E6712: Lecture 8

March 31, 2004

1 Stochastic Processes - Review

$X(t)$ - ensemble of random variables indexed by the parameter t.

$x(t)$ - single realization or sample function.

RVs $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are characterized by their joint PDF $p(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$.

Stationary random process:

$$p(x_{t_1}, x_{t_2}, \ldots, x_{t_n}) = p(x_{t_1+\tau}, x_{t_2+\tau}, \ldots, x_{t_n+\tau})$$

1.1 Statistical Averages

The mean:

$$m(t_i) = E(X_{t_i}) = \int_{-\infty}^{\infty} x_{t_i} p(x_{t_i}) dx_{t_i}$$

If the process is stationary, the mean is constant in time.

Autocorrelation function:

$$E(t, t+\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_t x_{t+\tau} p(x_t, x_{t+\tau}) dx_t dx_{t+\tau}).$$

If the process is *strict-sense* stationary, we have $E(t, t+\tau) = \phi(\tau) = \phi(-\tau)$.

If $E(t, t+\tau) = \phi(\tau) = \phi(-\tau)$ holds, the process is said to be *wide-sense* stationary.

Autocovariance:

Gaussian random process:

Power Spectral Density:

$$\Phi(f) = \int_{-\infty}^{\infty} \phi(\tau) e^{-j2\pi f \tau} d\tau$$

White noise is a random process with the autocorrelation function and PSD

$$\phi(\tau) = \frac{1}{2}\mathcal{N}_0 \delta(\tau) \quad \text{and} \quad \Phi(f) = \frac{1}{2}\mathcal{N}_0.$$

2 Two Probability Distributions

2.1 Rayleigh Distribution

Consider the noise process $n(t)$

$$\begin{aligned} n(t) =& r(t) e^{j\phi(t)} \\ =& x(t) + jy(t) \end{aligned}$$

where $r(t)$ is the magnitude or envelope and $\phi(t)$ is the phase, $x(t)$ is the in-phase, and $y(t)$ is the quadrature component.

If both random processes $x(t)$ and $y(t)$ are statistically independent Gaussian distributed with the same variance and zero mean, then their joint joint probability density function is

$$\begin{aligned} P(x,y) =& P(x)P(y) \\ =& \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \end{aligned}$$

Transform differential areas by using $dxdy = rdrd\phi$ gives the joint probability density function of $r(t)$ and $phi(t)$ as

$$P(r,\phi) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad r \in [0,\infty),\ \phi \in [-\pi,\pi],$$

We get

$$P(r) = \int_{-\pi}^{\pi} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} d\phi = \boxed{\frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}} \leftarrow \textit{Rayleigh distribution}$$

$$P(\phi) = \int_{0}^{\infty} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} dr = \boxed{\frac{1}{2\pi}} \leftarrow \textit{uniform distribution}$$

Note that the random variables r and ϕ are statistically independent:

$$P(r,\phi) = P(r)P(\phi).$$

2.2 Rice Distribution

Consider the noise process $n(t)$

$$\begin{aligned} n(t) =& A\cos(2\pi f_0 t) + r(t)e^{j\phi(t)} \\ =& x(t) + jy(t) \end{aligned}$$

3 Characterization of Fading Channels

One of the causes of fading is multipath (more than one propagation path):

Thus, the received signal is a superposition of of a number of signals arriving from different paths.

Let

$$s(t) = \Re\{u(t)e^{j2\pi f_c t}\}$$

be the transmitted bandpass signal.

Assume that there is a discrete number of propagation paths, each characterized by some random delay $\tau_n(t)$ and some attenuation $\alpha_n(t)$.

Then the received signal is

$$\begin{aligned} x(t) =& \sum_n \alpha_n(t) s(t - \tau_n(t)) \\ =& \Re\Big\{ \sum_n \alpha_n(t) u(t - \tau_n(t)) e^{j2\pi f_c (t - \tau_n(t))} \Big\} \\ =& \Re\Big\{ \underbrace{\Big[\sum_n \alpha_n(t) e^{j2\pi f_c \tau_n(t)} u(t - \tau_n(t)) \Big]}_{r(t)} e^{j2\pi f_c t} \Big\} \end{aligned}$$

Consequently, the received baseband equivalent signal is

$$r(t) = \sum_n \alpha_n(t) e^{j2\pi f_c \tau_n(t)} u(t - \tau_n(t))$$

This is equivalent to a passage through a time variant channel with impulse response

$$c(\tau; t) = \sum_n \alpha_n(t) e^{-j2\pi f_c \tau_n(t)} \delta(\tau - \tau_n(t))$$

We have

$$r(t) = \int_{-\infty}^{\infty} c(\tau; t) u(t - \tau) d\tau$$

For a continuum of paths, the received bandpass signal is

$$x(t) = \int_{-\infty}^{\infty} \alpha(\tau; t) s(t - \tau) d\tau$$

where $\alpha(\tau; t)$ denotes the attenuation of signal components at delay τ and at time instant t.

We have

$$x(t) = \Re\Big\{ \int_{-\infty}^{\infty} \alpha(\tau; t) u(t - \tau) e^{j2\pi f_c (t - \tau)} \Big\}$$

The equivalent low-pass time-variant impulse response is

$$c(\tau;t) = \alpha(\tau;t)e^{-j2\pi f_c \tau}$$

Consider the case of sending an unmodulated carrier of frequency f_c through a channel with discrete multipath components.

The received bandpass signal

$$r(t) = \sum_n \alpha_n(t)e^{j2\pi f_c \tau_n(t)} = \sum_n \alpha_n(t)e^{\theta_n(\tau)}$$

$\alpha_n(t)$ changes slowly, $\theta_n(\tau)$ changes fast.

If there is a large number of multipath components, then we can invoke the central limit theorem to conclude that the impulse response $c(\tau;t)$ is a complex Gaussian process in t.

Rayleigh Fading: When $c(\tau;t)$ is modelled as a complex *zero-mean* Gaussian process, then its envelope $|c(\tau;t)|$ is Rayleigh distributed.

Ricean Fading: When $c(\tau;t)$ is modelled as a complex *non zero-mean* Gaussian process, *i.e.,* when there is a fixed direct propagation path between the transmitter and the receiver.

The autocorrelation function for a zero-mean WSS process $c(\tau;t)$ is

$$\varphi_c(\tau,\tau';\Delta t) = \frac{1}{2}E\{c^*(\tau;t)c(\tau';t+\Delta t)\} = \frac{1}{2}\varphi_c(\tau;\Delta t)\delta(\tau-\tau')$$

For the last equality, we assume that the attenuation and the phase shift of the channel associated with the path delay τ is uncorrelated with the attenuation and the phase shift of the channel associated with the path delay τ'

For $\Delta t = 0$, $\varphi_c(\tau_1;0) = \varphi_c(\tau)$ is known as the *multipath intensity profile of the channel.*

T_m is the *multipath spread of the channel.* It determines the delay up to which the signals are still considered to be correlated.

$\varphi_c(\tau)$ can be measured by sending a very narrow pulse through the channel and correlating the received signal with a delayed version of itself.

Another way to characterize the time variant multipath channel is

$$C(f;t) = \int_{-\infty}^{\infty} c(\tau;t)e^{-j2\pi f\tau}d\tau$$

and

$$\begin{aligned}\phi_c(f, f'; \Delta t) =& \frac{1}{2} E\{C^*(f;t)C(f';t+\Delta t)\} \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{c^*(\tau;t)c(\tau';t+\Delta t)\}e^{j2\pi(f\tau - f'\tau')}d\tau d\tau' \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\varphi_c(\tau;\Delta t)e^{j2\pi(f-f')}d\tau \\ &= \varphi_c(\Delta f, \Delta t)\end{aligned}$$

For $\Delta t = 0$,

$(\Delta f)_c \approxeq 1/T_m$ is the coherence bandwidth of the channel.

When $(\Delta f)_c$ is small compared to the bandwidth of the of the signal, then the channel is *frequency selective*. Otherwise is frequency nonselective.

Time variations of the channel, *i.e.,* change of $\phi_c(\Delta f; \Delta t)$ as a function of Δt.

We define

$$S_c(\Delta f; \lambda) = \int_{-\infty}^{\infty}\phi_c(\Delta f; \Delta t)e^{-j2\pi\lambda\Delta t}d\Delta t$$

When $\Delta f = 0$, we have the *Doppler power spectrum*

$$S_c(\lambda) = \int_{-\infty}^{\infty}\phi_c(\Delta t)e^{-j2\pi\lambda\Delta t}d\Delta t$$

The range of values of λ over which $S(\lambda)$ is non-zero is B_d, the *Doppler spread of the channel.*

The *coherence time*:

$$(\Delta t)_c \approxeq \frac{1}{B_d}$$

Large coherence time means small channel time variations.

"Good" channels have large $\Delta t)_c$ and large $\Delta f)_c$.

Scattering function:

$$\begin{aligned} S(\tau;\lambda) &= \int_{-\infty}^{\infty} \phi_c(\tau;\Delta t) e^{-j2\pi\lambda\Delta t} d\Delta t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_c(\Delta f;\Delta t) e^{j2\pi\tau\Delta f} e^{-j2\pi\lambda\Delta t} d\Delta f d\Delta t \\ &= \int_{-\infty}^{\infty} S_c(\Delta f;\lambda) e^{-j2\pi\tau\Delta f} d\Delta f \end{aligned}$$

4 The Effect of Signal Properties on the Choice of Channel Model

Let $u(t)$ be the baseband signal and $U(f)$ its Fourier transform

Ignoring additive noise, we have

$$\begin{aligned} r(t) &= \int_{-\infty}^{\infty} c(\tau;t) u(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} C(f;t) U(f) e^{-j2\pi ft} dt \end{aligned}$$

If $u(t)$ has a bandwidth W such that $W > (\Delta f)_c$, the channel is frequency selective.

If

$$W \approxeq \frac{1}{T} \ll \frac{1}{T_m} \approxeq (\Delta f)_c$$

the channel id frequency-nonselective. In other words, all the all the frequency components of $U(f)$ undergo the same attenuation and phase shift in transmission through the channel.

$C(f;t)$ is approximately constant over W:

$$r(t) = C(0;t)u(t)$$

$C(0;t) = \alpha(t)e^{-j\phi(t)}$