E6712: Lecture 10

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1 Carrier and Symbol Synchronization

Coherent demodulation requires the use of an appropriately phased carrier at the receiver.

Both coherent and non-coherent demodulation requires symbol synchronization.

Carrier phase and symbol timing are *estimated* by the receiver from the incoming signal.

1.1 Signal Parameter Estimation

Assume that the channel delays the signal $s(t) = \Re[s_l(t)e^{j2\pi f_c t}]$, and corrupts it by the AWGN:

$$r(t) = s(t-\tau) + n(t).$$

The received signal may be expressed as

$$\begin{aligned} r(t) &= \Re\{[s_l(t-\tau)e^{j\phi} + z(t)]e^{j2\pi f_c t}\} \\ &= s(t;\phi,\tau) + n(t) \end{aligned}$$

$\{\phi, \tau\}$ is a signal parameter vector which we in general denote by Ψ.

The MAP estimate of Ψ is the value that maximizes the *a posteriori probability*

$$p(\Psi|r) = \frac{p(r|\Psi)p(\Psi)}{p(r)}$$

The MAP estimate of Ψ is the value that maximizes $p(r|\Psi)$ or the (log)-likelihood function.

The likelihood function:

$$\Lambda(\Psi) = \exp\Big\{-\frac{1}{\mathcal{N}_0}\int_{T_0}[r(t) - s(t;\Psi)]^2 dt\Big\}$$

We take partial derivatives to obtain necessary conditions the MAP estimate $\widehat{\Psi}$ has to satisfy.

The *bias* of an estimate is defined as

$$E(\widehat{\Psi}) - \Psi$$

When $E(\widehat{\Psi}) = \Psi$, the estimate is *unbiased*.

The variance of the estimate:

$$\sigma_{\widehat{\Psi}} = E[\widehat{\Psi}^2] - \{E[\widehat{\Psi}]\}^2$$

Cramèr-Rao lower-bound on the mean-square error:

$$E\big\{[\widehat{\Psi} - \Psi]^2\big\} \geq \frac{\Big[\frac{\partial}{\partial\Psi}E[\widehat{\Psi}]\Big]^2}{E\Big\{\Big[\frac{\partial}{\partial\Psi}\ln p(\widehat{\Psi}|\Psi)\Big]^2\Big\}}$$

1.2 Carrier Phase Estimation

$$\begin{aligned}\Lambda(\phi) &= \exp\Big\{-\frac{1}{\mathcal{N}_0}\int_{T_0}[r(t)-s(t;\phi)]^2dt\Big\}\\ &= \exp\Big\{-\frac{1}{\mathcal{N}_0}\int_{T_0}r^2(t)dt+\frac{2}{\mathcal{N}_0}\int_{T_0}r(t)s(t;\phi)dt-\frac{1}{\mathcal{N}_0}\int_{T_0}s(t;\phi)^2dt\Big\}\\ &= C\exp\Big\{\frac{2}{\mathcal{N}_0}\int_{T_0}r(t)s(t;\phi)dt\Big\},\end{aligned}$$

where C is a constant that does not depend on ϕ.

Examples

$$r(t) = A\underbrace{\cos(2\pi f_c t+\phi)}_{s(t;\phi)}$$

The likelihood is a function of A:

$$\Lambda(\phi;A) = C\exp\Big\{\frac{A}{\mathcal{N}_0}\int_{T_0}r(t)\cos(2\pi f_c t+\phi)dt\Big\}$$

1) Determining the phase of the unmodulated carrier (A is a constant):

A necessary condition for a maximum

$$\frac{d\Lambda(\phi;A)}{d\phi}=0$$

yields

$$\int_T r(t)\sin(2\pi f_c t+\hat{\phi}_{\mathrm{ML}})dt = 0$$

or, equivalently,

$$\hat{\phi}_{\mathrm{ML}} = -\arctan\frac{\int_T r(t)\sin 2\pi f_c t\,dt}{\int_T r(t)\cos 2\pi f_c t\,dt}$$

2) BPSK: $A\pm 1$ with equal probability $\Longrightarrow$

$$p(A) = \frac{1}{2}\delta(A-1)+\frac{1}{2}\delta(A+1)$$

The expected value of the likelihood function $\Lambda(\phi;A)$:

$$\begin{aligned}\bar{\Lambda}(\phi) &= \int\Lambda(\phi;A)p(A)dA\\ &= \ln\cosh C\Big\{\frac{2}{\mathcal{N}_0}\int_{T_0}r(t)\cos(2\pi f_c t+\phi)dt\Big\}\end{aligned}$$

Approximation used in practice:

$$\ln\cosh x = \begin{cases} x^2/2, & |x|\ll 1,\\ |x|, & |x|\gg 1.\end{cases}$$

3) A is a Gaussian RV:

$$p(A) = \frac{1}{\sqrt{2\pi}}e^{-A^2/2}$$

The expected value of the likelihood function $\Lambda(\phi; A)$:

$$\begin{aligned}\bar{\Lambda}(\phi) &= \int \Lambda(\phi; A) p(A) dA \\ &= C \exp\Big\{ \Big[\frac{2}{\mathcal{N}_0} \int_{T_0} r(t) \cos(2\pi f_c t + \phi) dt \Big]^2 \Big\}\end{aligned}$$

The corresponding log-likelihood:

$$\bar{\lambda}(\phi) = \Big[\frac{2}{\mathcal{N}_0} \int_{T_0} r(t) \cos(2\pi f_c t + \phi) dt \Big]^2$$

Phase-locked loop (PLL) consists of a multiplier, a loop filter and a voltage-controlled oscillator (VCO).

Assume that the input to the PLL is $\cos(2\pi f_c t + \phi)$

The input to the VCO is f_c and $\hat{\phi}$, and the output is $\sin(2\pi f_c t + \hat{\phi})$

The input to the loop filter is the product:

$$\begin{aligned}e(t) &= \cos(2\pi f_c t + \phi) \sin(2\pi f_c t + \hat{\phi}) \\ &= \frac{1}{2} \sin(\hat{\phi} - \phi) + \frac{1}{2} \sin(4\pi f_c t + \hat{\phi} + \phi)\end{aligned}$$

The loop filter is a low-pass filter which removes the component at $2f_c$.

It is usually a simple first order low-pass:

$$G(s) = \frac{1 + \tau_2 s}{1 + \tau_1 s}, \qquad \tau_1 \gg \tau_2.$$

1.3 Decision Directed Loops

$s(t; \phi)$ carries information symbols $\{I_n\}$ which is known.

Received low-pass signal:

$$r(t) = e^{-j\phi} \underbrace{\sum_n I_n g(t - nT)}_{s_l(t)} + z(t)$$

energy of $s_l(t)$:

$$\int \Big[\sum_n I_n g(t - nT) \Big]^2 = \sum_n I_n^2$$

The likelihood function:

$$\Lambda(\phi) = C \exp\Big\{ \Re\Big[\frac{2}{\mathcal{N}_0} \int_{T_0} r(t) s_l^*(t) e^{j\phi} dt \Big] \Big\}$$

The log-likelihood:

$$\lambda(\phi) = \Re\Big[e^{j\phi}\sum_n I_n^* \underbrace{\int_{nT}^{(n+1)T} r(t)g^*(t-nT)dt}_{y_n}\Big]$$

$$= \cos\phi\cdot\Re\Big[\sum_n I_n^* y_n\Big] + \sin\phi\cdot\Im\Big[\sum_n I_n^* y_n\Big]$$

$$\therefore$$

$$\frac{d\lambda(\phi)}{d\phi} = -\sin\phi\cdot\Re\Big[\sum_n I_n^* y_n\Big] + \cos\phi\cdot\Im\Big[\sum_n I_n^* y_n\Big]$$

$$\therefore$$

$$\hat{\phi}_{\mathrm{ML}} = -\arctan\frac{\Im\Big[\sum_n I_n^* y_n\Big]}{\Re\Big[\sum_n I_n^* y_n\Big]} \quad \leftarrow \quad \boxed{\text{open-loop solution}}$$

Recursive estimation via PLL.

1.4 Symbol Timing Estimation

The output of the demodulator is sampled at the symbol rate.

symbol synchronization or *timing recovery*

Decision-directed:

The received signal is

$$r(t) = s(t;\tau) + n(t),$$

where $s(t;\tau) = \sum_n I_n g(t-nT-\tau)$.

The log-likelihood function:

$$\lambda(\tau;\{I_n\}) \propto \int_{T_0} r(t)s^*(t;\tau)dt$$

$$= \sum_n I_n \int_{T_0} r(t)g^*(t-nT-\tau)dt$$

$$= \sum_n I_n^* y_n(\tau),$$

where $y_n(\tau) = \int_{T_0} r(t)g^*(t-nT-\tau)dt$.

Open loop solution:

$$\sum_n I_n^* \frac{d}{d\tau} y_n(\tau) = 0$$

Non-decision-directed:

First find the expected value of $\Lambda(\tau;\{I_n\})$ over $\{I_n\}$.

In case of binary PAM, where $I_n = \pm 1$, with equal probability, we have

$$\bar{\Lambda}(\tau) = \ln\cosh C y_n(\tau) \approx \boxed{\frac{1}{2}C^2 y_n^2(\tau)}$$