E6712: Midterm Exam - Solutions

Given: March 24, 2004

1. Consider the following binary detection problem on a 3-dimensional vector channel. The two possible signals are

$$s_1 = (\sqrt{E}, \sqrt{E}, \sqrt{E}) \text{ and } s_2 = -s_1.$$

The received signal is given by

$$r = s_i + n, \quad i \in \{1, 2\},$$

where n is a 3-dimensional noise vector whose components are independent Gaussian random variables with zero mean and variances σ_1^2, σ_2^2, and σ_3^2. Assume that the prior probabilities of the two messages are $P(s_1)$ and $P(s_2)$.

(a) Derive an optimum receiver decision rule.

(b) Derive an expression for this receiver's error probability, and its Chernoff bound.

Solution

(a) The optimum receiver decision rule:

$$\frac{P(s_1|r)}{P(s_2|r)} \underset{2}{\overset{1}{\gtrless}} 1 \Leftrightarrow \log \frac{P(s_1|r)}{P(s_2|r)} \underset{2}{\overset{1}{\gtrless}} 0$$

Next,

$$\begin{aligned} \log\Big[\frac{P(s_1|r)}{P(s_2|r)}\Big] &= \log\Big[\frac{P(s_1)}{P(s_2)} \cdot \frac{P(r|s_1)}{P(r|s_2)}\Big] \\ &= \log \frac{P(s_1)}{P(s_2)} + \log \frac{P(r|s_1)}{P(r|s_2)} \end{aligned}$$

and

$$P(r|s_i) = \prod_{j=1}^{3} \frac{1}{\sqrt{2\pi}\sigma_j^2} \exp\Big(-\frac{(r_j - s_{ij})^2}{2\sigma_j^2}\Big),$$

where $s_{11} = s_{12} = s_{13} = \sqrt{E}$ and $s_{21} = s_{22} = s_{23} = -\sqrt{E}$. Therefore,

$$\begin{aligned} \log \frac{P(r|s_1)}{P(r|s_2)} &= \sum_{j=1}^{3} [-(r_j - \sqrt{E})^2 + (r_j + \sqrt{E})^2]/2\sigma_j^2 \\ &= \sum_{j=1}^{3} \frac{2\sqrt{E}}{\sigma_j^2} r_j \end{aligned}$$

consequently, the optimum receiver decision rule becomes:

$$\boxed{\log \frac{P(s_1)}{P(s_2)} + 2\sqrt{E} \sum_{j=1}^{3} \frac{r_j}{\sigma_j^2} \underset{2}{\overset{1}{\gtrless}} 0}$$

(b) $P_e = P(s_1)P(e|s_1) + P(s_2)P(e|s_2)$. Consider $P(e|s_1)$:

$$P(e|s_1) = P\Big(\log\frac{P(s_1)}{P(s_2)} + 2\sqrt{E}\sum_{j=1}^{3}\frac{r_j}{\sigma_j^2} \le 0|s_1\Big)$$

Now, $\gamma_1 \triangleq \log\frac{P(s_1)}{P(s_2)} + 2\sqrt{E}\sum_{j=1}^{3}\frac{r_j}{\sigma_j^2}|s_1$ is a Gaussian random variable with the mean and variance given by

$$E(\gamma_1) = \log\frac{P(s_1)}{P(s_2)} + 2E\sum_{j=1}^{3}\frac{1}{\sigma_j^2} \quad \text{and} \quad \sigma_{\gamma_1}^2 = 4E\sum_{j=1}^{3}\frac{1}{\sigma_j^2}$$

Consider $P(e|s_2)$:

$$P(e|s_2) = P\Big(\log\frac{P(s_1)}{P(s_2)} + 2\sqrt{E}\sum_{j=1}^{3}\frac{r_j}{\sigma_j^2} \ge 0|s_2\Big)$$

Now, $\gamma_2 \triangleq \log\frac{P(s_1)}{P(s_2)} - 2\sqrt{E}\sum_{j=1}^{3}\frac{r_j}{\sigma_j^2}|s_2$ is a Gaussian random variable with the mean and variance given by

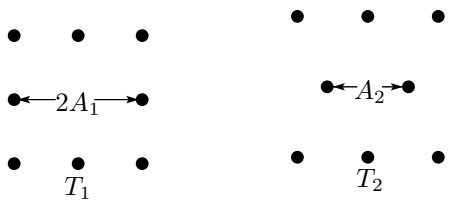
$$E(\gamma_2) = \log\frac{P(s_1)}{P(s_2)} - 2E\sum_{j=1}^{3}\frac{1}{\sigma_j^2} \quad \text{and} \quad \sigma_{\gamma_2}^2 = 4E\sum_{j=1}^{3}\frac{1}{\sigma_j^2}$$

Therefore,

$$\begin{aligned} P_e &= P(s_1)Q\left(\frac{E(\gamma_1)}{\sigma_{\gamma_1}}\right) + P(s_2)Q\left(\frac{E(\gamma_2)}{\sigma_{\gamma_2}}\right) \\ &= P(s_1)Q\left(\frac{2E\sum_{j=1}^{3}\frac{1}{\sigma_j^2} + \log\frac{P(s_1)}{P(s_2)}}{2\sqrt{E\sum_{j=1}^{3}\frac{1}{\sigma_j^2}}}\right) + P(s_2)Q\left(\frac{2E\sum_{j=1}^{3}\frac{1}{\sigma_j^2} - \log\frac{P(s_1)}{P(s_2)}}{2\sqrt{E\sum_{j=1}^{3}\frac{1}{\sigma_j^2}}}\right) \end{aligned}$$

(c) To get the Chernoff bound on P_e, use $Q(x) \le \exp(-x^2/2)$ in the equations above.

2. Consider the two 8-point QAM signal constellations shown in the figure.



(a) Suppose that $A_1 = A_2 = A$, *i.e.,* that the constellations have the same minimum distance. Determine the average transmitted power for each constellation assuming that the signal points are equally probable.

(b) Suppose that the constellation # 1 is in use, $A_1 = A$, and that the point T_1 has been transmitted. Derive the union bound to the probability of making an error under these assumptions.

(c) Suppose that the constellation # 2 is in use, $A_2 = A$, and that the point T_2 has been transmitted. Derive the union bound to the probability of making an error under these assumptions.

(d) Suppose that $A_1 = A$. Determine A_2 so that the two constellations have the same power.

(e) Some of the most common concerns a physical layer designer may have are the transmission power, error probability performance, and complexity of the receiver. In the light of each of these concerns, which one of the above two 8-point QAM signal constellations would you choose?

Solution

(a) The average transmitted power of constellation # 1 is

$$P_1 = \frac{1}{8}(4 \cdot A^2 + 4 \cdot 2A^2)\frac{3}{2}A^2$$

The average transmitted power of constellation # 2 is

$$P_2 = \frac{1}{8}\Big[2 \cdot \Big(\frac{A}{2}\Big)^2 + 2 \cdot \Big(\frac{A\sqrt{3}}{2}\Big)^2 + 4 \cdot \Big(\Big(\frac{A\sqrt{3}}{2}\Big)^2 + A^2\Big)\Big] = \frac{9}{8}A^2$$

(b) $2 \cdot Q\left(\sqrt{\frac{A^2}{2\mathcal{N}_0}}\right) + 2 \cdot Q\left(\sqrt{\frac{2A^2}{2\mathcal{N}_0}}\right) + 1 \cdot Q\left(\sqrt{\frac{4A^2}{2\mathcal{N}_0}}\right) + 2 \cdot Q\left(\sqrt{\frac{5A^2}{2\mathcal{N}_0}}\right) < 7 \cdot Q\left(\sqrt{\frac{A^2}{2\mathcal{N}_0}}\right)$

(c) $4 \cdot Q\left(\sqrt{\frac{A^2}{2\mathcal{N}_0}}\right) + 1 \cdot Q\left(\sqrt{\frac{3A^2}{2\mathcal{N}_0}}\right) + 2 \cdot Q\left(\sqrt{\frac{4A^2}{2\mathcal{N}_0}}\right) < 7 \cdot Q\left(\sqrt{\frac{A^2}{2\mathcal{N}_0}}\right)$

(d)

$$\frac{3}{2}A^2 = \frac{9}{8}A_2^2 \quad \Rightarrow \quad A_2 = \frac{2}{\sqrt{3}}A$$

(e) Complexity of decoding is similar. The error probability performance should be determined by simulation. When the constellations have the same power, constellation # 2 has larger minimum distance but it also have a larger number of nearest neighbors. Which neighbors matter depends on the operating SNR.

3. Consider the following partial response channel:

$$v_k = I_k - I_{k-1} + \eta_k,$$

where $\{\eta_k\}$ is discrete AWGN with zero mean and spectral density $\mathcal{N}_0/2$. The user data $D_k \in \{0, 1\}$ is binary, and the channel can take only bipolar signal values $I_k \in \{-1, 1\}$.

(a) Suppose that user data is directly mapped into the channel sequence as

$$I_k = 2D_k - 1$$

Suppose that the user sequence $\{D_k\}$ is the all-zero sequence, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is the most likely incorrect sequence $\{\widehat{D_k}\}$?

(b) Suppose that user data is mapped into the channel sequence in two steps as follows:

i. $P_k = D_k + P_{k-1} \mod 2$

ii. $I_k = 2P_k - 1$

Suppose again that the user sequence $\{D_k\}$ is the all-zero sequence, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is now the most likely incorrect sequence $\{\widehat{D_k}\}$?

(c) (bonus) Can we use the Tomlinson-Harashima precoding for this channel?

Solution

(a) The minimum (merged) distance on the bipolar channel with transfer function $(1-D)$ is given by

$$d^2_{<>} = \min_{\varepsilon(D)\neq 0} \|(1-D)\varepsilon(D)\|^2,$$

where the squared norm of a polynomial is defined as the sum of its squared coefficients, and $\varepsilon(D)$ is the polynomial corresponding to a finite length normalized error sequence:

$$\boldsymbol{\varepsilon} = \{\varepsilon_{n_0}, \varepsilon_1, \ldots, \varepsilon_{n_l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1, 0, 1\},\ \varepsilon_{n_0} = 1, \varepsilon_{n_l-1} \neq 0$$

We have

$$\begin{aligned} d^2_{<>} &= \min_{\varepsilon(D)\neq 0} \|(1-D)\sum_{i=0}^{l-1} \varepsilon_{n_i} D^i\|^2 \\ &= \min_{\varepsilon(D)\neq 0} \|\varepsilon_{n_0} + \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}}) D^i + \varepsilon_{n_{l-1}} D^l\|^2 \\ &= \min_{\varepsilon(D)\neq 0} (1 + \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}})^2 + 1) \\ &= 2 + \min_{\varepsilon(D)\neq 0} \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}})^2 \geq 2 \end{aligned}$$

where the equality holds iff

$$\varepsilon(D) = \sum_{i=0}^{l-1} D^i.$$

If the user sequence $\{D_k\}$ is the all-zero sequence, then the most likely incorrect sequence $\{\widehat{D_k}\}$ will have 1s starting at time n_0 and ending at time n_{l-1}.

(b) If the user sequence $\{D_k\}$ is the all-zero sequence, sequence $\{P_k\}$ is also an all-zero sequence. From (a), we know that then the most likely incorrect sequence $\{\widehat{P_k}\}$ will have 1s starting at time n_0 and ending at time n_{l-1}. Since $\widehat{D_k} = \widehat{P_k} + \widehat{D_{k-1}} \mod 2$, the decoded sequence $\{\widehat{D_k}\}$ will have a 1 at time n_0 and a 1 at time n_l. The rest of $\{\widehat{D_k}\}$ is decoded correctly as 0.

(c) There are two problems with Tomlinson-Harashima precoding on this channel:

i. The transmitted signal

$$x_k = I_k + x_{k-1} \mod 4$$

will take values in $\{-2, 1, 0, 1, 2\}$ but the channel can take only bipolar signal values in $\{-1, 1\}$.

ii. The average power doubles.