Lecture 5, E6712

Last time we studied the following M-ary modulation scheme:

$$\sqrt{E_m}s_m(t) = \Re\{A_m \cdot g(t) \cdot e^{j\theta_m} \cdot e^{j2\pi f_c t}\}, \quad 1 \le m \le M.$$

The receiver has to estimate the phase of the carrier! That can be done by sending training symbols where θ is known. Or we can use differential encoding.

Differentially Encoded PSK (DE-PSK)

Does not require the estimation of the carrier phase.

Differential encoding:

$$\begin{array}{rllll} \text{original phases in PSK}: & \theta(0) & \theta(1) & \theta(2) & \dots \\ \text{transmitted phases in DE-PSK}: & \bar{\theta}(0) & \bar{\theta}(1) & \bar{\theta}(2) & \dots \end{array}$$

where $\bar{\theta}(k) = \bar{\theta}(k-1) + \theta(k) \mod 2\pi$ and $\theta(-1) = 0$.

Decoding: $\theta(k) = \bar{\theta}(k) - \bar{\theta}(k-1) \mod 2\pi$.

Example: BPSK

original phases:		0	π	π	π	0	π	0	π	π
transmitted phases:	0	0	π	0	π	π	0	0	π	0
received phases:	π	π	0	π	0	0	π	π	0	π
decoded phases:		0	π	π	π	0	π	0	π	π

The receiver:

Performance

Coherently demodulate $\bar{\theta}$, then decode $\theta(k) = \bar{\theta}(k) - \bar{\theta}(k-1) \mod 2\pi$.

Probability of error is $P\{(\hat{\bar{\theta}}(k) \neq \bar{\theta}(k), \hat{\bar{\theta}}(k-1) = \bar{\theta}(k-1)) \bigcup (\hat{\bar{\theta}}(k) = \bar{\theta}(k), \hat{\bar{\theta}}(k-1) \neq \bar{\theta}(k-1))\}$

$P^e_{DE-BPSK} = 2P^e_{BPSK}(1 - P^e_{BPSK}) \approx 2P^e_{BPSK}$, where $P^e_{BPSK} = Q(\sqrt{2E_b/\mathcal{N}_0})$.

For DE-QPSK, we have independent two independent components: independent and quadrature. Thus,

$$\begin{aligned} P^e_{DE-QPSK} &= 1 - (1 - P^e_{DE-BPSK})^2 \\ &= P^e_{DE-BPSK}(2 - P^e_{DE-BPSK}) \approx 2P^e_{DE-BPSK} \\ &= 4P^e_{BPSK} \end{aligned}$$

DPSK

DE-PSK also allows a form of demodulation that does not require the estimation of the carrier phase.

We have

$$r(k) = \sqrt{E_s} e^{j(\theta(k)-\phi)} + n_k$$
$$r(k-1) = \sqrt{E_s} e^{j(\theta(k-1)-\phi)} + n_{k-1}$$

Consider $r(k) \cdot r^*(k-1)$:

$$\begin{aligned} r(k) \cdot r^*(k-1) =& (\sqrt{E_s} e^{j(\theta(k)-\phi)} + n_k)(\sqrt{E_s} e^{j(-\theta(k-1)+\phi)} + n_{k-1}) \\ =& E_s e^{j(\theta(k)-\theta(k-1))} && \leftarrow \text{signal} \\ &+ n_k \sqrt{E_s} e^{-j(\theta(k-1)-\phi)} + n_{k-1}\sqrt{E_s} e^{j(\theta(k)-\phi)} && \leftarrow \text{noise} \\ &+ n_k n_{k-1} && \leftarrow \text{negligible noise} \end{aligned}$$

The receiver:

DE-PSK decoded in this manner is called differential PSK or DPSK.

Performance? What is the signal to noise ratio?

$$\text{SNR}_{\text{DPSK}} = \frac{E_s^2}{2 \cdot \frac{\mathcal{N}_0}{2} \cdot E_s} = \frac{1}{2}\text{SNR}_{\text{PSK}}$$

However, for binary DPSK, we have

$$P_e = \frac{1}{2} e^{-E_b/\mathcal{N}_0}$$

The decision rule for binary DPSK is

$$\Re\{r_k r^*(k-1)\} \underset{\pi}{\overset{0}{\gtrless}} 0$$

or equivalently,

$$\|r_k + r_{k-1}\|^2 \underset{\pi}{\overset{0}{\gtrless}} \|r_k - r_{k-1}\|^2$$

Noncoherent Demodulation - AWGN

Last time we studied the following M-ary modulation scheme:

$$\sqrt{E_m}s_m(t) = \Re\{A_m \cdot g(t) \cdot e^{j\theta_m} \cdot e^{j2\pi f_c t}\}, \quad 1 \le m \le M.$$

Demodulation schemes requiring the carrier phase are called coherent. Those that do not need the phase are called incoherent. Incoherent modulation is simpler (less expensive) to implement, but has poorer performance.

Suppose that the receiver knows (or can estimate) f_c, but not the phase of the received signal. In general, we have

$$\begin{aligned} r(t) &= \Re\{\tilde{s}(t)e^{j(2\pi f_c t+\theta)}\} + n(t) \\ &= s(t;\theta) + n(t) \end{aligned}$$

where $s(t;\theta) \in \{s_1(t;\theta)\ldots, s_M(t;\theta)\}$, $\int_0^T |s_m(t;\theta)|^2 dt = E_m$, and θ is a RV with PDF $f(\theta)$.

Conditional likelihood:

$$\begin{aligned} \Lambda_m(\theta) &\triangleq p(r(t), t \in [0,T] | s_m(t), t \in [0,T], \theta) \\ &\propto \exp\Big(\frac{2}{\mathcal{N}_0}\int_0^T r(t)s_m(t;\theta)dt - \frac{E_m}{\mathcal{N}_0}\Big) \end{aligned}$$

Likelihood:

$$\begin{aligned} \Lambda_m = &\triangleq p(r(t), t \in [0,T] | s_m(t), t \in [0,T]) \\ &= \int_{-\pi}^{\pi} \Lambda_m(\theta) f(\theta) d\theta \\ &\propto \int_{-\pi}^{\pi} \exp\Big(\frac{2}{\mathcal{N}_0}\int_0^T r(t)s_m(t;\theta)dt - \frac{E_m}{\mathcal{N}_0}\Big) f(\theta)d\theta \\ &= \exp\Big(-\frac{E_m}{\mathcal{N}_0}\Big) \cdot \int_{-\pi}^{\pi} \exp\Big(\frac{2}{\mathcal{N}_0}\int_0^T r(t)s_m(t;\theta)dt\Big) f(\theta)d\theta \\ &= \exp\Big(-\frac{E_m}{\mathcal{N}_0}\Big) \cdot \int_{-\pi}^{\pi} \exp\Big(\frac{2}{\mathcal{N}_0}\Re\Big\{e^{j\theta}\underbrace{\int_0^T r(t)\tilde{s}_m(t)e^{j2\pi f_c t}dt}_{L_m = |L_m|e^{j\varphi_m}}\Big\}\Big) f(\theta)d\theta \\ &= \exp\Big(-\frac{E_m}{\mathcal{N}_0}\Big) \cdot \int_{-\pi}^{\pi} \exp\Big(\frac{2}{\mathcal{N}_0}|L_m|\cos(\varphi_m + \theta)\Big) f(\theta)d\theta \end{aligned}$$

Special case – uniform phase $f(\theta) = 1/(2\pi)$ on $[-\pi, \pi]$:

$$\Lambda_m = \exp\Big(-\frac{E_m}{\mathcal{N}_0}\Big) \cdot \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\Big(\frac{2}{\mathcal{N}_0}|L_m|\cos(\phi)\Big)d\phi}_{I_0\left(\frac{2}{\mathcal{N}_0}|L_m|\right)}$$

where

$$I_0 \triangleq \frac{1}{\pi}\int_0^{\pi} \exp(x\cos\theta)d\theta.$$

$$\boxed{\Lambda_m = \exp\Big(-\frac{E_m}{\mathcal{N}_0}\Big) \cdot I_0\Big(\frac{2}{\mathcal{N}_0}|L_m|\Big)}$$

Example: Equal energy symbols

$$\hat{m} = \max_{1 \le m \le M} |L_m|^2$$

Computation of L_m

$$L_m \triangleq \int_0^T r(t) \underbrace{\tilde{s}_m(t) e^{j2\pi f_c t}}_{s_m(t)+j\hat{s}_m(t)} dt$$

$\Rightarrow$

$$|L_m|^2 = \left(\int_0^T r(t) s_m(t) dt\right)^2 + \left(\int_0^T r(t) \hat{s}_m(t) dt\right)^2$$

Performance

Consider ON-OFF signaling $\{s_1(t) = 0,\ s_2(t)\}$.

We have

$$\lambda_1 \triangleq \log \Lambda_1 = \log I(0) = 0$$
$$\lambda_2 \triangleq \log \Lambda_2 = \log I(2|L_2|/\mathcal{N}_0) - E/\mathcal{N}_0$$

The decision rule:

$$\lambda_2 \underset{s_1}{\overset{s_2}{\gtrless}} \lambda_1$$

$$\Leftrightarrow \log I(2|L_2|/\mathcal{N}_0) \underset{s_1}{\overset{s_2}{\gtrless}} E/\mathcal{N}_0$$

$$\Leftrightarrow |L_2| \underset{s_1}{\overset{s_2}{\gtrless}} \sqrt{\eta}$$

Suppose that $s_1(t) = 0$ was transmitted. What is the distribution of $|L_2|$?

Recall that $L_2 = \int_0^T r(t)\tilde{s}_2(t)e^{j2\pi f_c t}dt = L_c + jL_s$, where L_c is the in-phase, and L_s is the quadrature component.

Note that L_c and L_s are statistically independent, Gaussian distributed with the same variance $\mathcal{N}_0/2$ and zero mean.

Therefore, $|L_2|$ is Rayleigh distributed:

$$P(|L_2| = l) = \boxed{\frac{2l}{\mathcal{N}_0} e^{-\frac{l^2}{\mathcal{N}_0}}} \leftarrow \textit{Rayleigh distribution}$$

Consequently,

$$\begin{aligned} P(e|s_1) &= P(|L_2| > \sqrt{\eta}|s_1) \\ &= \int_{\sqrt{\eta}}^{\infty} \frac{2l}{\mathcal{N}_0} e^{-\frac{l^2}{\mathcal{N}_0}} = \exp(-\eta/\mathcal{N}_0) \end{aligned}$$

Suppose that $s_2(t)$ was transmitted. What is the distribution of $|L_2|$?

Recall that $L_2 = \int_0^T r(t)\check{s}_2(t)e^{j2\pi f_c t}dt = L_c + jL_s$, where L_c is the in-phase, and L_s is the quadrature component.

Note that L_c and L_s are statistically independent, Gaussian distributed with the same variance $\mathcal{N}_0/2$ but different means.

Therefore $|L_2|$ is Rice distributed with the mean E_b^2 and variance $\mathcal{N}_0 E_b/2$. Consequently,

$$\begin{aligned} P(e|s_2) =& P(|L_2| \leq \sqrt{\eta}|s_2) \\ =& 1 - P(|L_2| > \sqrt{\eta}|s_2) \\ =& 1 - Q\left(\sqrt{\frac{2E_b}{\mathcal{N}_0}}, \sqrt{\frac{2\eta}{\mathcal{N}_0}}\right) \leftarrow \textit{Marcum Q-function} \end{aligned}$$

Examples with no phase error: BPSK, binary orthogonal signals.