E6712: Lecture 11

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1 Carrier and Symbol Synchronization

1.1 Effect of Additive Noise on the Phase Estimation

Assume that the PLL is tracking the phase of the following signal:

$$A\cos(2\pi f_c t + \phi)$$

The actual received signal is given by

$$\begin{aligned} r(t) &= \Re\{[s(t) + z(t)]e^{j(2\pi f_c t+\phi)}\} \\ &= \Re\{[s(t) + n_c(t) + jn_s(t)]e^{j(2\pi f_c t+\phi)}\} \\ &= A\cos(2\pi f_c t + \phi) + n_c\cos(2\pi f_c t + \phi) - n_s\sin(2\pi f_c t + \phi) \end{aligned}$$

The detector signal:

$$r(t)\cdot A\cos(2\pi f_c t + \hat{\phi}) = \frac{1}{2}[A^2\cos\Delta\phi + An_c\cos\Delta\phi - An_s\sin\Delta\phi + \text{double-frequency terms}]$$

The PLL signal:

$$r(t)\cdot \sin(2\pi f_c t + \hat{\phi}) = \frac{1}{2}[A\sin\Delta\phi + n_c\sin\Delta\phi - An_s\cos\Delta\phi + \text{double-frequency terms}]$$

After the loop filtering, we have

$$\sin\Delta\phi + n_2(t),$$

where

$$n_2(t) = \frac{n_c(t)}{A}\sin\Delta\phi - \frac{n_c(t)}{A}\cos\Delta\phi$$

Therefore,

$$n_2(t) \sim \mathcal{N}(0,\sigma_2^2), \qquad \sigma_2^2 = B_{\text{eq}}\left(\frac{\sin^2\Delta\phi}{A^2}\frac{\mathcal{N}_0}{2} + \frac{\cos^2\Delta\phi}{A^2}\frac{\mathcal{N}_0}{2}\right) = B_{\text{eq}}\frac{\mathcal{N}_0}{2A^2}$$

Signal-strength issues.

1.2 Decision Directed Loops

$s(t;\phi)$ carries information symbols $\{I_n\}$ which is known.

Received low-pass signal:

$$r(t) = e^{-j\phi}\underbrace{\sum_n I_n g(t-nT)}_{s_l(t)} + z(t)$$

energy of $s_l(t)$:

$$\int\Big[\sum_n I_n g(t-nT)\Big]^2 dt = \sum_n I_n^2$$

The likelihood function:

$$\Lambda(\phi) = C \exp\Big\{\Re\Big[\frac{2}{\mathcal{N}_0}\int_{T_0} r(t)s_l^*(t)e^{j\phi}dt\Big]\Big\}$$

The log-likelihood:

$$\begin{aligned}\lambda(\phi) &= \Re\Big[e^{j\phi}\sum_n I_n^* \underbrace{\int_{nT}^{(n+1)T} r(t)g^*(t-nT)dt}_{y_n}\Big] \\ &= \cos\phi\cdot\Re\Big[\sum_n I_n^* y_n\Big] + \sin\phi\cdot\Im\Big[\sum_n I_n^* y_n\Big] \\ &\therefore \\ \frac{d\lambda(\phi)}{d\phi} &= -\sin\phi\cdot\Re\Big[\sum_n I_n^* y_n\Big] + \cos\phi\cdot\Im\Big[\sum_n I_n^* y_n\Big] \\ &\therefore \\ \hat{\phi}_{\mathrm{ML}} &= -\arctan\frac{\Im\Big[\sum_n I_n^* y_n\Big]}{\Re\Big[\sum_n I_n^* y_n\Big]} \quad \leftarrow \quad \boxed{\text{open-loop solution}}\end{aligned}$$

Recursive estimation via PLL.

1.3 Non-Decision Directed Loops

Data are treated as random variables.

We base the estimation on the average $\Lambda(\phi)$.

We may know the distribution (PAM) or assume Gaussian data.

The log-likelihood is quadratic under the Gaussian assumption and approximately quadratic for PAM.

Squaring loop:
a non-decision directed loop widely used in practice:

Consider the PAM:

$$s^2(t) = A^2(t)\cos^2(2\pi f_c t + \phi) = \frac{1}{2}A^2(t) + \frac{1}{2}A^2(t)\cos(4\pi f_c t + 2\phi)$$

Use a band-pass filter and frequency divider.

See Fig. 6.2-14.

Costas (1956) loop:

The received signal is multiplied by $\sin(2\pi f_c t + \hat{\phi})$ (which is the output of VCO) and $\cos(2\pi f_c t + \hat{\phi})$:

$$y_c(t) = r(t)\cdot\cos(2\pi f_c t + \hat{\phi}) = \frac{1}{2}[(A(t) + n_c(t))\cos\Delta\phi + n_s(t)\sin\Delta\phi + \text{double-frequency terms}]$$

$$y_s(t) = r(t)\cdot\sin(2\pi f_c t + \hat{\phi}) = \frac{1}{2}[(A(t) + n_c(t))\sin\Delta\phi - n_s(t)\cos\Delta\phi + \text{double-frequency terms}]$$

After passing through a low-pass filter, we multiply $y_c(t)$ and $y_s(t)$ to get

$$e(t) = \frac{1}{8}\{[A(t) + n_c(t)]^2 - n_s(t)\} \sin 2\Delta\phi - \frac{1}{4} n_s(t)[A(t) + n_c(t)] \cos 2\Delta\phi.$$

which after filtering drives the VCO.

See Fig. 6.2-15.

Carrier recovery for M-ary PSK

Transmitted phase:

$$\hat{\theta} = \frac{2\pi}{M}(m-1)$$

Received signal:

$$r(t) = A\cos(2\pi f_c t + \phi + \theta) + n(t)$$

Use an M-th power-low device.

Use a band-pass filter and frequency divider.

See Fig. 6.2-16.

1.4 Joint Estimation of Carrier Phase and Symbol Timing

The log-likelihood function:

$$\lambda(\phi, \tau) = \Re\left\{\frac{1}{\mathcal{N}_0}\int_{T_0} r(t)s^*(t;\phi,\tau)dt\right\}$$

where $s(t;\phi,\tau) = e^{-j\phi}\sum_n I_n g(t - nT - \tau)$. Thus,

$$\begin{aligned}
\lambda(\phi,\tau) &= \Re\left\{\frac{e^{j\phi}}{\mathcal{N}_0}\sum_n I_n \int_{T_0} r(t)g^*(t-nT-\tau)dt\right\} \\
&= \Re\left\{\frac{e^{j\phi}}{\mathcal{N}_0}\sum_n I_n^* y_n(\tau)\right\} \\
&= \Re\Big\{\frac{e^{j\phi}}{\mathcal{N}_0}\Big[\underbrace{\Re\Big\{\sum_n I_n^* y_n(\tau)\Big\}}_{A(\tau)} + j\underbrace{\Im\Big\{\sum_n I_n^* y_n(\tau)\Big\}}_{B(\tau)}\Big]\Big\}
\end{aligned}$$

where and $y_n(\tau) = \int_{T_0} r(t)g^*(t - nT - \tau)dt$.

Necessary conditions:

$$\frac{\partial\lambda(\phi,\tau)}{\partial\phi} = 0 \qquad \frac{\partial\lambda(\phi,\tau)}{\partial\tau} = 0$$

We have

$$\begin{aligned}
0 &= \frac{\partial\lambda(\phi,\tau)}{\partial\phi} = -A(\tau)\sin\phi - B(\tau)\cos\phi \\
0 &= \frac{\partial\lambda(\phi,\tau)}{\partial\tau} = \frac{\partial A(\tau)}{\partial\tau}\cos\phi - \frac{\partial B(\tau)}{\partial\tau}\sin\phi,
\end{aligned}$$

and thus

$$\hat{\phi}_{\mathrm{ML}} = -\arctan \frac{\Im\Big[\sum_n I_n^* y_n(\hat{\tau}_{\mathrm{ML}})\Big]}{\Re\Big[\sum_n I_n^* y_n(\hat{\tau}_{\mathrm{ML}})\Big]}$$

$$A(\tau)\frac{\partial A(\tau)}{\partial \tau} + B(\tau)\frac{\partial B(\tau)}{\partial \tau}|_{\tau=\hat{\tau}_{\mathrm{ML}}} = 0.$$

See Fig. 6.4-1.