E6712: Homework 1

Due date: Feb. 11, 2004

1. Consider the following binary detection problem discussed in class:

$$r = s_i + n, \quad i \in \{1, 2\},$$

where n is an N-dimensional noise vector whose components are independent identically distributed Gaussian random variables with zero mean and unit variance, and the two possible signals are the following N-dimensional vectors:

$$s_1 = (m, m, \ldots, m) \text{ and } s_2 = (0, 0, \ldots 0).$$

Assume that the prior probabilities of the two messages are equal.

   (a) Evaluate analytically the error probability performance of an optimal receiver for $N = 1$ and $m \in \{1, 2, 3, 4, 5, 6\}$, and for $N = 4$ and $m \in \{1, 2, 3, 4, 5, 6\}$. Plot the error probability v.s. m on a single graph for both $N = 1$ and $N = 4$. using the logarithmic scale for the error probability axis.

   (b) Write a computer program (see the note below) to evaluate by simulation the error probability performance for the same parameters as in a). Plot the results on the same graph as for a) for comparison.

   (c) Compute the Chernoff bound on the probability of error evaluated analytically in a) and by simulation in b), and plot on the same graph as for a) and b).

2. Consider the same signaling scheme as in Problem 1) for arbitrary N, but with a suboptimal receiver which operates as follows. After each of the N symbol intervals, the receiver decides whether an m or a 0 was sent according to the rule that is optimal for $N = 1$. After the 4 symbol intervals, the receivers declares message s_1 if the majority of its per-symbol decisions were m, and message s_2 if the majority of its per-symbol decisions were 0. If there was a tie in the per-symbol decisions (in the case of an even N), the receiver decides either s_1 or s_2, each with probability 1/2.

   (a) Derive an expression for this receiver's error probability, and its Chernoff bound.

   (b) Write a computer program to evaluate by simulation the error probability performance for $N = 4$ and $m \in \{1, 2, 3, 4, 5, 6\}$. Evaluate analytically the error probability performance for the same parameters based on the expression derived in a). Compute the Chernoff bound. Plot the error probability (simulated, computed, and the Chernoff bound) v.s. m on a single graph.

   (c) How does the performance of this receiver differ from the performance of the optimal receiver?

3. On-Off Keying (OOK) is a modulation scheme suitable for the optical direct detection channel. Under OOK, information is conveyed by turning the laser on for T seconds to send the symbol 1 and turning the laser off for T seconds to send the symbol 0. The information available at the receiver is the number of photons observed in a symbol interval. This number is modelled as a Poisson random variable with mean $(\lambda_s + \lambda_n)T$ when 1 is sent and $\lambda_n T$ when 0 is sent.

   (a) Derive the rule on which an optimal receiver for this channel should base its decision.

   (b) Derive an expression for the exact error-probability of this receiver.

Note: Generate messages s_1 and s_2 randomly, each with probability 1/2. To generate two independent, zero mean, unit variance, Gaussian random variables y_1 and y_2, first generate two random variables u and v uniform on $[0, 1)$. Then compute y_1 and y_2 as

$$y_1 = \sqrt{-2\ln(u)}\cos(2\pi v) \quad \text{and} \quad y_2 = \sqrt{-2\ln(u)}\sin(2\pi v).$$