E6712: Final Exam - Solutions

Given: April 28, 2004
Due: May 5, 2004

1. Pulse-position modulation (PPM) is a modulation scheme suitable for the deep space channel. Under M-ary PPM, a signaling interval of $T = M \cdot T_s$ seconds is partitioned into M equal-length slots. Message i is sent by turning the laser on during the i-th slot and keeping it off for the rest of the signaling interval. The information available at the receiver is the number of photons observed in each of the M slots. This number is modelled as a Poisson random variable with mean $(\lambda_s + \lambda_n)T_s$ when the laser is on and mean $\lambda_n T_s$ when the laser is off.

   (a) Derive the rule on which an optimal receiver for this channel should base its decision.

   (b) Derive the Union-Chernoff bound on the error-probability of this receiver.

Solution

(a) Conditioned on the i-th signal being transmitted, the received data is a vector $\boldsymbol{X}$ of independent Poisson random variables X_k having mean $\lambda_n T_s$ when $k \neq i$ and $(\lambda_n + \lambda_s)T_s$ when $k = i$. We have

$$\begin{aligned} P(\boldsymbol{X} = \boldsymbol{x}|i) &= \prod_{k=1}^{M} P(X_k = x_k|i) \\ &= e^{-(\lambda_n+\lambda_s)} \frac{[(\lambda_n + \lambda_s)T_s]^{x_i}}{x_i!} \prod_{k\neq i}^{M} e^{-(\lambda_n T_s)} \frac{(\lambda_n)^{x_k}}{x_k!} \\ &= e^{-\lambda_s T_s} \cdot \boxed{\left(1 + \frac{\lambda_s}{\lambda_n}\right)^{x_i}} \cdot \prod_{k=i}^{M} e^{-(\lambda_n)} \frac{(\lambda_n Ts)^{x_k}}{x_k!} \end{aligned}$$

The exponential and the product terms above are positive and do not depend on i. The remaining term implies

$$\arg\max_i P(\boldsymbol{X} = \boldsymbol{x}|i) = \boxed{\arg\max_i \{x_i : i = 1, \ldots, M\}},$$

giving the optimum decision rule.

(b) Clearly, the conditional probability of error is not a function of the transmitted signal. Thus, we can assume that signal 1 is transmitted. The union bound then yields (PPM is an orthogonal signaling scheme):

$$P(e) = (M-1)P_2(e),$$

where $P_2(e)$ is the the probability of mistaking transmitted signal 1 to (say) signal 2. This binary error probability can be upper-bounded using the Chernoff bound as follows:

$$P_2(e) = P(X_1 - X_2 \leq 0|1) \leq E[e^{s(X_1-X_2)}|1] = E[e^{sX_1}|1]E[e^{-sX_2}|1], \quad s < 0,$$

We have

$$E[e^{sX_1}|1] = \sum_{k=0}^{\infty} e^{-(\lambda_n+\lambda_s)T_s} \frac{[(\lambda_n + \lambda_s)T_s]^k}{k!} e^{sk} = e^{-[(\lambda_n+\lambda_s)T_s](1-e^s)}.$$

Similarly,

$$E[e^{-sX_2}|1] = \sum_{k=0}^{\infty} e^{-\lambda_n T_s} \frac{(\lambda_n T_s)^k}{k!} e^{-sk} = e^{-(\lambda_n T_s)(1-e^{-s})}.$$

Thus,

$$E[e^{sX_1}|1]E[e^{-sX_2}|1] = e^{-(\lambda_n T_s)(1-e^{-s})-[(\lambda_n+\lambda_s)T_s](1-e^{s})}.$$

Minimizing the exponent above with respect to s, we obtain

$$s_{\min} = \frac{1}{2}\ln\Big(\frac{\lambda_n}{\lambda_n+\lambda_s}\Big) < 0.$$

Substituting in the bound and simplifying, we obtain

$$\boxed{P(e) = (M-1)e^{-\frac{1}{2}[\sqrt{(\lambda_n+\lambda_s)T_s}-\sqrt{\lambda_n T_s}]^2}}.$$

2. Consider the following partial response channel:

$$v_k = I_k + I_{k-1} + \eta_k,$$

where $\{\eta_k\}$ is discrete AWGN with zero mean and spectral density $\mathcal{N}_0/2$. The user data $D_k \in \{0,1\}$ is binary, and the channel can take only bipolar signal values $I_k \in \{-1,1\}$.

(a) Suppose that user data is directly mapped into the channel sequence as

$$I_k = 2D_k - 1$$

Suppose that the user sequence $\{D_k\}$ is the all-zero sequence, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is the most likely incorrect sequence $\{\widehat{D_k}\}$?

(b) Is this detector prone to error-propagation? If yes, how would you organize precoding to alleviate this problem?

(c) How would your answers to the questions (a) and (b) differ if the received signal was

$$w_k = I_k - I_{k-2} + \eta_k.$$

Solution

(a) The minimum (merged) distance on the bipolar channel with transfer function $(1-D)$ is given by

$$d^2_{<>} = \min_{\varepsilon(D)\neq 0} \|(1+D)\varepsilon(D)\|^2,$$

where the squared norm of a polynomial is defined as the sum of its squared coefficients, and $\varepsilon(D)$ is the polynomial corresponding to a finite length normalized error sequence:

$$\boldsymbol{\varepsilon} = \{\varepsilon_{n_0}, \varepsilon_1, \ldots, \varepsilon_{n_l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1,0,1\},\ \varepsilon_{n_0} = 1, \varepsilon_{n_l-1} \neq 0$$

Since the user sequence is the all-zero sequence, we have $\varepsilon_i \in \{0,1\}$, $\varepsilon_{n_0} = 1$, and $\varepsilon_{n_l-1} = 1$, and thus

$$\begin{aligned} d^2_{<>} &= \min_{\varepsilon(D)\neq 0} \|(1+D)\sum_{i=0}^{l-1} \varepsilon_{n_i} D^i\|^2 \\ &= \min_{\varepsilon(D)\neq 0} \|\varepsilon_{n_0} + \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}}) D^i + \varepsilon_{n_{l-1}} D^l\|^2 \\ &= \min_{\varepsilon(D)\neq 0} (1 + \sum_{i=1}^{l-1} (\varepsilon_{n_i} + \varepsilon_{n_{i-1}})^2 + 1) \\ &= 2 + \min_{\varepsilon(D)\neq 0} \sum_{i=1}^{l-1} (\varepsilon_{n_i} + \varepsilon_{n_{i-1}})^2 \geq 4 \end{aligned}$$

where the equality holds iff

$$\varepsilon(D) = 1 + D^{l-1}.$$

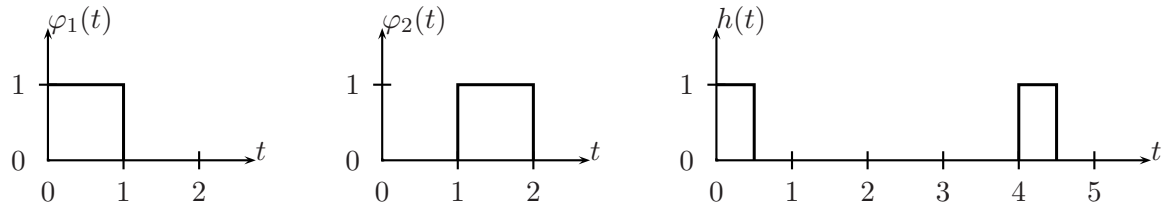
(b) The user data should be mapped into the channel symbols as follows: sequence in two steps as follows:

i. $P_k = D_k + P_{k-1} \mod 2$ or $P_k = D_k + P_{k-1} \mod 2$

ii. $I_k = 2P_k - 1$

(c) See Problem 3 of the Midterm.

3. A communication system transmits one of two equally likely messages by means of the signals $s_1(t) = \sqrt{E_b}\varphi_1(t)$ and $s_2(t) = \sqrt{E_b}\varphi_2(t)$ through a multipath channel with impulse response $h(t)$. Waveforms $\varphi_1(t)$, $\varphi_2(t)$, and $h(t)$ are shown below:



(a) What kind of detector is optimal if one symbol is transmitted every $2ms$?

(b) Which transmission rate would you choose to insure the best error-rate performance?

Solution

Although the symbol duration is $2ms$, because of the multipath, the duration of the corresponding output signal is $6.5ms$ which we obtain as the sum the input signal duration and the channel impulse response duration. To answer the questions, we need to plot the output waveforms.

(a) If one signal is transmitted every $2ms$, there will be intersymbol interference and the optimal receiver is the MLSE.

(b) One symbol should be sent every $6.5ms$ to avoid the ISI, which in this case gives the best error-rate performance.

4. One of M equally likely messages is transmitted over a uniformly random-phase, AWGN channel by means of M orthogonal signals, each of energy E_s. The noise power density is $\mathcal{N}_0/2$.

   (a) Draw a block diagram of the optimum receiver.

   (b) Show that when the random channel phase θ is equal to γ, the conditional error probability is given by

   $$P\{e|\theta = \gamma\} = 1 - E_{XY}\Big\{\big[1 - e^{-(X^2+Y^2)/2}\big]^{M-1}\Big\},$$

   where X an Y are independent Gaussian random variables with unit variances and means equal to $\sqrt{2E_s/\mathcal{N}_0}\cos\gamma$ and $\sqrt{2E_s/\mathcal{N}_0}\sin\gamma$, respectively.

   (c) Derive an expression for $P(e)$.

Solution

See Section 5.4.3 in the text.