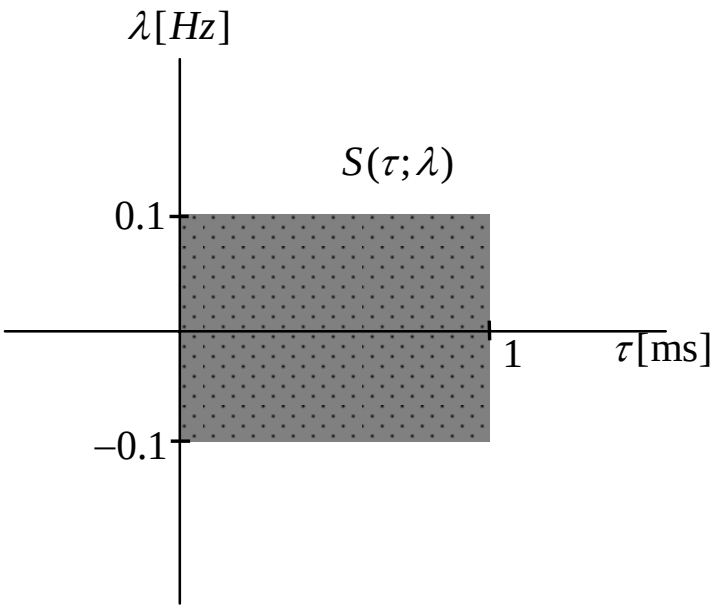
E6712: Homework 4 - Solution

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1. (14.1) $S(\tau;\lambda)$ is nonzero for the range of values $0 \leq \tau \leq 1$ ms and $-01.Hz \leq \lambda \leq 0.1Hz$. $S(\tau;\lambda)$ is approximately uniform in the two variables.



(a) Give numerical values for the following parameters:

i. The multipath spread of the channel
(*Solution.*)
$$T_m = 10^{-3} \text{ sec}$$

ii. The Doppler spread of the channel
(*Solution.*)
$$B_d = 0.2 \text{ Hz}$$

iii. The coherence time of the channel
(*Solution.*)
$$(\Delta t)_c \cong \frac{1}{B_d} = 5 \text{ sec}$$

iv. The coherence bandwidth of the channel
(*Solution.*)
$$(\Delta f)_c \cong \frac{1}{T_m} = 1000 \text{ Hz}$$

v. The spread factor of the channel
(*Solution.*)
$$T_m B_d = 0.2 \cdot 10^{-3} = 2 \cdot 10^{-4}$$

(b) Explain the meaning of the following, taking into consideration the answers given in (a):

i. The channel is frequency-nonselective
(*Solution.*)

$$W \ll (\Delta f)_c = 1000 \text{ Hz}$$

Here, W is the bandwidth of the transmitted signal
If a channel is frequency-nonselective to a signal, the signal experiences multiplicative distortion.

ii. The channel is slowly fading.
(*Solution.*)

$$T \ll (\Delta t)_c = 5 \text{ sec}$$

Here, T is the signaling interval. In case of the slowly fading channel, the channel attenuation and phase shift are fixed for the signaling interval, T.

iii. The channel is frequency selective.
(*Solution.*)

$$W > (\Delta f)_c = 1000 \text{ Hz}$$

In this case, the signal is severely distorted by the channel.

(c) Suppose that we have a frequency allocation (bandwidth) of 10kHz and we wish to transmit at a rate of 100 bits/s over this channel. Design a binary communication system with frequency diversity. In particular, specify

i. The type of modulation.
(*Solution.*) Binary PSK

ii. The number of subchannels
(*Solution.*) 10 (It comes from the frequency separation of 1000Hz based on the coherence frequency.)

iii. The frequency separation between adjacent carriers.
(*Solution.*) The separation between successive carriers must be equal or greater than the coherence bandwidth $(\Delta f)_c$, 1000. Here, assume the frequency separation as 1000 Hz.

iv. The signaling interval used in your design.
(*Solution.*) $T = 0.01$ sec
Justify your choice of parameters.
(*Solution.*) The bandwidth of the signal $W = 1/T = 100Hz$ is much smaller than the given coherence frequency, $(\Delta f)_c = 1000Hz$. The channel is frequency non-selective for this signal.
By the way, the channel is slowly fading because $T = 0.01 \ll (\Delta t)_c = 5$.

2. (14.2) Consider a binary communication system for transmitting a binary sequence over a fading channel. The modulation is orthogonal FSK with third-order frequency diversity (L=3). The demodulator consists of matched filters followed by squarelaw detectors. Assume that the FSK carriers fade independently and identically according to a Rayleigh envelope distribution. The additive noises on the diversity signals are zero-mean Gaussian with autocorrelation functions $1/2E[z_k^*(t)z_k(t+\tau)] = N_0\delta(\tau)$. The noise processes are mutually statistically independent.

(a) The transmitted signal may be viewed as binary FSK with square-law detection, generated by a repetition code of the form

$$1 \rightarrow C_1 = [1\ 1\ 1], \quad 0 \rightarrow C_0 = [0\ 0\ 0]$$

Determine the error rate performance P_{2h} for a hard-decision decoder following the square-law-detected signals.

(*Solution.*)

Assume coherent detection of binary FSK signal under Rayleigh fading channel. In case of hard-decision, the decision of signal will be made for each channel. The error probability for each channel comes from equation (14.4-14) in the textbook.

$$P_2 = \left[\frac{1}{2}(1-\mu)\right]^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left[\frac{1}{2}(1+\mu)\right]^k$$

Inserting $L = 1$ and $\mu = \sqrt{\frac{\bar{\gamma}_c}{2+\bar{\gamma}_c}}$ for one channel,

$$P_{2c} = \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}_c}{2+\bar{\gamma}_c}}\right)$$

The final decision will be made based on the hamming distance from the sent signal. Therefore, the error probability of hard-decision receiver is:

$$P_2 = P_{2c}^3 + P_{2c}^2(1 - P_{2c})$$

(b) Evaluate P_{2h} for $\bar{\gamma}_c = 100$ and 1000.
(*Solution.*)

$$P_{2c,100} = \frac{1}{2}\left(1 - \sqrt{\frac{100}{2+100}}\right) = 4.926 \cdot 10^{-3}$$
$$P_{2h,100} = 2.4268 \cdot 10^{-5}$$
$$P_{2c,1000} = \frac{1}{2}\left(1 - \sqrt{\frac{1000}{2+1000}}\right) = 4.992 \cdot 10^{-4}$$
$$P_{2h,100} = 2.4925 \cdot 10^{-7}$$

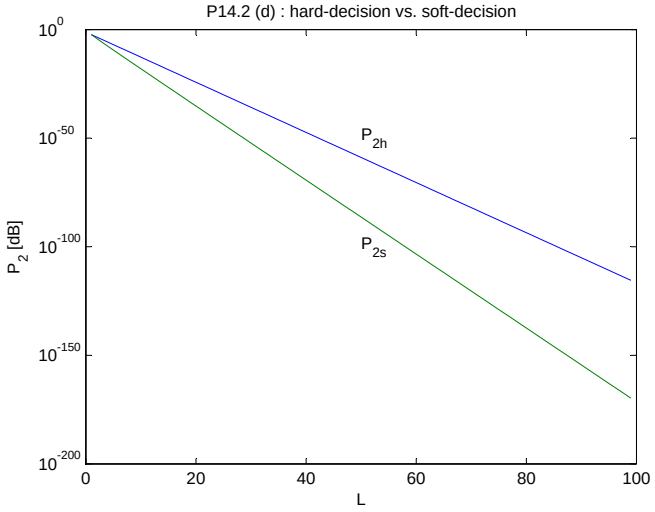
(c) Evaluate the error rate P_{2s} for $\bar{\gamma}_c = 100$ and 1000 if the decoder employs soft-decision decoding.
(*Solution.*)
The equation (14.4-14) can be used with $L = 3$ and $\mu = \sqrt{\frac{\bar{\gamma}_c}{2+\bar{\gamma}_c}}$. Or, for the large value of $\bar{\gamma}_c$, equation (14.4-22) can be used. Here, we use the latter.

$$P_{2s} = \left(\frac{1}{2\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$
$$P_{2s,100} = \left(\frac{1}{2 \cdot 100}\right)^3 \binom{2 \cdot 3 - 1}{3} = 1.25 \cdot 10^{-6}$$
$$P_{2s,1000} = \left(\frac{1}{2 \cdot 1000}\right)^3 \binom{2 \cdot 3 - 1}{3} = 1.25 \cdot 10^{-9}$$

(d) Consider the generalization of the result in (a). If a repetition code of block length L (L odd) is used, determine the error probability P_{2h} of the hard-decision decoder and compare that with P_{2s}, the error rate of the soft-decision decoder. Assume $\bar{\gamma}_c \gg 1$
(*Solution.*)
If the hard-decision error probability of each channel is P_{2c}, the error probability of overall communication system is:

$$P_2 h = \sum_{k=0}^{\frac{L-1}{2}} P_{2c}^{L-k}(1-P_{2c})^k = P_{2c}^L \sum_{k=0}^{\frac{L-1}{2}} \left(\frac{1}{P_{2c}} - 1\right)^k$$

The error probability for soft-decision is given in (c).
Below plot compares P_{2h} and P_{2s}. As a reward of system complexity, the soft-decision achieves lower error probability.



3. (14.3) Suppose that the binary signal $\pm s_l(t)$ is transmitted over a fading channel and the received signal is

$$r_l(t) = \pm a s_l(t) + z(t), \quad 0 \le t \le T$$

where $z(t)$ is zero-mean white Gaussian noise with autocorrelation function

$$\phi_{zz}(t) = N_0 \delta(\tau)$$

The energy in the transmitted signal is $\varepsilon = \frac{1}{2}\int_0^T |s_l(t)|^2 dt$. The channel gain a is specified by the probability density function

$$p(a) = 0.1\delta(a) + 0.9\delta(a-2)$$

(a) Determine the average probability of error P_2 for the demodulator that employs a filter matched to $s_l(t)$.
(*Solution.*) Let $s_1(t) = -s_l(t)$ and $s_2(t) = +s_l(t)$. Then, the error probability is that of binary antipodal signal with symbol energy, $2a^2\varepsilon$, and noise energy, N_0.

$$P_e(a) = Q\left(\sqrt{\frac{2a^2\varepsilon}{N_0}}\right)$$

P_e is a function of a discrete random variable a. The P_2 is a expected value of P_e:

$$\begin{aligned} P_2 &= \sum_{j=\{0,2\}} P_e(j) \cdot Pr(a=j) \\ &= Q(0)\cdot 0.1 + Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right)\cdot 0.9 \\ &= 0.05 + 0.9Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right) \end{aligned}$$

(b) What value does P_2 approach as ε/N_0 approaches infinity?
(*Solution.*) Because $Q(\infty)$ is 0, P_2 approaches 0.05.

(c) Suppose that the same signal is transmitted on two statistically *independently fading* channels with gains a_1 and a_2, where

$$p(a_k) = 0.1\delta(a_k) + 0.9\delta(a_k - 2), \qquad k = 1, 2$$

The noises on the two channels are statistically independent and identically distributed. The demodulator employs a matched filter for each channel and simply adds the two filter outputs to form the decision variable. Determine the average P_2.

(*Solution.*) This channel can be interpreted as a non-fading, fixed channel which consists of following transmitted signals with the corresponding probability:

$$\begin{aligned}
&s_1(t) = 0, s_2(t) = 0, && \text{with } p = 0.1 \cdot 0.1 \\
&s_1(t) = 0, s_2(t) = +2s_l(t), && \text{with } p = 0.1 \cdot 0.9 \\
&s_1(t) = -2s_l(t), s_2(t) = 0, && \text{with } p = 0.9 \cdot 0.1 \\
&s_1(t) = -2s_l(t), s_2(t) = +2s_l(t), && \text{with } p = 0.9 \cdot 0.9
\end{aligned}$$

If a_k is *not known* to the receiver, the same decision criteria is used for all differently faded signals. Assuming probability of $s_1(t)$ and $s_2(t)$ is the same, the error probability is:

$$\begin{aligned}
P_e &= 0.01 \cdot Q(0) + 0.18 \cdot \left[0.5Q(0) + 0.5Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right)\right] + 0.81 \cdot Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right) \\
&= 0.05 + 0.9Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right)
\end{aligned}$$

If a_k is *known* to the receiver, the decision criteria will be changed respectively and the performance will be changed.

$$\begin{aligned}
P_e &= 0.01 \cdot Q(0) + 0.18 \cdot Q\left(\sqrt{\frac{2\varepsilon}{N_0}}\right) + 0.81 \cdot Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right) \\
&= 0.005 + 0.18 \cdot Q\left(\sqrt{\frac{2\varepsilon}{N_0}}\right) + 0.81 \cdot Q\left(\sqrt{\frac{8\varepsilon}{N_0}}\right)
\end{aligned}$$

(d) For the case in (c) what value does P_2 approach as ε/N_0 approaches infinity?

(*Solution.*) If a_k is known to the receiver, P_e approaches 0.005. Otherwise, it approaches 0.05.

4. (14.4) A multipath fading channel has a multipath spread of $T_m = 1s$ and a Doppler spread $B_d = 0.01Hz$. The total channel bandwidth at band-pass available for signal transmission is $W = 5Hz$. To reduce the effects of intersymbol interference, the signal designer selects a pulse duration $T = 10s$.

(a) Determine the coherence bandwidth and the coherence time.

$$(\Delta f)_c \cong \frac{1}{T_m} = 1Hz$$

$$(\Delta t)_c \cong \frac{1}{B_d} = 100sec$$

(b) Is the channel frequency selective? Explain.

(*Solution.*) Yes. Because signal bandwidth W, 5Hz, is much greater than the coherence bandwidth, 1Hz.

(c) Is the channel fading slowly or rapidly? Explain.

(*Solution.*) It is slowly fading because the signal duration, 10 sec, is much smaller than the coherence time, 100sec.

(d) Suppose that the channel is used to transmit binary data via (antipodal) coherently detected PSK in a frequency diversity mode. Explain how you would use the available channel bandwidth to obtain frequency diversity and determine how much diversity is available.

(*Solution.*) Suppose that every channel is separated by 1 Hz based on the coherence bandwidth, 1 Hz. Thus there are 5 diversity channels.

From the pulse duration of $10sec$, we can get the low-pass bandwidth of the signal as $0.1Hz$. It means that each diversity channel can deliver ten same signals parallelly.

(e) For the case in (d), what is the *approximate* SNR required per diversity to achieve an error probability of 10^{-6}?

(*Solution.*) For the coherent detection of binary PSK signal, we can use equation (14.4-18) from the textbook.

$$P_2 \approx \left(\frac{1}{4\bar{\gamma}_c}\right)^L \binom{2L-1}{L}$$

By inserting $L = 5$,

$$P_2 \approx \left(\frac{1}{4\bar{\gamma}_c}\right)^5 \binom{9}{5} = \frac{0.123}{(\bar{\gamma}_c)^5} < 10^{-6}$$

$$\bar{\gamma}_c > 10.424$$

(f) Suppose that a wideband signal is used for transmission and a RAKE-type receiver is used for demodulation. How many taps would you use in the RAKE receiver?

(*Solution.*) The number of taps is $\lfloor T_m W \rfloor + 1$. With $T_m = 1$ and $W = 5$, we get 6 taps.

(g) Explain whether or not the RAKE receiver can be implemented as a coherent receiver with maximal ratio combining.

(*Solution.*) Because $T \ll (\Delta t)_c$, a signal experiences constant multiplicative distortion in a pulse duration, T. Therefore, a coherent receiver can be used here.

(h) If binary orthogonal signals are used for the wideband signal with square-law postdetection combining in the RAKE receiver, what is the *approximate* SNR required to achieve an error probability of 10^{-6}? (Assume that all taps have the same SNR.)

(*Solution.*) By using the equation (14.4-28) with $\bar{\gamma}_c$ replaced by $\frac{1}{2}\bar{\gamma}_c$,

$$P_2 \approx \binom{9}{5}\left(\frac{1}{\bar{\gamma}_c}\right)^5 = 10^{-6}$$

$$\bar{\gamma}_c = 41.694$$

5. (14.5) In the binary communication system shown in Figure P14.5, $z_1(t)$ and $z_2(t)$ are statistically independent white Gaussian noise processes with zero-mean and identical autocorrelation functions $\phi_{zz}(\tau) = N_0\delta(\tau)$. The sampled values U_1 and U_2 represent the *real parts* of the matched filter outputs. For example, if $s_l(t)$ is transmitted, then we have

$$U_1 = 2\varepsilon + N_1$$
$$U_2 = N_1 + N_2$$

where ε is the transmitted signal energy and

$$N_k = \Re\left[\int_0^T s_l^*(t) z_k(t) dt\right], \quad k = 1, 2$$

It is apparent that U_1 and U_2 are correlated Gaussian variables while N_1 and N_2 are independent Gaussian variables. Thus,

$$p(n_1) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{n_1^2}{2\sigma^2}\right)$$
$$p(n_2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{n_2^2}{2\sigma^2}\right)$$

where the variance of N_k is $\sigma^2 = 2\varepsilon N_0$.

(a) Show that the joint probability density function for U_1 and U_2 is

$$p(u_1, u_2) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{\sigma^2} \left[(u_1 - 2\varepsilon)^2 - u_2(u_1 - 2\varepsilon) + \frac{1}{2}u_2^2 \right] \right\}$$

if $s(t)$ is transmitted and

$$p(u_1, u_2) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{\sigma^2} \left[(u_1 + 2\varepsilon)^2 - u_2(u_1 + 2\varepsilon) + \frac{1}{2}u_2^2 \right] \right\}$$

if $-s(t)$ is transmitted and

(*Solution.*) We know that linearly transformed random variables can be represented as an equation of original random variables by using Jacobian. For example,

$$V = g_1(X, Y) \text{ and } W = g_2(X, Y) \quad \text{, V, W, X and Y are all random variables}$$

$$f_{V,W}(v, w) = \frac{f_{X,Y}(h_1(v, w), h_2(v, w))}{|J(x, y)|}$$

$$\text{where } J(x, y) = \det \begin{bmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \text{ and } x = h_1(v, w) \text{ and } y = h_2(v, w)$$

If $s_l(t)$ is sent, $J(n_1, n_2) = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$, $n_1 = u_1 - 2\varepsilon$ and $n_2 = u_2 - u_1 + 2\varepsilon$.

From $p(n_1, n_2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{n_1^2 + n_2^2}{2\sigma^2}\right)$,

$$\begin{aligned} p(u_1, u_2) &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(u_1 - 2\varepsilon)^2 + (u_2 - u_1 + 2\varepsilon)^2}{2\sigma^2} \right) \\ &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{2(u_1 - 2\varepsilon)^2 + u_2^2 - 2u_2(u_1 - 2\varepsilon)}{2\sigma^2} \right) \\ &= \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{\sigma^2} \left[(u_1 - 2\varepsilon)^2 - u_2(u_1 - 2\varepsilon) + \frac{1}{2}u_2^2 \right] \right\} \end{aligned}$$

If $-s_l(t)$ is sent, $J(n_1, n_2) = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$, $n_1 = u_1 + 2\varepsilon$ and $n_2 = u_2 - u_1 - 2\varepsilon$.

$$\begin{aligned} p(u_1, u_2) &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(u_1 + 2\varepsilon)^2 + (u_2 - u_1 - 2\varepsilon)^2}{2\sigma^2} \right) \\ &= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{2(u_1 + 2\varepsilon)^2 + u_2^2 - 2u_2(u_1 + 2\varepsilon)}{2\sigma^2} \right) \\ &= \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{1}{\sigma^2} \left[(u_1 + 2\varepsilon)^2 - u_2(u_1 + 2\varepsilon) + \frac{1}{2}u_2^2 \right] \right\} \end{aligned}$$

(b) Based on the likelihood ratio, show that the optimum combination of U_1 and U_2 results in the decision variable

$$U = U_1 + \beta U_2$$

where β is constant. What is the optimum value of β?

(*Solution.*)

$$\Lambda = \frac{P(s_l(t))}{P(-s_l(t))} = \frac{P(u_1, u_2| - s_l(t) \text{ sent})}{P(u_1, u_2|s_l(t) \text{ sent})} = \exp\left\{ -\frac{1}{\sigma^2} \left[8\varepsilon u_1 - 4\varepsilon u_2 \right] \right\}$$

$$\ln \Lambda = -\frac{8\varepsilon}{\sigma^2} \left[u_1 - \frac{1}{2}u_2 \right]$$

Therefore, $\beta = -\frac{1}{2}$.

(c) Suppose that $s_l(t)$ is transmitted. What is the probability density function of U?

(*Solution.*) Because U_1 and U_2 are Gaussian, $U = U_1 - \frac{1}{2}U_2$ is also Gaussian. The mean is 2ε and the variance is εN_0.

$$p(u) = \frac{1}{\sqrt{2\pi\varepsilon N_0}} \exp\left(-\frac{(u-2\varepsilon)^2}{2\varepsilon N_0}\right)$$

(d) What is the probability error assuming that $s_l(t)$ was transmitted? Express your answer as a function for the **SNR** ε/N_0.

(*Solution.*) When $s_l(t)$ was sent, the signal will be detected as $-s_l(t)$ if U is smaller than 0.

$$P_2 = P(U < 0) = \int_{-\infty}^{0} p(u)du = Q\left(\sqrt{\frac{4\varepsilon}{N_0}}\right)$$

(e) What is the loss in performance if only $U = U_1$ is the decision variable?

(*Solution.*) If only U_1 is used, P_2 is $Q\left(\sqrt{\frac{2\varepsilon}{N_0}}\right)$. We can see that 3dB lower error probability when using U_1 and U_2 together.