Lecture 2, E6712

Example 1 Last time we looked into the following problem of binary detection:

$$r = s_i + n, \quad i \in \{1, 2\},$$

where n is an N-dimensional noise vector whose components are independent identically distributed Gaussian random variables with zero mean and variance σ^2, and the two possible signals are the following N-dimensional vectors:

$$s_1 = (m, m, \ldots, m) \text{ and } s_2 = (0, 0, \ldots, 0).$$

We derived the optimum decision rule to be

$$\boxed{\sum_j r_j \underset{m_0}{\overset{m_1}{\gtrless}} \frac{mN}{2}}$$

For the special case $N = 1$, we had

$$r \underset{m_0}{\overset{m_1}{\gtrless}} \frac{m}{2}$$

Note that the receiver does not have to know the variance of the noise.

We next derive an expression for the probability of error of this receiver for $N = 1$:

$$P_e = \frac{1}{2}P(e|m_1) + \frac{1}{2}P(e|m_2) = P(e|m_1) = P(e|m_2).$$

Now, $P(e|m_2) = P(r > m/2|m_2)$, and given m_2, we know that r is a Gaussian zero-mean random variable with variance σ^2. We have

$$\begin{aligned} P(e|m_2) &= P(r > m/2|m_2) \\ &= \int_{m/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx \qquad \text{set } y = x/\sigma \\ &= \int_{m/(2\sigma)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= Q\left(\frac{m}{2\sigma}\right), \end{aligned}$$

where

$$Q(x) = \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{y^2}{2}} dy$$

Similarly, $P(e|m_1) = P(r < m/2|m_1)$, and given m_1, we know that r is a Gaussian random variable

with mean m and variance σ^2. We have

$$\begin{aligned}
P(e|m_1) =& P(r < m/2|m_1) \\
=& \int_{-\infty}^{m/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad \text{set } y = -(x-m)/\sigma \\
=& -\int_{\infty}^{m/(2\sigma)} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
=& Q\left(\frac{m}{2\sigma}\right)
\end{aligned}$$

Most of the time we use bounds on the probability of error.

Chernoff Bound

Let n be a random variable with probability distribution p_n. We need to bound the tail probability

$$Pr[n \geq \epsilon] = \int_{\epsilon}^{\infty} p_n(x)dx.$$

We define

$$M_n(s) = E(e^{sx}) = \int_{-\infty}^{\infty} e^{sx} p_n(x)dx.$$

For $s \in \mathbb{R}$ and $s \geq 0$, we have

$$\begin{aligned}
M_n(s) =& \int_{-\infty}^{\infty} e^{sx} p_n(x)dx \\
\geq& \int_{\epsilon}^{\infty} e^{sx} p_n(x)dx \\
\geq& \int_{\epsilon}^{\infty} e^{s\epsilon} p_n(x)dx \\
=& e^{s\epsilon} \int_{\epsilon}^{\infty} p_n(x)dx \\
=& e^{s\epsilon} Pr[n \geq \epsilon]
\end{aligned}$$

We have the following bound:

$$\boxed{Pr[n \geq \epsilon] \leq e^{-s\epsilon} M_n(s)}.$$

Now, $e^{-s\epsilon}M_n(s)$ should be minimized with respect to $s > 0$ to get the tightest bound of this form.

$M_n(s)$ is usually easy to compute. Consider $n \sim \mathcal{N}(0, \sigma^2)$. We have

$$\begin{aligned}
M_n(s) =& \int_{-\infty}^{\infty} e^{sx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \\
=& e^{\frac{s}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-s\sigma^2)^2}{2\sigma^2}} dx \\
=& e^{\frac{s}{2\sigma^2}}
\end{aligned}$$

and thus

$$e^{-s\epsilon}M_n(s) = \exp(-s\epsilon + \frac{1}{2}\sigma^2 s^2)$$

The minimum of LHS is achieved for $s_{min} = \epsilon/\sigma^2$, giving

$$Q(\epsilon) = Pr[n \geq \epsilon] \leq e^{-s\epsilon}M_n(s) \leq \exp\Big(-\frac{\epsilon^2}{2\sigma^2}\Big).$$

Therefore, we have

$$Q\left(\frac{m}{2\sigma}\right) \leq \exp\left(-\frac{m^2}{8\sigma^2}\right)$$

When dealing with discrete random variables, we define

$$M_n(s) = E(e^{sx}) = \sum_i e^{si} p_n(i)$$

For the Binomially distributed random variable X, we have

$$\begin{aligned} E[e^{sX}] &= \sum_{i=1}^{N} \binom{N}{i} e^{si} p^i (1-p)^{N-i} \\ &= [e^s p + (1-p)]^N, \end{aligned}$$

and thus

$$P(X \geq \varepsilon) \leq e^{-\varepsilon s} [e^s p + (1-p)]^N.$$

Let the right hand side equation be $g(s)$. Then, $g(s)$ can be minimized over s by differentiation:

$$\begin{aligned} \frac{dg(s)}{ds} =& -\varepsilon e^{-\varepsilon s} [e^s p + (1-p)]^N + N e^{-\varepsilon s} [e^s p + (1-p)]^{(N-1)} e^s p \\ =& e^{-\varepsilon s} [e^s p + (1-p)]^{N-1} \Big[N e^s p - \varepsilon (e^s p + (1-p)) \Big] \end{aligned} \tag{1}$$

Now,

$$\begin{aligned} \frac{dg(s)}{ds} = 0 \Leftrightarrow \\ -\varepsilon [e^s p + (1-p)] + N e^s p = 0 \Leftrightarrow \\ (N-\varepsilon) p e^s = \varepsilon (1-p) \Leftrightarrow \\ e^s = \boxed{z_{min} \triangleq \frac{\varepsilon}{N-\varepsilon} \frac{1-p}{p}} \end{aligned}$$

From (1), we see that

$$\frac{dg(s)}{ds} = \begin{cases} < 0, & e^s < z_{min}; \\ = 0, & e^s = z_{min}; \\ > 0, & e^s > z_{min}. \end{cases}$$

Therefore $g(s)$ has a minimum at $e^s = z_{min}$. The Chernoff bound was derived under the assumption that $s > 0$. Therefore, we should check if $z_{min} > 1$. This definitely holds for our case of interest: $\varepsilon > N/2$ and $p < 1/2$.

Finally, the Chernoff bound for a Binomially distributed random variable X is

$$P(X \geq \varepsilon) \leq \Big[\frac{1-p}{p} \frac{\varepsilon}{N-\varepsilon} \Big]^{-\varepsilon} \Big[(1-p) \frac{N}{N-\varepsilon} \Big]^N$$

Vector Channel – *General Case*

We now derive the optimal decision rule and the error probability for the vector channel

$$\mathbf{r} = \sqrt{E_m} \mathbf{s}_m + \mathbf{n},$$

where

$\mathbf{r} = (r_1, r_2, \ldots, r_N)$

$\mathbf{n} = (n_1, n_2, \ldots, n_N)$ are i.i.d. $\sim (0, \mathcal{N}_0/2)$

$\mathbf{s_m} = (s_{m1}, s_{m2}, \ldots, s_{mN})$ are unit energy and equiprobable.

We have shown that under this conditions, the optimal detector finds

$$\arg\max_m p(\mathbf{r}|\mathbf{s}_m).$$

We have

$$\begin{aligned}
p(\mathbf{r}|\mathbf{s_m}) &= \prod_{j=1}^{N} p(r_j|s_{mj}) \\
&= \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big[-\frac{(r_j - \sqrt{E_m}s_{mj})^2}{\mathcal{N}_0}\Big] \\
&\propto \exp\Big[-\sum_{j=1}^{N}(r_j - \sqrt{E_m}s_{mj})^2/\mathcal{N}_0\Big] \\
&= \exp\Big[-\sum_{j=1}^{N} r_j^2\Big] \cdot \exp\Big[-E_m + 2\sum_{j=1}^{N}\sqrt{E_m}s_{mj}r_j\Big]
\end{aligned}$$

and therefore, the optimal detector can equivalently find

$$\arg\max_m \Big[-E_m + 2\sum_{j=1}^{N}\sqrt{E_m}s_{mj}r_j\Big].$$