Lecture 3, E6712

Vector Channel – *Probability of the Receiver Error*

Last time, we derived the optimal decision rule for the following vector channel:

$$\mathbf{r} = \sqrt{E_m}\mathbf{s}_m + \mathbf{n},$$

where

$$\begin{aligned}
&\mathbf{r} = (r_1, r_2, \ldots, r_N)\\
&\mathbf{n} = (n_1, n_2, \ldots, n_N) \text{ are i.i.d. } \sim (0, \mathcal{N}_0/2)\\
&\mathbf{s_m} = (s_{m1}, s_{m2}, \ldots, s_{mN}) \text{ are unit energy and equiprobable.}
\end{aligned}$$

The optimal detector finds

$$\arg\min_m \Big[\sum_{j=1}^{N} (r_j - \sqrt{E_m}s_{mj})^2\Big],$$

or, equivalently,

$$\arg\max_m \Big[-E_m + 2\sum_{j=1}^{N} \sqrt{E_m}s_{mj}r_j\Big].$$

With what probability will this detector be making an error? Suppose we are told that that the transmitted signal is either $\mathbf{s}_m$ or $\mathbf{s}_k$. We will be making an error if we find that

$$-E_m + \sum_{j=1}^{N} \sqrt{E_m}s_{mj}r_j > -E_k + \sum_{j=1}^{N} \sqrt{E_k}s_{kj}r_j$$

given that $\mathbf{s}_k$ was transmitted. Probability of this happening is

$$P\Big\{-E_m + 2\sum_{j=1}^{N} \sqrt{E_m}s_{mj}r_j + E_k - 2\sum_{j=1}^{N} \sqrt{E_k}s_{kj}r_j > 0\Big\}$$

or equivalently

$$P\Big\{\underbrace{\sum_{j=1}^{N} (\sqrt{E_m}s_{mj} - \sqrt{E_k}s_{kj})r_j}_{X} > E_m - E_k\Big\}$$

Now, given that $\mathbf{s}_k$ was transmitted, X is a Gaussian random variable with mean

$$\mu = E[X|\mathbf{s}_k] = 2\sum_{j=1}^{N} (\sqrt{E_m}s_{mj} - \sqrt{E_k}s_{kj})\sqrt{E_k}s_{kj}$$

and variance

$$\sigma^2 = E[(X-\mu)^2|\mathbf{s}_k] = 4\sum_{j=1}^{N} (\sqrt{E_m}s_{mj} - \sqrt{E_k}s_{kj})^2 \cdot \mathcal{N}_0/2.$$

For a Gaussian random variable with mean μ and variance σ^2, we have

$$\begin{aligned} P(X > l) &= \int_l^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad \text{set } y = (x-\mu)/\sigma \\ &= -\int_{(l-\mu)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ &= Q\left(\frac{l-\mu}{\sigma}\right). \end{aligned}$$

We have

$$E_m - E_k - 2\sum_{j=1}^{N} (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj}) \sqrt{E_k} s_{kj} = \sum_{j=1}^{N} (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj})^2$$

and therefore

$$P\Big\{ \sum_{j=1}^{N} (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj}) r_j > E_m - E_k \Big\} = Q\left(\sqrt{\frac{\sum_j (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj})^2}{2\mathcal{N}_0}} \right)$$

We have a lower (two message case or genie) bound on the probability of error:

$$P_e > P(s_k \to s_m) = Q\left(\sqrt{\frac{\sum_j (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj})^2}{2\mathcal{N}_0}} \right)$$

The Union Bound on the Probability of Error

Let s_k be transmitted

$$\begin{aligned} P_e &= P\Big\{ \bigcup_{m \neq k} (s_k \to s_m) \Big\} \\ &\leq \sum_{i \neq m} P(s_k \to s_m) \end{aligned}$$

We define the minimum distance as

$$d_{min}^2 = \min_{m,k} \sum_j (\sqrt{E_m} s_{mj} - \sqrt{E_k} s_{kj})^2$$

We further have

$$P_e \leq \sum_{i \neq m} P(s_k \to s_j) \leq (M-1) Q\left(\sqrt{\frac{d_{min}^2}{2\mathcal{N}_0}} \right)$$

Examples of Signal Constellations - *Which should we choose?*

antipodal signaling (BPSK) 1 bit per symbol: $E_1 = E_2 = E_s$, $s_1 = (-1, 0)$, $s_2 = (1, 0)$.
we have $E_b = E_s$, $d_{min}^2 = 4E_s$ and

$$P(s_1 \to s_2) = Q\left(\sqrt{\frac{2E_b}{\mathcal{N}_0}} \right).$$

binary orthogonal signaling: $E_1 = E_2 = E_s$, $s_1 = (1, 0)$, $s_2 = (0, 1)$
we have $E_b = E_s$, $d_{min}^2 = 2E_s$ and

$$P(s_1 \to s_2) = Q\left(\sqrt{\frac{E_b}{\mathcal{N}_0}} \right).$$

QPSK signaling 2 bits per symbol: $E_m = E_s$, $s_1 = (1,1)$, $s_2 = (-1,1)$, $s_3 = (-1,-1)$, $s_4 = (1,-1)$,
we have $E_b = E_s/2$, $d^2_{min} = 4E_s$,

$$P(s_m \to s_i) = Q\left(\sqrt{\frac{4E_b}{\mathcal{N}_0}}\right) \quad \text{and} \quad P_e \le 2 \cdot Q\left(\sqrt{\frac{4E_b}{\mathcal{N}_0}}\right) + Q\left(\sqrt{\frac{8E_b}{\mathcal{N}_0}}\right).$$

orthogonal signaling: $E_m = E_s$, $s_m = e_m$, note that M=N,
we have $E_b = E_s/\log M$, $d^2_{min} = 2E_s$,

$$P(s_1 \to s_2) = Q\left(\sqrt{\frac{E_s \cdot \log M}{\mathcal{N}_0}}\right) \quad \text{and} \quad P_e \le (M-1) \cdot Q\left(\sqrt{\frac{E_s \cdot \log M}{\mathcal{N}_0}}\right).$$

simplex signals:
we start with a set of M orthogonal signals $\mathbf{s}_m$, and obtain the corresponding set of simplex signals

$$\mathbf{s}'_m = \mathbf{s}_m - \bar{\mathbf{s}} \quad \text{where} \quad \bar{\mathbf{s}} = \frac{1}{M}\sum_{m=1}^{M} \mathbf{s}_m$$

The signals are not orthogonal, but they are equally correlated and have the same energy

$$\rho_{mi} = -\frac{1}{M-1} \qquad E'_s = E_s\left(1 - \frac{1}{M}\right)$$

Waveform Channels

There are M signal waveforms $\{s_m(t), m = 1,2,\ldots,M\}$. The symbol interval has duration T and the waveforms have unit energy $\int_0^T s_m^2(t)dt = 1$. The received signal is given by

$$r(t) = \sqrt{E_m}s_m(t) + n(t), \ 0 \le t < T,$$

where $n(t)$ is a realization of a stochastic process. What should the optimal receiver do?

To deal with the problem, we follow the following procedure:

1. Expand all involved random and deterministic waveforms using a *suitable* set of complete orthonormal functions.
2. Truncate the possibly infinite series expansions into finite sums with N terms.
3. Derive the optimal receiver for the resulting vector channel.
4. Take the limit as $N \to \infty$ to obtain the corresponding waveform-based receiver.

What are the issues here?