Lecture 11, E6712

Binary Signaling Over Frequency-Nonselective Channels – *Review*

On a frequency-nonselective channel, the received signal is given by

$$r(t) = \alpha(t)e^{-j\phi(t)}g(t) + z(t)$$

Assume slow fading $\therefore$ $\alpha(t)$ is constant and $\phi(t)$ is constant and can be estimated $\therefore$ *coherent* detection is possible for

$$r(t) = \alpha e^{-j\phi}g(t) + z(t)$$

Do we need to know α?

Error probability performance for BPSK:

$$P_e(\alpha) = Q\left(\sqrt{\frac{2\alpha^2 E_b}{\mathcal{N}_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{\alpha^2 E_b}{\mathcal{N}_0}}\right)$$

$\therefore$

$$P_e(\gamma_b) = Q(\sqrt{2\gamma_b}) \quad \text{where} \quad \gamma_b = \frac{\alpha^2 E_b}{\mathcal{N}_0}$$

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b$$

The distribution of α is Rayleigh; the distribution of α^2 and thus γ_b is chi-square with two degrees of freedom:

$$p(\gamma_b) = \frac{1}{\overline{\gamma}_b}e^{-\gamma_b/\overline{\gamma}_b},$$

where

$$\overline{\gamma}_b = \frac{E_b}{\mathcal{N}_0}E(\alpha^2)$$

We have

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}_b}{1+\overline{\gamma}_b}}\right)$$

On channels for which the fading is too rapid for phase estimation, we can use DPSK which requires phase stability over only two consecutive signaling intervals.

Error probability performance for DPSK:

$$P_e(\gamma_b) = \frac{1}{2}e^{-\gamma_b}$$

$$P_e = \int P_e(\gamma_b)p(\gamma_b)d\gamma_b$$

$\therefore$

$$P_e = \frac{1}{2(1+\overline{\gamma}_b)}$$

Other fading models.

Diversity Techniques for Fading Multipath Channels

Errors occur when the channel attenuation is large *i.e.*, when the channel is in deep fade.

Suppose that p is the probability that the signal fades below some critical value.

Solution: Transmit L copies of the same information over independently fading channels.

The probability that all signal components will fade simultaneously is p^L

Frequency diversity: The same signal is transmitted simultaneously over L different carriers with separation larger or equal to the coherence bandwidth $(\Delta F)_c$ of the channel.

Time diversity: The same signal is transmitted in L different time slots with separation larger or equal to the coherence time $(\Delta t)_c$ of the channel.

Space diversity: Use multiple antennas.

Optimum Receiver – Binary Signaling

The received signal

$$r_k(t) = \alpha_k e^{-j\phi_k} s_{km}(t) + z_k(t), \quad k = 1, \ldots, L, \quad m = 1, 2,$$

Assume that the the two messages are equiprobable, and that the L copies of the signal receive *independent* disturbance.

We have

$$\begin{aligned} p(r(t)|\alpha, \phi, m) &= \prod_{k=1}^{L} p(r_k(t)|\alpha_k, \phi_k, m) \\ &\propto \prod_{k=1}^{L} \exp\Big\{-\frac{1}{\mathcal{N}_0}\int |r_k(t) - \alpha_k e^{-j\phi_k} s_{km}(t)|^2 dt\Big\} \\ &= \exp\Big\{-\frac{1}{\mathcal{N}_0}\sum_{k=1}^{L}\int |r_k(t) - \alpha_k e^{-j\phi_k} s_{km}(t)|^2 dt\Big\} \\ &= \exp\Big\{-\frac{2}{\mathcal{N}_0}\Re\Big\{\sum_{k=1}^{L}\alpha_k e^{j\phi_k}\underbrace{\int r_k(t)s_{km}^*(t)dt}_{l_{km}}\Big\} + \frac{1}{\mathcal{N}_0}\sum_{k=1}^{L}\alpha_k^2\int |s_{km}(t)|^2 dt\Big\} \end{aligned}$$

For BPSK we have

$$s_k(t) \triangleq s_{k1}(t) = -s_{k2}(t) \quad \Longrightarrow \quad l_k \triangleq l_{k1} = -l_{k2}$$

and

$$\int |s_{km}(t)|^2 dt = E_b.$$

the optimum decision rule becomes (maximal ratio combiner)

$$\Re\Big\{\sum_{k=1}^{L}\alpha_k e^{j\phi_k} l_k\Big\} \underset{2}{\overset{1}{\gtrless}} 0$$

where

$$\begin{aligned} l_k &= \int r_k(t) s_k^*(t) dt \\ &= \int (\alpha_k e^{-j\phi_k} s_k(t) + z(t)) s_k^*(t) dt \\ &= \alpha_k e^{-j\phi_k} E_b + \underbrace{\int z(t) s_k^*(t) dt}_{N_k} \end{aligned}$$

To analyze the receiver performance, consider the random variable U:

$$\begin{aligned} U &\triangleq \Re\Big\{\sum_{k=1}^{L} \alpha_k e^{j\phi_k} l_k\Big\} \\ &= E_b \sum_{k=1}^{L} \alpha_k^2 + \sum_{k=1}^{L} \alpha_k \underbrace{\Re\{N_k\}}_{N_{kr}} \end{aligned}$$

U is Gaussian given $\{\alpha_k\}$ and $\{\phi_k\}$:

$$\mu_U = E\{U|\{\alpha_k\},\{\phi_k\}\} = E_b \sum_{k=1}^{L} \alpha_k^2 \qquad \sigma_U^2 = \sigma^2\{U|\{\alpha_k\},\{\phi_k\}\} = \frac{\mathcal{N}_0}{2} E_b \sum_{k=1}^{L} \alpha_k^2$$

Therefore

$$P(e|\{\alpha_k\},\{\phi_k\}) = Q(\mu_U/\sigma_U) = Q(\sqrt{2\gamma_L}), \qquad \gamma_L = \frac{E_b}{\mathcal{N}_0} \sum_{k=1}^{L} \alpha_k^2$$

To find $P(e)$, we must take an expectation with respect to $\{\alpha_k\}$ or γ_L. Note that, since the receiver requires knowledge of both $\{\alpha_k\}$ and $\{\phi_k\}$, its actual error probability will be higher than the computed one.

The characteristic function of γ_L:

$$\begin{aligned} \Psi_{\gamma_L}(j\omega) &= E\left[e^{j\omega\gamma_L}\right] = E\left[e^{j\omega \frac{E_b}{\mathcal{N}_0} \sum_{k=1}^{L} \alpha_k^2}\right] \\ &= \Pi_{k=1}^{L} E\left[e^{j\omega \frac{E_b}{\mathcal{N}_0} \alpha_k^2}\right]^2 \\ &= \frac{1}{(1 - j\omega\bar{\gamma}_L)^L} \qquad \bar{\gamma}_L = \frac{E_b}{\mathcal{N}_0} \sum_{k=1}^{L} E(\alpha_k^2). \end{aligned}$$

This is the characteristic function of a chi-square-distributed random variable with $2L$ degrees of freedom.

Therefore

$$p(\gamma_L) = \frac{1}{(L-1)!\bar{\gamma}_L} \gamma_L^{L-1} e^{-\gamma_L/\bar{\gamma}_L}$$

The probability of error is given by

$$\begin{aligned} P_e &= \int P_e(\gamma_L) p(\gamma_L) d\gamma_L \\ &= [0.5(1-\mu)]^L \sum \binom{L-1+k}{k} [0.5(1+\mu)]^k, \qquad \mu \triangleq \sqrt{\frac{\bar{\gamma}_L}{1+\bar{\gamma}_L}} \end{aligned}$$

When $\overline{\gamma}_L \gg 1$, we take

$$0.5(1+\mu) \approxeq 1, \quad \text{and} \quad 0.5(1-\mu) \approxeq \frac{1}{\overline{\gamma}_L}$$

and thus

$$P_e \approxeq \left(\frac{1}{\overline{\gamma}_L}\right)^L \binom{2L-1}{L}$$

L is known as the *diversity gain* of the system.

Signaling over a Frequency Selective, Slowly Fading Channel

When bandwidth $W \gg (\Delta f)_c$ is available to the user, in a number of FDM subchannels having a separation of at least $(\Delta f)_c$ to achieve frequency diversity.

Alternatively, we can employ a wideband signal covering W. The channel is still slowly fading since $T \ll (\Delta t)_c$.

The equivalent lowpass signal $s(t)$ is bandlimited to $|f| < .5W$.

By the sampling theorem, we have

$$s(t) = \sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) \frac{\sin[\pi W(t-n/W)]}{\pi W(t-n/W)}$$

The Fourier transform of $s(t)$ is

$$S(f) = \begin{cases} \frac{1}{W}\sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right) e^{-j2\pi fn/W}, & |f| \le \frac{1}{2}W \\ 0, & |f| > \frac{1}{2}W \end{cases}$$

The noiseless received signal is

$$\begin{aligned} r(t) =& \int C(f;t)S(f)e^{j2\pi ft}df \\ =& \frac{1}{W}\sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right)\int C(f;t)e^{j2\pi f(t-n/W)}df \\ =& \frac{1}{W}\sum_{n=-\infty}^{\infty} s\left(\frac{n}{W}\right)c(t-n/W;t) \\ =& \frac{1}{W}\sum_{n=-\infty}^{\infty} s\left(t-\frac{n}{W}\right)c(n/W;t) \end{aligned}$$

Define time-variable channel coefficients as

$$c_n(t) = \frac{1}{W}c(n/W;t)$$

Therefore

$$r(t) = \sum_{n=-\infty}^{\infty} c_n(t)s\left(t-\frac{n}{W}\right)$$

We deduce that the time-variant impulse response for the channel is

$$c(\tau;t) = \sum_{n=-\infty}^{\infty} c_n(t)\delta\left(\tau - \frac{n}{W}\right)$$

and the corresponding time-variant transfer function is

$$C(f;t) = \sum_{n=-\infty}^{\infty} c_n(t)e^{-j2\pi fn/W}$$

There are $L\lfloor T_m W\rfloor + 1$ significant terms, and thus

$$r(t) = \sum_{n=1}^{L} c_n(t)s\left(t - \frac{n}{W}\right)$$

This is a delay line model of frequency-selective channels.

The RAKE Demodulator

We consider the problem of digital signaling over a frequency-selective channel modelled by a tapped delay line with statistically independent time-variant tap weights $c_n(t)$. Consider a binary system:

$$r(t) = \underbrace{\sum_{n=1}^{L} c_n(t)s_i\left(t - \frac{n}{W}\right)}_{v_i(t)} + z(t) \qquad i = 1,2$$

The decision variables are

$$\begin{aligned} U_i =& \Re\left[\int r(t)v_i^*(t)dt\right] \\ =& \Re\left[\sum_{n=1}^{L}\int r(t)c_n^*(t)s_i^*i\left(t - \frac{n}{W}\right)dt\right] \qquad i = 1,2 \end{aligned}$$

Note that the receiver attempts to collect energy from all the received paths within the span of the delay line. The name comes from the similarity to a garden rake.

To evaluate the performance, we assume that the fading is sufficiently slaw to allow us to estimate $c_k(t)$ which are constant during one symbol interval. Because the wide-band signals $s(t)$ are usually generated from pseudo-random sequences, we have

$$\int_0^t s_i(t - n/W)s_j^*(t - k/W)dt \approx 0 \text{ for } k \neq n, \qquad i = 1,2$$

Suppose that the transmitted signal is $s_1(t)$. We have

$$U_i = \Re\left[\sum_{n=1}^{L}|c_n|^2\int_0^t s_1(t - n/W)s_i^*(t - n/W)dt + \sum_{n=1}^{L}c_n^*\int_0^t z(t)s_i^*(t - n/W)dt\right] \qquad i = 1,2$$

In the case of antipodal signaling, we need only one decision variable:

$$U \triangleq \Re\left[E_b\sum_{k=1}^{L}|c_n|^2 + \sum_{k=1}^{L}c_n^*N_k\right]$$

Note the similarity with the maximal ratio combiner.