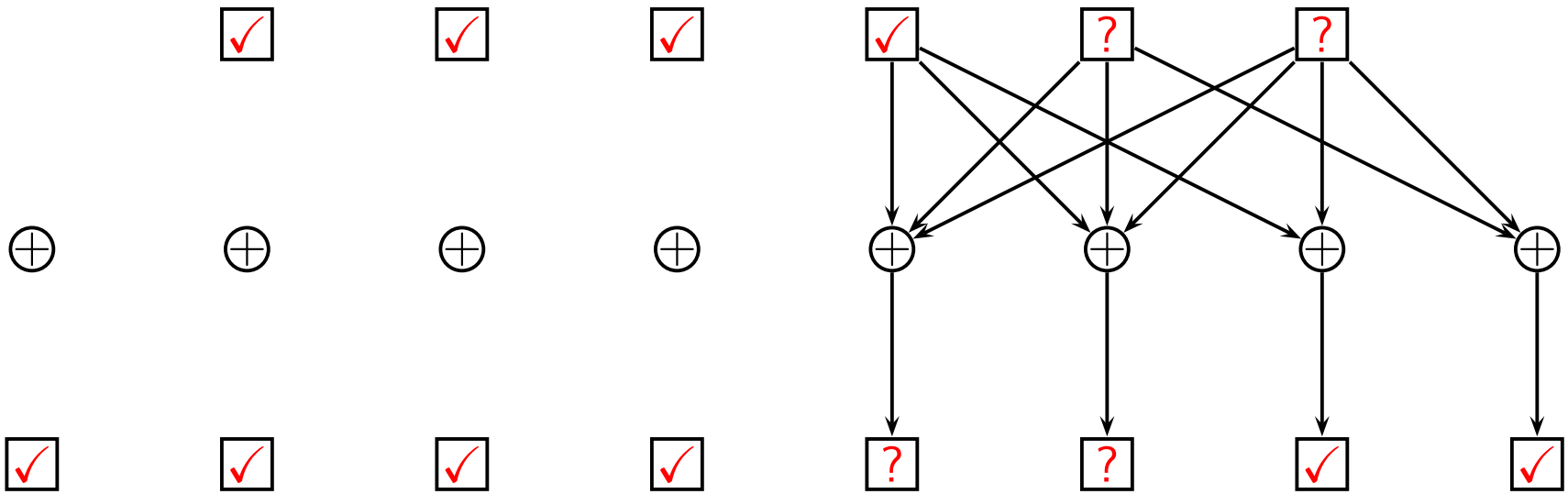
LT CODES – Decoding



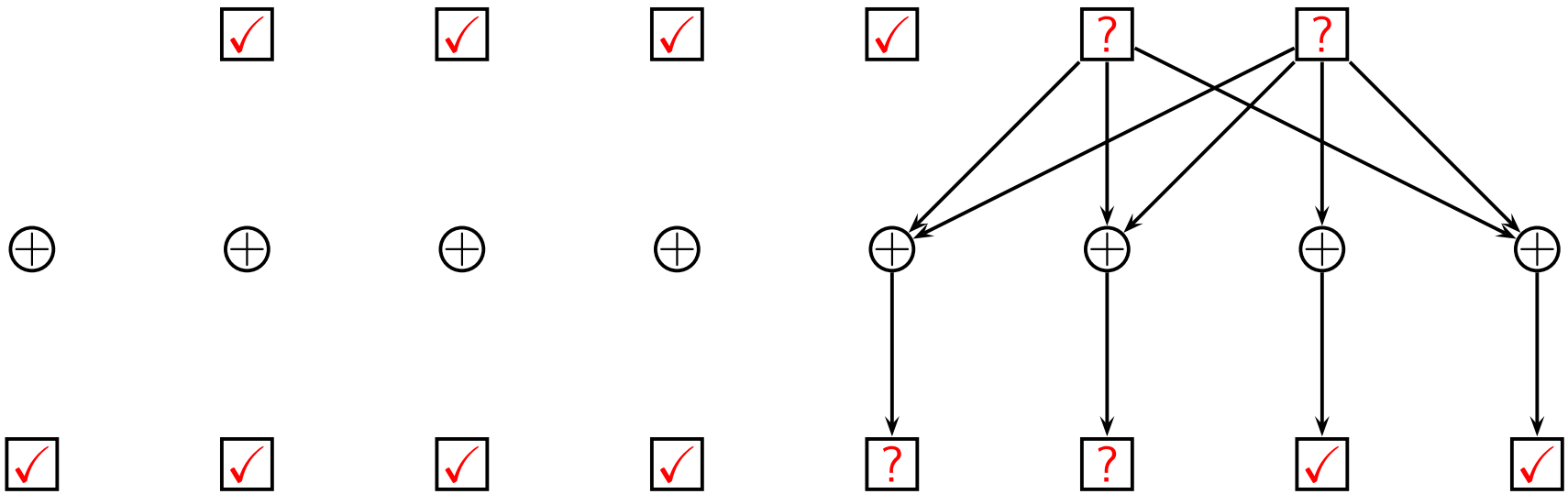
LT CODES – Decoding



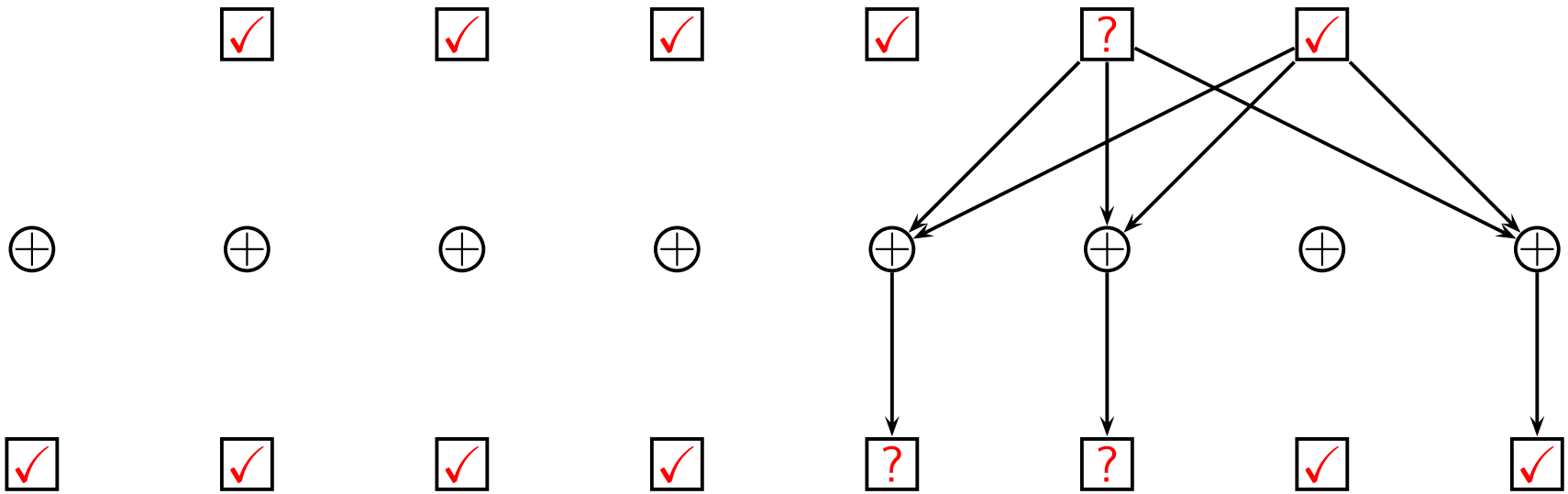
LT CODES – Decoding



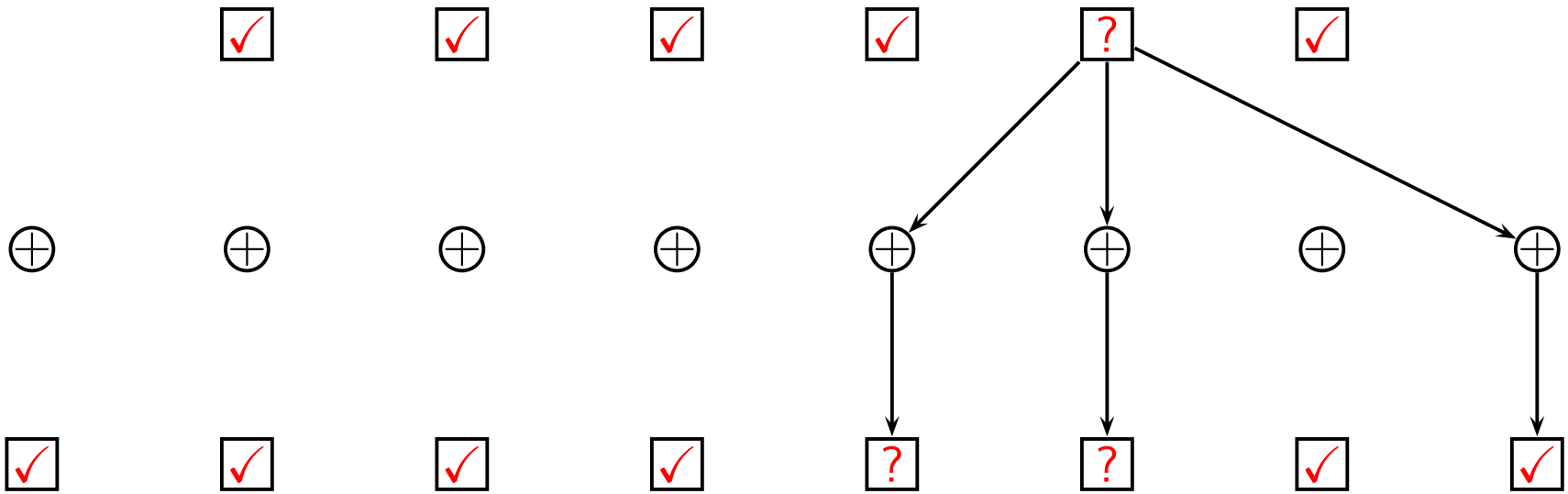
LT CODES – Decoding



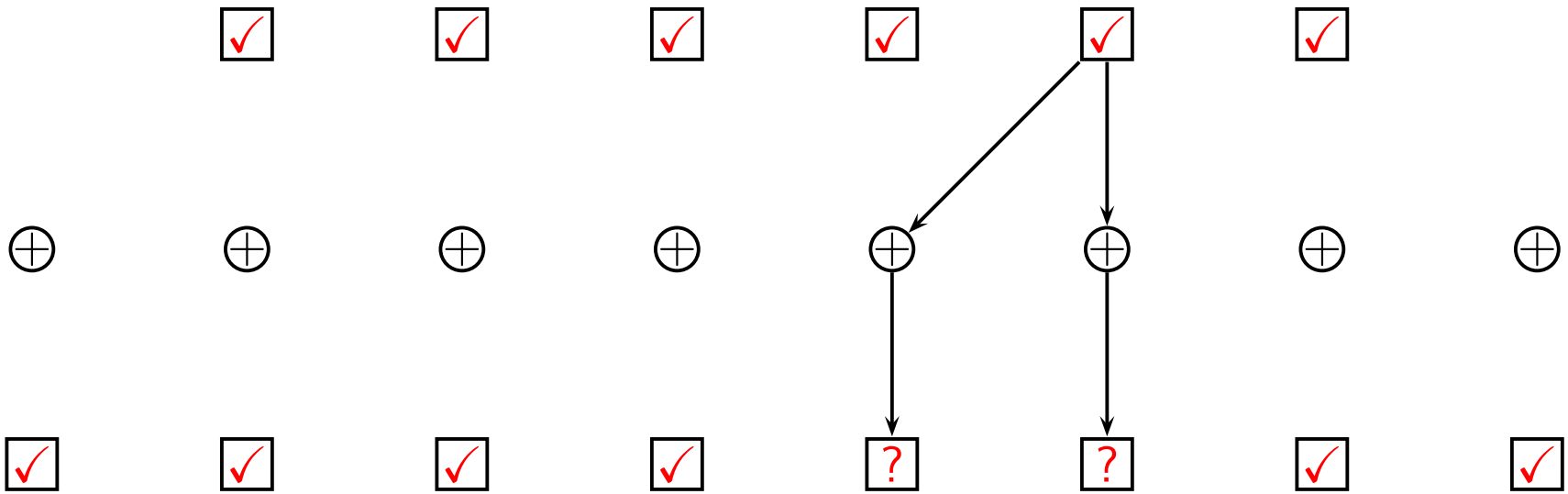
LT CODES – Decoding



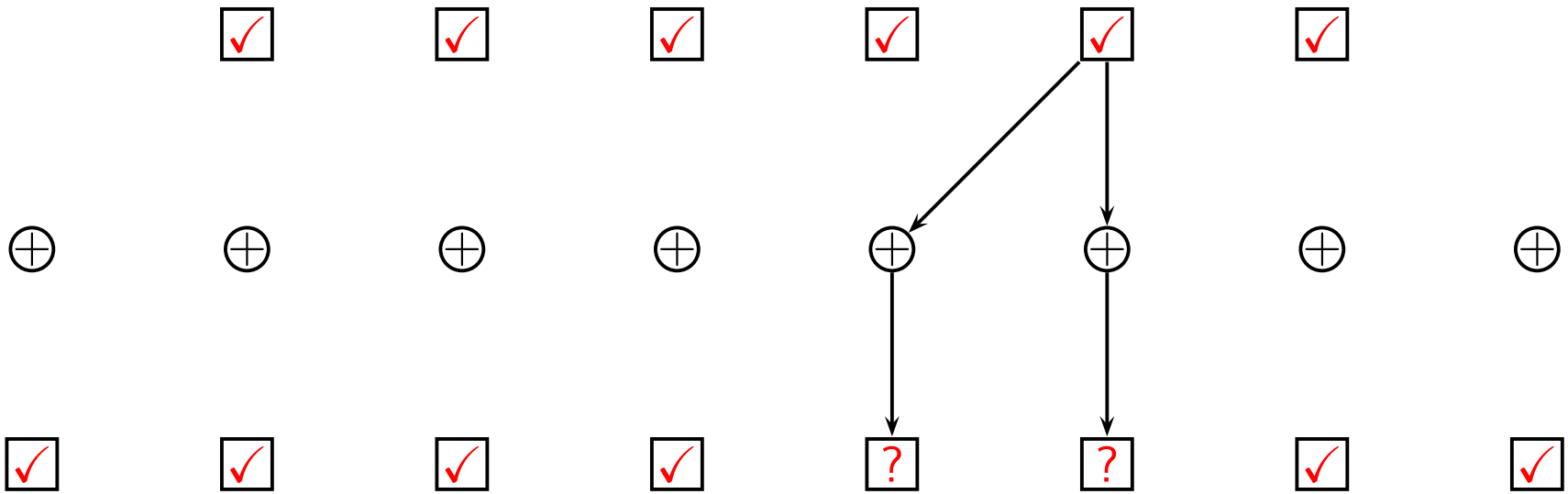
LT CODES – Decoding



LT CODES – Decoding



LT CODES – Decoding



- Code design:

  Ω has to be chosen to enable simple decoding under erasures.

- Raptor codes, fountain codes.

HARQ WITH RAPTOR CODES

- k information bits are pre-coded by an (n, k) code.
- An LDPC code ensemble will be used to model the precode.
- For transmission i,
  - N_i Raptor bits are generated by LT of the n LDPC bits.
  - γ_j is the Bhattacharyya noise parameter for transmission j.

HARQ WITH RAPTOR CODES

- k information bits are pre-coded by an (n, k) code.
- An LDPC code ensemble will be used to model the precode.
- For transmission i,
  - N_i Raptor bits are generated by LT of the n LDPC bits.
  - γ_j is the Bhattacharyya noise parameter for transmission j.
- Suppose that m transmissions have taken place.
- The union bound on the average ML decoder word error probability:

$$P_W^{\mathcal{R}} \leq \sum_{w_1=1}^{N_1} \cdots \sum_{w_m=1}^{N_m} \overline{B}_{w_1 \ldots w_m} \prod_{j=1}^{m} \gamma_j^{w_m},$$

where

$$\overline{B}_{w_1 \ldots w_m} = \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \prod_{i=1}^{m} \binom{N_i}{w_i} p(h/n)^{w_i} \left(1 - p(h/n)\right)^{N_i - w_i}.$$

HARQ PERFORMANCE

- The union bound on the average ML decoder word error probability:

$$\overline{P}_W^{\mathcal{R}} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \prod_{i=1}^{m} \underbrace{\left[1 - p(d/n)(1-\gamma_i)\right]}^{N_i}$$

where $p(d/n) = \frac{1}{2} - \frac{1}{2}\sum_h \Omega_h (1-2d/n)^h$.

HARQ PERFORMANCE

- The union bound on the average ML decoder word error probability:

$$\overline{P}_W^{\mathcal{R}} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \prod_{i=1}^{m} \Big[\underbrace{1 - p(d/n)(1-\gamma_i)}_{\geq \gamma_i} \Big]^{N_i}$$

where $p(d/n) = \frac{1}{2} - \frac{1}{2} \sum_h \Omega_h (1 - 2d/n)^h$.

HARQ PERFORMANCE

- The union bound on the average ML decoder word error probability:

$$\overline{P}_W^{\mathcal{R}} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \prod_{i=1}^{m} \Big[\underbrace{1 - p(d/n)(1-\gamma_i)}_{\geq \gamma_i} \Big]^{N_i}$$

  where $p(d/n) = \frac{1}{2} - \frac{1}{2}\sum_h \Omega_h (1-2d/n)^h$.

- Compare with the punctured scheme counterpart:

$$\overline{P}_W^{[\mathcal{C}](n)} \leq \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \overline{\gamma}_m^d.$$

HARQ PERFORMANCE

- There exist θ_0, $0 < \theta_0 < 1$ and

$$\pi_\Omega \triangleq \min_{\theta_0 < \theta \leq 1} \frac{p(\theta)}{\theta} \qquad c_0^{[\mathcal{C}]} \triangleq \limsup_{n\to\infty} \max_{\lceil n\theta_0 \rceil < d \leq n} \frac{\log \overline{A}_d^{[\mathcal{C}](n)}}{d}.$$

s.t.

$$\begin{aligned}\overline{P}_W^{\mathcal{R}} \leq & \sum_{d=1}^{n} \overline{A}_d^{[\mathcal{C}](n)} \prod_{i=1}^{m} \left[1 - p(d/n)(1-\gamma_i)\right]^{N_i} \\ \leq & O(n^{-\frac{1}{2}}) + \sum_{d=\lceil n\theta_0 \rceil + 1}^{n} \exp\left\{ d \left[c_0^{[\mathcal{C}]} - \pi_\Omega \sum_{i=1}^{m} \frac{1-\gamma_i}{R_{li}} \right] \right\},\end{aligned}$$

where $R_{li} = n/N_i$.

THE m-th TRANSMISSION

- The tradeoff between the rate and the power:

$$\frac{1-\gamma_m}{R_{lm}} > \frac{c_0^{[\mathcal{C}]}}{\pi_\Omega} - \sum_{j=1}^{m-1} \frac{1-\gamma_j}{R_{lj}}.$$

THE m-th TRANSMISSION

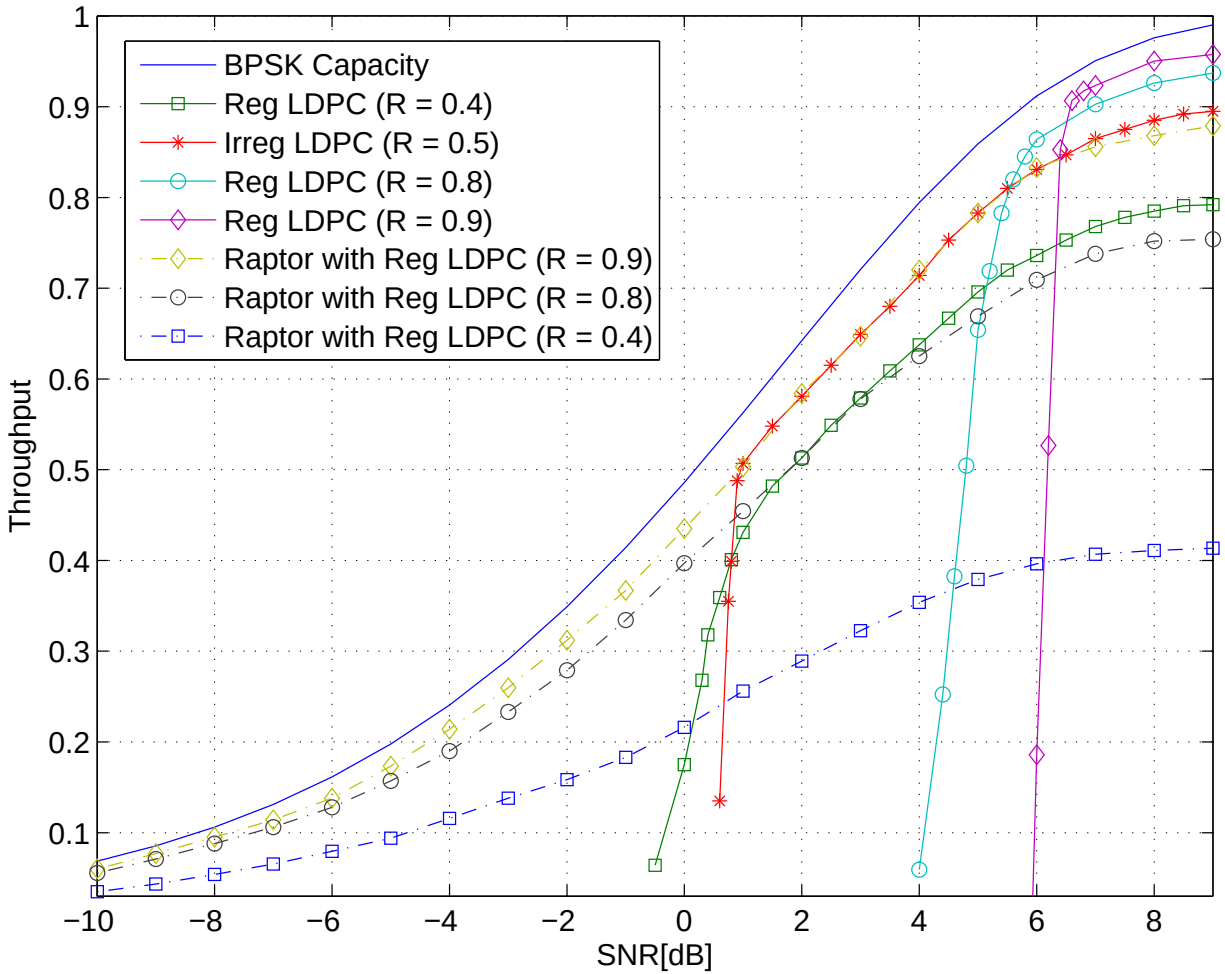
- The tradeoff between the rate and the power:

$$\frac{1-\gamma_m}{R_{lm}} > \frac{c_0^{[\mathcal{C}]}}{\pi_\Omega} - \sum_{j=1}^{m-1} \frac{1-\gamma_j}{R_{lj}}.$$

- Compare with the punctured scheme counterpart:

$$\alpha_m(1-\gamma_m) > 1 - \exp(-c_0^{[\mathcal{C}]}) - \sum_{j=1}^{m-1} \alpha_j(1-\gamma_j).$$

Punctured vs Rateless Codes for Hybrid ARQ



Punctured vs Rateless Codes for Hybrid ARQ

- Rateless schemes win when
  - the information about the range of SNR is not available,
  - the SNR range is wide.
- Punctured schemes win when
  - the SNR range is known and narrow,
  - changing the mother code (precode) is not possible.

Punctured vs Rateless Codes for Hybrid ARQ

- Rateless schemes win when
  - the information about the range of SNR is not available,
  - the SNR range is wide.
- Punctured schemes win when
  - the SNR range is known and narrow,
  - changing the mother code (precode) is not possible.
- Encoding/Decoding complexity.

Punctured vs Rateless Codes for Hybrid ARQ

- Rateless schemes win when
  - the information about the range of SNR is not available,
  - the SNR range is wide.
- Punctured schemes win when
  - the SNR range is known and narrow,
  - changing the mother code (precode) is not possible.
- Encoding/Decoding complexity.
- Hybrid, hybrid ARQ schemes?