Lecture 4, E6712

Stochastic Processes

$\{X_t\}$ - ensemble of random variables indexed by the parameter t.

$x(t)$ - single realization or sample function.

Characterization of random processes?

RVs $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ are characterized by their joint PDF $p(x_{t_1}, x_{t_2}, \ldots, x_{t_n})$ for all n and any choice of t_i-s.

The mean:

$$m(t_i) = E(X_{t_i}) = \int_{-\infty}^{\infty} x_{t_i} p(x_{t_i}) dx_{t_i}$$

Autocorrelation function:

$$R_x(t,u) = E(X_t X_u) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_t X_u p(X_t, X_u) dX_t dX_u.$$

Covariance:

$$K_x(t,u) = E[(X_t - m_x(t))(X_u - m_x(u))] = R_x(t,u) - m_x(t)m_x(u)$$

White noise is a random process with the autocorrelation function $\frac{1}{2}\mathcal{N}_0\delta(t-u)$.

Series Representation of Sample Functions of Random Processes

We know how to represent a deterministic waveform in terms of a series expansion. How about sample functions of random processes? Let us start by choosing an arbitrary complete orthonormal (CON) set:

$$\varphi_j(t),\ j = 1, 2, \ldots,\ \int_0^T \varphi_i(t)\varphi_j(t)dt = \delta_{ij}$$

To expand $x(t)$, we write

$$x(t) = \text{l.i.m.}_{N\to\infty} \sum_{j=1}^{N} x_j \varphi_j(t) \text{ where } x_j \triangleq \int_0^T x(t)\varphi_j(t)dt,$$

where the convergence is the limit in the mean (l.i.m.). Note that an ordinary limit is not useful because this would require establishing conditions on the process to guarantee that *every sample function* can be represented in this manner. Because of the stochastic nature of $x(t)$, the convergence is chosen to be in the mean-square sense

$$\lim_{N\to\infty} E\left[\left(x(t) - \sum_{j=1}^{N} x_j \varphi_j(t)\right)^2\right] = 0.$$

Note that the x_j are random variables. We should choose the CON that leads to *uncorrelated* coefficients x_i, namely, we would like to have

$$E[(x_i - m_i)(x_j - m_j)] = \lambda_i \delta_{ij}, \text{ where } E(x_i) \triangleq m_i.$$

What does this requirement imply about the CON? Assume for simplicity that $m_i = 0$ for all i.

$$\begin{aligned}\lambda_i \delta_{ij} = E(x_i x_j) &= E\Big[\int_0^T x(t)\varphi_i(t)dt \int_0^T x(u)\varphi_j(u)du\Big] \\ &= \int_0^T \varphi_i(t) \cdot \underbrace{\int_0^T K_x(t,u)\varphi_j(u)du}_{f(t)}\, dt\end{aligned}$$

This equality holds if and only if $\varphi_i(t)$

$$\lambda_i \varphi_i(t) = \int_0^T K_x(t,u)\varphi_i(u)du \tag{1}$$

The expansion with such $\varphi_i(t)$ is known as the Karhunen-Loève expansion. Equation (1) as the eigenequation. The functions $\varphi_i(t)$ are the eigenfunctions and λ_i are the eigenvalues of $K_x(t,u)$. It can be easily verified that

$$K_x(t,u) = \sum_i \lambda_i \varphi_i(t)\varphi_i(u),$$

which is known as the Mercer's theorem.

We need to show that equation (1) has a solution for some useful class of random processes. Let $x(t)$ be an uncorrelated zero-mean process having covariance $\frac{1}{2}\mathcal{N}_0\delta(t-u)$. Then

$$\int_0^T K_x(t,u)\varphi_j(u)du = \int_0^T \frac{1}{2}\mathcal{N}_0\delta(t-u)\varphi_j(u)du = \frac{1}{2}\mathcal{N}_0\varphi_j(t).$$

Thus the integral equation (1) is satisfied by any ON set of functions, and the eigenvalues are all equal to $\mathcal{N}_0/2$.

For our waveform channel problem, we will further assume that the noise is zero-mean, Gaussian, white, and WSS:

Wide-sense stationary (WSS): the mean does not depend on time and $R_x(t,u) = R_x(t-u) = R_x(\tau)$.

White or uncorrelated $R_x(t,u) = \frac{1}{2}\mathcal{N}_0\delta(t-u)$

Gaussian: We will say that $x(t)$ is a Gaussian random process if

$$y = \int_0^T$$

is a Gaussian random variable for any function $g(u)$ s.t $E(y^2) < \infty$.

Optimum Receiver for the AWGN Channel

M signal waveforms $\{s_m(t), m = 1, 2, \ldots, M\}$; symbol interval of duration T $\int_0^T s_m^2(t)dt = 1$

Additive white Gaussian noise channel:

$$r(t) = \sqrt{E_m}s_m(t) + n(t),\ 0 \le t < T.$$

Design a receiver that is optimal is the sense that it minimizes the probability of making an error.

Reduce the problem to the one of the vector channel by expanding each waveform over a CON.

CON: $\int_0^T \varphi_i(t)\varphi_j(t)dt = \delta_{ij}$

$$r(t) = \lim_{N\to\infty} \sum_{j=1}^{N} r_j\varphi_j(t) \qquad r_j = \int_0^T r(t)\varphi_j(t)dt \qquad \mathbf{r} = (r_1, r_2, \ldots, r_N)$$

$$n(t) = \lim_{N\to\infty} \sum_{j=1}^{N} n_j\varphi_j(t) \qquad n_j = \int_0^T n(t)\varphi_j(t)dt \qquad \mathbf{n} = (n_1, n_2, \ldots, n_N)$$

$$s_m(t) = \lim_{N\to\infty} \sum_{j=1}^{N} s_{mj}\varphi_j(t) \qquad s_{mj} = \int_0^T s_m(t)\varphi_j(t)dt \qquad \mathbf{s_m} = (s_{m1}, s_{m2}, \ldots, s_{mN})$$

What can we say about the statistics of $(n_1, n_2, \ldots, n_N)$?

$$E(n_j) = \int_0^T E[n(t)]\varphi_i(t)dt$$

$$\begin{aligned} E(n_i n_j) &= \int_0^T \int_0^T E[n(t)n(\tau)]\varphi_i(t)\varphi_j(\tau)dtd\tau \\ &= \frac{\mathcal{N}_0}{2}\int_0^T \int_0^T \delta(t-\tau)\varphi_i(t)\varphi_j(\tau)dtd\tau \\ &= \frac{\mathcal{N}_0}{2}\int_0^T \varphi_i(t)\varphi_j(t)dt = \frac{\mathcal{N}_0}{2}\delta_{ij} \end{aligned}$$

Optimal receiver for the vector channel $\mathbf{r} = \sqrt{E_m}\mathbf{s_m} + \mathbf{n}$:

Recall that $p(\mathbf{r}|\mathbf{s_m}) \propto \exp[-\sum_{j=1}^{N}(r_j - \sqrt{E_m}s_{mj})^2/\mathcal{N}_0]$,

$$\hat{m} = \arg \max_{1\le m\le M} \left\{ \sqrt{E_m}\sum_{j=1}^{N} r_j s_{mj} - \frac{E_m}{2}\sum_{j=1}^{N} s_{mj}^2 + \frac{\mathcal{N}_0}{2}\ln P(s_m) \right\}$$

As $N \to \infty$,

$$\hat{m} = \arg \max_{1\le m\le M} \left\{ \sqrt{E_m}\int_0^T r(t)s_m(t)dt - \frac{E_m}{2} + \frac{\mathcal{N}_0}{2}\ln P(s_m) \right\}$$

Matched filter has the impulse response $h_m(t) = s_m(T-t)$, and output $u(\tau)$:

$$\begin{aligned} u(\tau) &= \int_0^T r(t)h_m(\tau - t)dt \\ &= \int_0^T r(t)s_m(T - \tau + t)dt \\ \Rightarrow u(T) &= \int_0^T r(t)s_m(t)dt \end{aligned}$$

The correlation and the matched filter receivers:

Basic Digital Modulation Schemes

Waveform (f_c is the carrier frequency, $g(t)$ is a low-pass waveform)

$$\begin{aligned}\sqrt{E_m}s_m(t) &= \Re\{A_m \cdot g(t) \cdot e^{j\theta_m} \cdot e^{j2\pi f_c t}\}, \quad 1 \le m \le M. \\ &= A_m g(t)\cos(2\pi f_c t + \theta_m) \\ &= A_m \cos\theta_m \cdot \underbrace{g(t)\cos(2\pi f_c t)}_{\varphi_1(t)} - A_m \sin\theta_m \cdot \underbrace{g(t)\sin(2\pi f_c t)}_{\varphi_2(t)}\end{aligned}$$

We shall assume that $g(t)$ is a rectangular pulse of duration T, and height $\sqrt{\frac{2}{T}}$, and T a multiple of $1/f_c$.

Therefor we have

$$\frac{2}{T}\int_0^T \cos^2(2\pi f_c t)dt = 1, \quad \frac{2}{T}\int_0^T \sin^2(2\pi f_c t)dt = 1, \quad \text{and} \quad \frac{2}{T}\int_0^T \sin(2\pi f_c t)\cos(2\pi f_c t)dt = 0$$

Thus $A_m = \sqrt{E_m}$.

PAM: $A_m = (2m - 1 - M)d_{min}$ carries the information, $\theta_m = 0$

PSK: $\theta_m = 2\pi(m-1)/M$ carries the information, $A_m = \sqrt{E}$; $d_{min} = 2\sqrt{E}\sin(\pi/M)$.

QAM: Both $A_m = \sqrt{A_{ms}^2 + A_{mc}^2}$ and $\theta_m = \arctan(A_{ms}/A_{mc})$ carry the information.

Gray mapping: