**E6712: Lecture 7**

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# 1 MLSE Error Probability Performance for $M$-ary PAM

The discrete time AWGN channel:

$$v_k = \sum_{n=0}^{L} I_n f_{n-k} + \eta_k.$$

The MLSE finds the sequence minimizing the log-likelihood function

$$\Omega(\{I_k\}) = \sum_{n=0}^{L} [v_k - \sum_{n} I_n f_{n-k}]^2$$

Assume that we know that either sequence $\{I_n\}$ or sequence $\{\tilde{I}_n\}$ is transmitted with equal probability.

Probability of error is then given by

$$P\Big[\Omega\big(\{\tilde{I}_n\}\big) < \Omega\big(\{I_n\}\big)\Big|\{I_n\}\Big]$$

For $M$-ary PAM, $\{I_n\}$ takes the values $\pm d, \pm 3d, \ldots, \pm(M-1)d$, and $2d$ is the distance between two successive levels.

The normalized error sequence corresponding to $\{I_n\}$ and $\{\tilde{I}_n\}$:

$$\varepsilon_n = \frac{1}{2d}(I_n - \tilde{I}_n)$$

In the binary (bipolar) case, we have $I_n \in \{-1, 1\}$ and $\varepsilon_n \in \{-1, 0, 1\}$:

$$\varepsilon_n = \frac{1}{2}(I_n - \tilde{I}_n)$$

We will be concerned with finite length error events (illustrate on a trellis) starting at some time $k$ and ending at time $k + l - L - 1$ s.t.:

1. $S_k = \tilde{S}_k$, where $S_k = (I_{k-1}, I_{k-2}, \ldots, I_{k-L})$ and $S_k = (\tilde{I}_{k-1}, \tilde{I}_{k-2}, \ldots, \tilde{I}_{k-L})$.

$$P\{S_k = \tilde{S}_k\} \leq 1$$

2. When information symbols $I_k, I_{k+1}, \ldots, I_{k+l-L-1}$ are added to the scaled error sequence $2d\boldsymbol{\varepsilon} = 2d\{\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{k+l-L-1}\}$, the result is an allowable symbol sequence. The probability of this event is

$$\prod_{i=0}^{l-L-1} \frac{M - |\varepsilon_i|}{M}$$

3. The sum of the branch metrics for the estimated path exceeds the sum of the branch metrics of the correct path:

$$\sum_{i=k}^{k+l-1} \left( v_i - \sum_{j=0}^{L} f_j \tilde{I}_{i-j} \right)^2 < \sum_{i=k}^{k+l-1} \left( v_i - \sum_{j=0}^{L} f_j I_{i-j} \right)^2$$

The probability of this event is

$$Q\left(\sqrt{\frac{2d^2}{\mathcal{N}_0}\delta^2(\boldsymbol{\varepsilon})}\right)$$

where $\delta^2(\boldsymbol{\varepsilon})$ is the distance between $\{I_n\}$ and $\{\tilde{I}_n\}$ on the channel (at the input of the MLSE):

$$\delta^2(\boldsymbol{\varepsilon}) = \sum_{i=k}^{k+l-1}\left(\sum_{j=0}^{L} f_j\varepsilon_{i-j}\right)^2$$

### 1.1 Error Events - Binary Example

Finite length normalized error sequence:

$$\boldsymbol{\varepsilon} = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1, 0, 1\},\ \varepsilon_0 = 1, \varepsilon_{l-1} \neq 0$$

The corresponding polynomial:

$$\varepsilon(D) = \sum_{i=0}^{l-1}\varepsilon_i D^i,$$

The minimum (merged) distance on the channel with transfer function $F(D)$ is defined by

$$d^2_{<>} = \min_{\varepsilon(D)\neq 0}\|F(D)\varepsilon(D)\|^2,$$

where the squared norm of a polynomial is defined as the sum of its squared coefficients.

The minimum distance is bounded from above by $\|F(D)\|^2$, the energy in the channel impulse response.

The *matched-filter bound* (MFB):

$$d^2_{<>} \leq d^2_{\text{MFB}} \stackrel{\text{def}}{=} \|h(D)\|^2.$$

A channel *achieves the MFB* if the error sequence of length $l = 1$, *i.e.*, $\varepsilon(D) = 1$, is in the set

$$\arg\min_{\varepsilon(D)\neq 0}\|h(D)\varepsilon(D)\|^2.$$

$F(D) = (1 - D)$ achieves the MFB.

$F(D) = (1 - D)(1 + D)^2$ does not achieve the MFB.

## 2 Linear Equalization

Recall that the sampled MF output are given by

$$y_k = \sum_n I_n x_{n-k} + \nu_k,$$

where $E[\nu_n\nu_m] = x_{n-m}\mathcal{N}_0/2$ and

$$\{x_n\} \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = F(z)F^*(z^{-1}).$$

We will assume that $F(z)$ has unit norm.

After sequence $\{y_n\}$ pases through the *noise-whitening filter* with response $1/F^*(z^{-1})$, we get

$$v_k = \sum_n I_n f_{n-k} + \eta_k,$$

where $\{\eta_k\}$ is AWGN.

MLSE complexity grows exponentially with the memory length (time dispersion) of the channel.

*Solution:* suboptimal symbol-by-symbol detectors.

### 2.1 Zero Forcing Equilazation

After sequence $\{v_n\}$ passes through the filter with response $1/F(z^{-1})$, we get no ISI but the noise is not white any more

## 3 Decision-Feedback Equalization

Principles and Block Diagram

Performance comparison with the MLSE

### 3.1 Equalization at the Transmitter – Tomlinson-Harashima Precoding

The equalizer can be placed at the transmitter if the channel is known and does not vary (wireline).

Pros and cons of a feedback (as in DFE) filter at the transmitter: no noise distortion, no penalty in SNR, no error propagation, larger dynamic range)

Tomlinson-Harashima precoding for $M$-ary PAM, $\{I_n\}$ takes the values $\pm d, \pm 3d, \ldots, \pm(M-1)d$.

The precoder output is

$$a_k = I_k - \sum_j f_j a_{k-j} \mod 2M$$

reduced to the desired range by performing modulo-$2M$ operation.

We can describe the process by introducing sequence $\{b_k\}$ s.t.

$$a_k = I_k - \sum_j f_j a_{k-j} + 2Mb_k$$