E6712: Lecture 6

March 3, 2004

1 Modulation Codes for Spectrum Shaping

1.1 Two Simple Rate $1/2$ Codes

- original data

  $$10001101$$

  requires a piece of track of length $8 \times \Delta$

- FM coded sequence

  $$\underline{1}1\underline{1}0\underline{1}0\underline{1}0\underline{1}1\underline{1}1\underline{1}0\underline{1}1$$

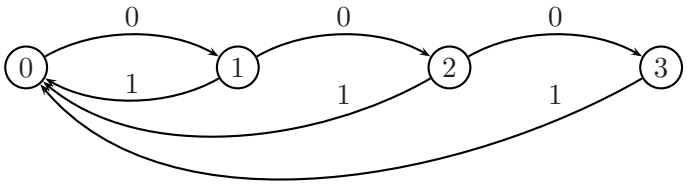
  requires a piece of track of length $2 \times 8 \times \Delta$

- MFM coded sequence

  $$\underline{0}1\underline{0}0\underline{1}0\underline{1}0\underline{0}1\underline{0}1\underline{0}0\underline{0}1$$

  requires a piece of track of length $2 \times 8 \times \Delta/2$

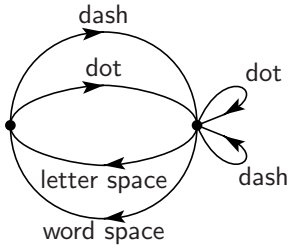
- $(1, 3)$ run-length limited constraint



- (d, k) run-length limited constraint: between two 1's, there are at least d and at most k 0's.

1.2 Constrained Codes

Shannon's representation of the constraints on telegraph symbols:



Finite state transition diagram (FSTD).

Shannon capacity (topological entropy) for FSTD F:

$$C(F) = \lim_{n \to \infty} \frac{\log_2(N(n;(F)))}{n} = \log_2 \lambda(A(F)),$$

where $A(F)$ is the constraint adjacency matrix, and $\lambda(A(F))$ is its maximum eigenvalue.

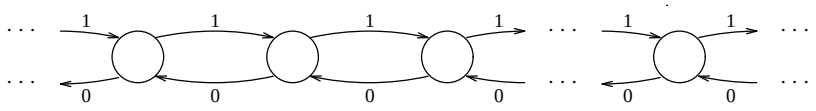
Code rate:

$$p/q \leq C(F).$$

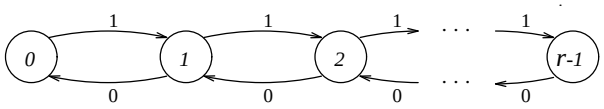
For the $(1,3)$ constraint:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \lambda = 1.4656 \quad C(1,3) = 0.5525$$

Canonical diagram for sequences with spectral null at 0:



Finite case:

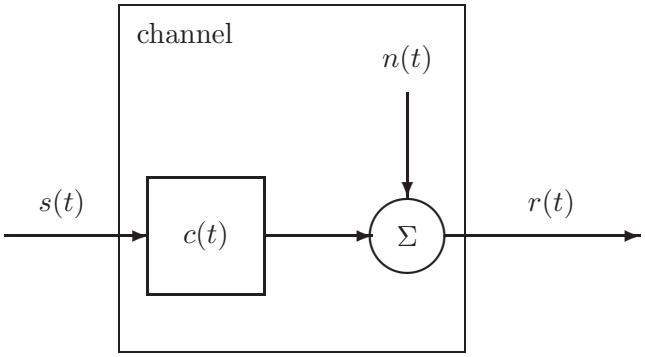


$$C(F_r) = \log_2 \left[2 \cos \frac{\pi}{r+1} \right].$$

Encoder/Decoder design: state splitting, sliding block decoders.

2 Channels with ISI and AWGN

Continuous-time channel model:



Sequence of symbols $\{I_n\}$ is transmitted or stored as a continuous time signal

$$s(t) = \sum_n I_n g(t - nT),$$

The received signal:

$$r(t) = \sum_n I_n h(t - nT) + z(t), \qquad z(t) \sim (0, \mathcal{N}_0/2).$$

The sampled MF output:

$$\begin{aligned} y_k &= \int_{-\infty}^{\infty} h(t - kT) r(t) dt \\ &= \sum_n I_m x_{k-n} + \nu_n \\ &= (I * x)_k + \nu_k, \end{aligned}$$

where

$$x_k = \int_{-\infty}^{\infty} h(t) h(t - kT) dt$$

and

$$\nu_k = \int_{-\infty}^{\infty} z(t) h(t - kT) dt$$

is discrete-time noise with the autocorrelation of ν_n and ν_m given by $x_{n-m}\mathcal{N}_0/2$.

Let

$$\{x_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = F(z)F^*(z^{-1}).$$

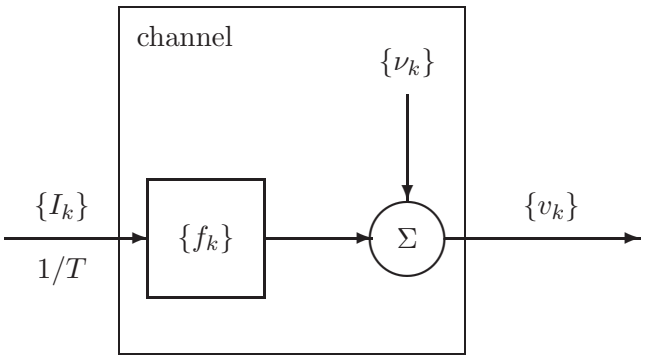
We will assume that $F(z)$ has unit norm.

After sequence $\{y_n\}$ passes through the *noise-whitening* filter with response $1/F^*(z^{-1})$, we get

$$v_k = \sum_n I_n f_{n-k} + \eta_k,$$

where $\{\eta_k\}$ is AWGN.

The equivalent discrete-time channel model:



3 The Optimal Viterbi Detection

Maximum likelihood sequence detection:

$$\max_{\{I_k\}\in\mathcal{C}} p(\{v_k\}|\{I_k\})$$

The log-likelihood function:

$$\Omega(\{I_k\}) = \sum_n [v_k - \sum_n I_n f_{n-k}]^2$$

Sequences $\{I_k\}$ are represented as paths through a trellis

We will use the $(1-D)$ channel as an example to explain principles, implementation issues, and performance of the Viterbi detection.

3.1 The Principles

The channel:

$$v_k = I_k - I_{k-1} + \eta_n$$

The log-likelihood function:

$$\Omega(\{I_k\}) = \sum_k \big[v_k - (I_k - I_{k-1})\big]^2.$$

Since $\{I_k\} \in \mathcal{C} \subseteq \{-1, 1\}^\infty$, we can equivalently minimize of the cost function given by

$$-\sum_k [(I_k - I_{k-1})v_k + I_k I_{k-1}].$$

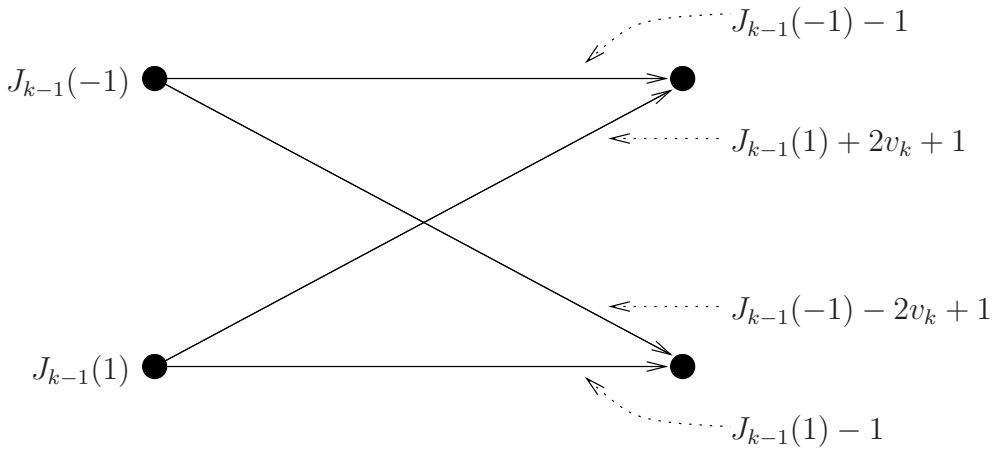
Sequences $\{I_k\}$ are represented as paths through a two state trellis where each state corresponds to the polarity of last symbol.

Draw the trellis.

At time k, as the sample v_k becomes available, the Viterbi algorithm finds two sequences: one minimizing the cost function over all possible sequences that end in state '1' and the other minimizing the cost function over all possible sequences that end in state '-1' at time k. We call these sequences *survivor sequences* at time k.

$J_k(s)$ the minimum cost up to time k and state s, $s \in \{-1, 1\}$:

$$J_k(s) = \min_{\{I_k\} \in \mathcal{C}} \sum_{\substack{n=-\infty \\ I_k = s}}^{k} [-(I_k - I_{k-1})v_k - I_k I_{k-1}].$$



The algorithm is based on the principle of optimality which says that a survivor sequence at time k must correspond to an extension of one of the survivor sequences at time $k-1$.

3.2 Implementation Issues

Common implementations of the Viterbi detector for $(1 - D)$ channel use the recursive *difference metric algorithm.*

The decision about extensions to both states at time k can be made based on single variable

$$\delta_k = \Delta J_{k-1} - v_k, \quad \text{where} \quad \Delta J_k = [J_k(-1) - J_k(1)]/2.$$

Condition	Update	Extension and Reliability
$\delta_k \leq -1$	$\Delta J_k = v_k + 1$	$\delta_k + 1$ $\delta_k - 1$
$-1 < \delta_k < 1$	$\Delta J_k = \Delta J_{n-1}$	$\delta_k + 1$ $-\delta_k + 1$
$\delta_k \geq 1$	$\Delta J_k = v_k - 1$	$-\delta_k - 1$ $-\delta_k + 1$