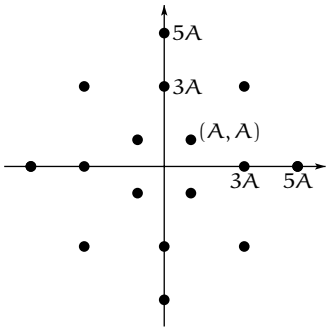
Communication Theory (E6712) May 10, 2005
Columbia University, Spring 2005

Final Exam

1. Compare the constellation shown in the figure below to a standard 16-QAM constellation and determine which one performs better in an AWGN channel and by how many dBs, for the same average energy per bit.



Solution

The average energy of the above constellation is

$$E = \frac{27}{2}A^2,$$

and its minimum squared distance is

$$d^2 = 4A^2.$$

Now, consider the 16-QAM with the nearest neighbor distance B. Then the average energy of the 16-QAM constellation is

$$E_{QAM} = \frac{5}{2}A^2,$$

and its minimum squared distance is

$$d^2_{QAM} = B^2.$$

The requirement $E = E_{QAM}$ implies

$$A^2 = \frac{5}{27}B^2$$

and, consequently,

$$d^2 = \frac{20}{27}\, d^2_{QAM}.$$

Therefore, the 16-QAM is better than the above constellation by

$$10 \log_{10} \frac{27}{20} = 1.30\,\text{dB}.$$

2. Consider the following binary signaling scenario: The two possible signals are

$$s_1(t) = \begin{cases} \sqrt{\frac{E}{T}}, & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad s_2(t) = -s_1(t).$$

The receiver is designed to be optimal for the AWGN memoryless channel with zero mean. However, in addition to the additive noise, there is a multipath, namely, the channel's response is $h(t) = \delta(t) + \alpha\delta(t-\tau)$, where α is a positive number and $0 \le \tau \le T$ is a delay. Derive an expression for the receiver's error probability as a function of α, τ, and the noise spectral density $\mathcal{N}_0/2$. Assume that the symbol is surrounded by other randomly and independently chosen transmitted symbols.

Solution

The receiver implements

$$l = \int_0^T r(t)s_1(t)dt \underset{-1}{\overset{1}{\gtrless}} 0,$$

where the received signal on the interval $[0, T]$ is of the form

$$r(t) = s(t) + \alpha s(t-\tau) + d\alpha s(t+T-\tau) + n(t),$$

and d takes values in $\{1, -1\}$ with equal probability. If $s_1(t)$ is transmitted, we have

$$\begin{aligned} l =& E + \alpha E \int_0^T s(t-\tau)s_1(t)dt + d\alpha E \int_0^T s(T+t-\tau)s_1(t)dt + \int_0^T n(t)s_1(t)dt \\ =& E\Big[1 + \alpha\big(1 - \frac{\tau}{T}\big) + d\alpha\frac{\tau}{T}\Big] + \eta, \end{aligned}$$

where η is a Gaussian random variable with zero mean and variance $E\mathcal{N}_0/2$. Consequently, l is with probability $1/2$ Gaussian with mean $E(1+\alpha)$ and variance $E\mathcal{N}_0/2$, and with probability $1/2$ Gaussian with mean $E(1+\alpha-2\alpha\tau/T)$ and variance $E\mathcal{N}_0/2$. Therefore, the receiver's error-probability is

$$P_e = \frac{1}{2}Q\left[\sqrt{\frac{2E}{\mathcal{N}_0}}(1+\alpha)\right] + \frac{1}{2}Q\left[\sqrt{\frac{2E}{\mathcal{N}_0}}(1+\alpha-2\alpha\tau/T)\right].$$

3. Consider the following vector channel: The transmitter modulates its equiprobable messages as two-dimensional vectors (s_1, s_2). The received vector (r_1, r_2) is given by

$$\begin{aligned} r_1 &= \sqrt{E} s_1 + n_1 \\ r_2 &= r_1 + \sqrt{E} s_2 + n_2 \end{aligned}$$

where n_1 and n_2 are independent zero-mean Gaussian random variables with variance $\mathcal{N}_0/2$. Derive an ML decision rule and an expression for its error probability for the two following cases:

(a) (s_1, s_2) take values in the set $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$,

(b) (s_1, s_2) take values in the set $\{(1,1),(1,-1)\}$.

Solution

The ML detector maximizes the posterior probability

$$\begin{aligned} p(r_1, r_2 | s_1, s_2) &= p(r_2 | r_1, s_1, s_2) \cdot p(r_1 | s_1, s_2) \\ &= \frac{1}{\sqrt{\pi \mathcal{N}_0}} \exp\left[-(r_2 - r_1 - \sqrt{E} s_2)^2 / \mathcal{N}_0\right] \cdot \frac{1}{\sqrt{\pi \mathcal{N}_0}} \exp\left[-(r_1 - \sqrt{E} s_1)^2 / \mathcal{N}_0\right] \end{aligned}$$

or, equivalently, the log-likelihood

$$-(r_2 - r_1 - \sqrt{E} s_2)^2 - (r_1 - \sqrt{E} s_1)^2,$$

which further reduces (by expanding the quadratics and dropping unnecessary terms and factors) to the maximization of

$$(r_2 - r_1) s_2 + r_1 s_1.$$

(a) The optimal decision rule for this constellation is

$$\boxed{s_1 \colon r_1 \underset{-1}{\overset{1}{\gtrless}} 0 \quad \text{and} \quad s_2 \colon r_2 - r_1 \underset{-1}{\overset{1}{\gtrless}} 0}$$

The detector makes and error in (s_1, s_2) if it detects incorrectly either s_1 or s_2 or both. The probability of error in detecting s_i is $Q(\sqrt{2E/\mathcal{N}_0}$. Note that the decision variables r_1 and $r_2 - r_1$ are independent. Therefore, the probability that the ML detector makes an error is the same as for QPSK:

$$P_e = 1 - \left[1 - Q\left(\sqrt{\frac{2E}{\mathcal{N}_0}}\right)\right]^2$$

(b) Here only s_2 needs to be detected:

$$\boxed{s_2 \colon r_2 \underset{-1}{\overset{1}{\gtrless}} r_1}$$

The probability of error is

$$P_e = Q\left(\sqrt{\frac{2E}{\mathcal{N}_0}}\right)$$