E6712: Lecture 3

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1 Vector Channel – Review

Optimal receiver for the vector channel $\mathbf{r} = \sqrt{E_m}\mathbf{s_m} + \mathbf{n}$, where

$\mathbf{r} = (r_1, r_2, \ldots, r_N)$, $\mathbf{n} = (n_1, n_2, \ldots, n_N)$ i.i.d. $(0, \mathcal{N}_0/2)$

$\mathbf{s_m} = (s_{m1}, s_{m2}, \ldots, s_{mN})$ unit energy equiprobable

$p(\mathbf{r}|\mathbf{s_m}) \propto \exp[-\sum_{j=1}^{N}(r_j - \sqrt{E_m}s_{mj})^2/\mathcal{N}_0]$

error probability for the two message case (genie bound):

$$P(s_m \to s_i) = Q\left(\sqrt{\frac{\sum_j (s_{mj} - s_{ij})^2}{2\mathcal{N}_0}}\right)$$

1.1 Examples of Signal Constellations

antipodal signaling (BPSK) 1 bit per symbol:

$$P(s_m \to s_i) = Q\left(\sqrt{\frac{2E_b}{\mathcal{N}_0}}\right)$$

orthogonal signaling:

$$P(s_m \to s_i) = Q\left(\sqrt{\frac{E_b}{\mathcal{N}_0}}\right)$$

QPSK signaling 2 bits per symbol:

$$P(s_m \to s_i) = Q\left(\sqrt{\frac{2E_b}{\mathcal{N}_0}}\right)$$

1.2 The Union Bound

$$d^2_{\min} = \min_{i,j} \sum_j (s_{mj} - s_{ij})^2$$

Let s_m be transmitted

$$P_e = P\Big\{\bigcup_{i \neq m}(s_m \to s_i)\Big\} \leq \sum_{i \neq m} P(s_m \to s_i)$$

$$\leq (M-1)Q\left(\sqrt{\frac{d^2_{\min}}{2\mathcal{N}_0}}\right)$$

2 Stochastic Processes

$X(t)$ - ensemble of random variables indexed by the parameter t.

$x(t)$ - single realization or sample function.

RVs $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are characterized by their joint PDF $p(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$.

Stationary random process:

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$$

2.1 Statistical Averages

The mean:

$$m(t_i) = E(X_{t_i}) = \int_{-\infty}^{\infty} x_{t_i} p(x_{t_i}) dx_{t_i}$$

If the process is stationary, the mean is constant in time.

Autocorrelation function:

$$E(t, t+\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_t x_{t+\tau} p(x_t, x_{t+\tau}) dx_t dx_{t+\tau}).$$

If the process is *strict-sense* stationary, we have $E(t, t+\tau) = \phi(\tau) = \phi(-\tau)$.

If $E(t, t+\tau) = \phi(\tau) = \phi(-\tau)$ holds, the process is said to be *wide-sense* stationary.

Autocovariance:

Gaussian random process:

Power Spectral Density:

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(\tau) e^{-j2\pi f\tau} d\tau$$

White noise is a random process with the autocorrelation function and PSD

$$\phi(\tau) = \frac{1}{2}\mathcal{N}_0 \delta(\tau) \quad \text{and} \quad \Phi(f) = \frac{1}{2}\mathcal{N}_0.$$

2.2 Response of a Linear Time-Invariant System to a Stationary Random Process

$h(t) \leftrightarrow H(f)$ – filter impulse response

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

Let the input $x(t)$ be a sample function of a stationary stochastic process $X(t)$ with m_x and $\phi_{xx}(\tau)$.

Then the output $y(t)$ is a sample function of a stationary stochastic process $Y(t)$ with

the mean value

$$m_y = \int_{-\infty}^{\infty} h(\tau) E[x(t-\tau)] d\tau = m_x H(0).$$

the autocorrelation function

$$\begin{aligned}\phi_{yy}(t, t+\tau) &= \int_{-\infty}^{\infty} h(\alpha) h^*(\beta) E[X(t-\alpha) X^*(t+\tau-\beta) d\alpha d\beta \\ &= \int_{-\infty}^{\infty} h(\alpha) h^*(\beta) \phi_{xx}(\tau+\alpha-\beta) d\alpha d\beta\end{aligned}$$

the power density spectrum

$$\Phi_{yy}(f) = \Phi_{xx}(f) |H(f)|^2$$

3 Optimum Receiver for the AWGN Channel

M signal waveforms $\{s_m(t), m = 1, 2, \ldots, M\}$; symbol interval of duration T $\int_0^T s_m^2(t) dt = 1$

Additive white Gaussian noise channel:

$$r(t) = \sqrt{E_m} s_m(t) + n(t), \ 0 \le t < T.$$

Design a receiver that is optimal is the sense that it minimizes the probability of making an error.

Reduce the problem to the one of the vector channel by expanding each waveform over a CON.

CON: $\int_0^T \varphi_i(t) \varphi_j(t) dt = \delta_{ij}$

$$\begin{aligned}
r(t) &= \lim_{N\to\infty} \sum_{j=1}^{N} r_j \varphi_j(t) & r_j &= \int_0^T r(t) \varphi_j(t) dt & \mathbf{r} &= (r_1, r_2, \ldots, r_N) \\
n(t) &= \lim_{N\to\infty} \sum_{j=1}^{N} n_j \varphi_j(t) & n_j &= \int_0^T n(t) \varphi_j(t) dt & \mathbf{n} &= (n_1, n_2, \ldots, n_N) \\
s_m(t) &= \lim_{N\to\infty} \sum_{j=1}^{N} s_{mj} \varphi_j(t) & s_{mj} &= \int_0^T s_m(t) \varphi_j(t) dt & \mathbf{s_m} &= (s_{m1}, s_{m2}, \ldots, s_{mN})
\end{aligned}$$

Statistics of $\mathbf{n}$:

$$E(n_j) = \int_0^T E[n(t)] \varphi_i(t) dt$$

$$\begin{aligned}E(n_i n_j) &= \int_0^T \int_0^T E[n(t) n(\tau)] \varphi_i(t) \varphi_j(\tau) dt d\tau \\ &= \frac{\mathcal{N}_0}{2} \int_0^T \int_0^T \delta(t-\tau) \varphi_i(t) \varphi_j(\tau) dt d\tau \\ &= \frac{\mathcal{N}_0}{2} \int_0^T \varphi_i(t) \varphi_j(t) dt = \frac{\mathcal{N}_0}{2} \delta_{ij}\end{aligned}$$

Optimal receiver for the vector channel $\mathbf{r} = \sqrt{E_m}\mathbf{s_m} + \mathbf{n}$:

Recall that $p(\mathbf{r}|\mathbf{s_m}) \propto \exp[-\sum_{j=1}^{N}(r_j - \sqrt{E_m}s_{mj})^2/\mathcal{N}_0]$,

$$\hat{m} = \arg \max_{1\le m\le M} \left\{ \sqrt{E_m}\sum_{j=1}^{N} r_j s_{mj} - \frac{E_m}{2}\sum_{j=1}^{N} s_{mj}^2 + \frac{\mathcal{N}_0}{2}\ln P(s_m) \right\}$$

As $N \to \infty$,

$$\hat{m} = \arg \max_{1\le m\le M} \left\{ \sqrt{E_m}\int_0^T r(t)s_m(t)dt - \frac{E_m}{2} + \frac{\mathcal{N}_0}{2}\ln P(s_m) \right\}$$

Matched filter: $h_m(t) = s_m(T-t)$.

Need to know signals? the channel?

4 Basic Digital Modulation Schemes

Waveform (f_c is the carrier frequency, $g(t)$ is a low-pass waveform)

$$\begin{aligned}\sqrt{E_m}s_m(t) &= \Re\{A_m \cdot g(t) \cdot e^{j\theta_m} \cdot e^{j2\pi f_c t}\}, \quad 1 \le m \le M. \\ &= A_m g(t)\cos(2\pi f_c t + \theta_m) \\ &= A_m \cos\theta_m \cdot g(t)\cos(2\pi f_c t) - A_m \sin\theta_m \cdot g(t)\sin(2\pi f_c t)\end{aligned}$$

We shall assume that $g(t)$ is a rectangular pulse of duration T, and height $\sqrt{\frac{2}{T}}$, and T a multiple of $1/f_c$.

$$\frac{2}{T}\int_0^T \cos^2(2\pi f_c t)dt = 1$$

Thus $A_m = \sqrt{E_m}$.

PAM: $A_m = (2m - 1 - M)d_{\min}$ carries the information, $\theta_m = 0$

PSK: $\theta_m = 2\pi(m-1)/M$ carries the information, $A_m = \sqrt{E}$; $d_{\min} = 2\sqrt{E}\sin(\pi/M)$.

QAM: Both $A_m = \sqrt{A_{ms}^2 + A_{mc}^2}$ and $\theta_m = \arctan(A_{ms}/A_{mc})$ carry the information.

Gray mapping: