Lecture 8, E6712

We consider the following discrete-time channel model:

$$\nu_k = \sum_n I_n f_{k-n} + \eta_k,$$

where $\{\eta_k\}$ is AWGN.

The Optimal Viterbi Detection

Maximum likelihood sequence detection:

$$\max_{\{I_k\}\in\mathcal{C}} p(\{\nu_k\}|\{I_k\})$$

The log-likelihood function:

$$\Omega(\{I_k\}) = \sum_n [\nu_k - \sum_n I_n f_{k-n}]^2$$

Sequences $\{I_k\}$ are represented as paths through a trellis

We will use the $(1-D)$ channel as an example to explain principles, implementation issues, and performance of the Viterbi detection.

The Principles

The channel:

$$\nu_k = I_k - I_{k-1} + \eta_n$$

The log-likelihood function:

$$\Omega(\{I_k\}) = \sum_k \left[\nu_k - (I_k - I_{k-1})\right]^2.$$

Since $\{I_k\} \in \mathcal{C} \subseteq \{-1, 1\}^\infty$, we can equivalently minimize of the cost function given by

$$-\sum_k [(I_k - I_{k-1})\nu_k + I_k I_{k-1}].$$

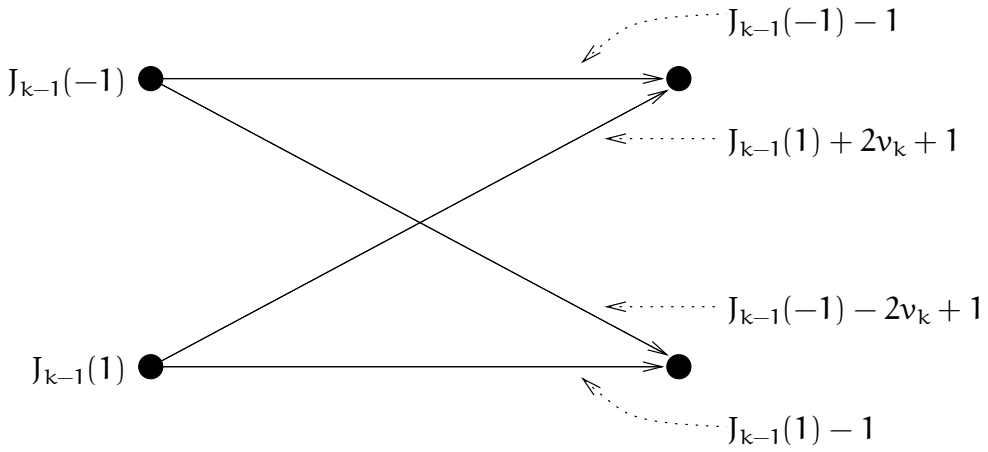
Sequences $\{I_k\}$ are represented as paths through a two state trellis where each state corresponds to the polarity of last symbol.

Draw the trellis.

At time k, as the sample ν_k becomes available, the Viterbi algorithm finds two sequences: one minimizing the cost function over all possible sequences that end in state '1' and the other minimizing the cost function over all possible sequences that end in state '-1' at time k. We call these sequences *survivor sequences* at time k.

$J_k(s)$ the minimum cost up to time k and state s, $s \in \{-1, 1\}$:

$$J_k(s) = \min_{\{I_k\} \in \mathcal{C}} \sum_{\substack{n=-\infty \\ I_k = s}}^{k} [-(I_k - I_{k-1})v_k - I_k I_{k-1}].$$



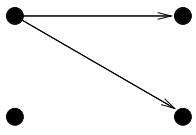
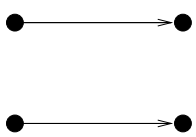
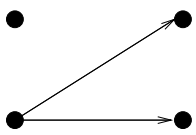
The algorithm is based on the principle of optimality which says that a survivor sequence at time k must correspond to an extension of one of the survivor sequences at time $k-1$.

Implementation Issues

Common implementations of the Viterbi detector for $(1-D)$ channel use the recursive *difference metric algorithm*.

The decision about extensions to both states at time k can be made based on single variable

$$\delta_k = \Delta J_{k-1} - v_k, \quad \text{where} \quad \Delta J_k = [J_k(-1) - J_k(1)]/2.$$

Condition	Update	Extension and Reliability
$\delta_k \leq -1$	$\Delta J_k = 2v_k - 1$	 $\delta_k + 1$ $\delta_k - 1$
$-1 < \delta_k < 1$	$\Delta J_k = \Delta J_{n-1}$	 $\delta_k + 1$ $-\delta_k + 1$
$\delta_k \geq 1$	$\Delta J_k = 2v_k + 1$	 $-\delta_k - 1$ $-\delta_k + 1$

MLSE Error Probability Performance for M-ary PAM

The discrete time AWGN channel:

$$v_k = \sum_{n=0}^{L} I_n f_{k-n} + \eta_k.$$

The MLSE finds the sequence minimizing the log-likelihood function

$$\Omega(\{I_k\}) = \sum_{n=0}^{L} [v_k - \sum_{n} I_n f_{k-n}]^2$$

Assume that we know that either sequence $\{I_n\}$ or sequence $\{\tilde{I}_n\}$ is transmitted with equal probability.

Probability of error is then given by

$$P\Big[\Omega(\{\tilde{I}_n\}) < \Omega(\{I_n\})\Big|\{I_n\}\Big]$$

For M-ary PAM, $\{I_n\}$ takes the values $\pm d, \pm 3d, \ldots, \pm(M-1)d$, and $2d$ is the distance between two successive levels.

The normalized error sequence corresponding to $\{I_n\}$ and $\{\tilde{I}_n\}$:

$$\varepsilon_n = \frac{1}{2d}(I_n - \tilde{I}_n)$$

In the binary (bipolar) case, we have $I_n \in \{-1, 1\}$ and $\varepsilon_n \in \{-1, 0, 1\}$:

$$\varepsilon_n = \frac{1}{2}(I_n - \tilde{I}_n)$$

We will be concerned with finite length error events (illustrate on a trellis) starting at some time k and ending at time $k + l - L - 1$ s.t.:

1. $S_k = \tilde{S}_k$, where $S_k = (I_{k-1}, I_{k-2}, \ldots, I_{k-L})$ and $S_k = (\tilde{I}_{k-1}, \tilde{I}_{k-2}, \ldots, \tilde{I}_{k-L})$.

$$P\{S_k = \tilde{S}_k\} \leq 1$$

2. When information symbols $I_k, I_{k+1}, \ldots, I_{k+l-L-1}$ are added to the scaled error sequence $2d\varepsilon = 2d\{\varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{k+l-L-1}\}$, the result is an allowable symbol sequence. The probability of this event is

$$\prod_{i=0}^{l-L-1} \frac{M - |\varepsilon_i|}{M}$$

3. The sum of the branch metrics for the estimated path exceeds the sum of the branch metrics of the correct path:

$$\sum_{i=k}^{k+l-1} \left(v_i - \sum_{j=0}^{L} f_j \tilde{I}_{i-j} \right)^2 < \sum_{i=k}^{k+l-1} \left(v_i - \sum_{j=0}^{L} f_j I_{i-j} \right)^2$$

The probability of this event is

$$Q\left(\sqrt{\frac{2d^2}{\mathcal{N}_0} \delta^2(\varepsilon)} \right)$$

where $\delta^2(\varepsilon)$ is the distance between $\{I_n\}$ and $\{\tilde{I}_n\}$ on the channel (at the input of the MLSE):

$$\delta^2(\varepsilon) = \sum_{i=k}^{k+l-1} \left(\sum_{j=0}^{L} f_j \varepsilon_{i-j} \right)^2$$

Therefore, the probability of error event $\boldsymbol{\varepsilon}$ is given by

$$P(\boldsymbol{\varepsilon}) = \prod_{i=0}^{l-L-1} \frac{M - |\varepsilon_i|}{M} Q\left(\sqrt{\frac{2d^2}{\mathcal{N}_0}\delta^2(\boldsymbol{\varepsilon})}\right).$$

Error Events - Binary Example

Finite length normalized error sequence:

$$\boldsymbol{\varepsilon} = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1, 0, 1\},\ \varepsilon_0 = 1, \varepsilon_{l-1} \neq 0$$

The corresponding polynomial:

$$\varepsilon(D) = \sum_{i=0}^{l-1} \varepsilon_i D^i,$$

The minimum (merged) distance on the channel with transfer function $F(D)$ is defined by

$$d^2_{<>} = \min_{\varepsilon(D)\neq 0} \|F(D)\varepsilon(D)\|^2,$$

where the squared norm of a polynomial is defined as the sum of its squared coefficients.

The minimum distance is bounded from above by $\|F(D)\|^2$, the energy in the channel impulse response.

The *matched-filter bound* (MFB):

$$d^2_{<>} \leq d^2_{MFB} \stackrel{\text{def}}{=} \|h(D)\|^2.$$

A channel *achieves the MFB* if the error sequence of length $l = 1$, *i.e.*, $\varepsilon(D) = 1$, is in the set

$$\arg \min_{\varepsilon(D)\neq 0} \|h(D)\varepsilon(D)\|^2.$$

$F(D) = (1 - D)$ achieves the MFB.

$F(D) = (1 - D)(1 + D)^3$ does not achieve the MFB.

Example: Consider the following partial response channel:

$$v_k = I_k - I_{k-1} + \eta_k,$$

where $\{\eta_k\}$ is discrete AWGN with zero mean and spectral density $\mathcal{N}_0/2$. The user data $D_k \in \{0, 1\}$ is binary, and the channel can take only bipolar signal values $I_k \in \{-1, 1\}$.

1. Suppose that user data is directly mapped into the channel sequence as

   $$I_k = 2D_k - 1$$

   Suppose that the user sequence $\{D_k\}$ is the all-zero sequence, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is the most likely incorrect sequence $\{\widehat{D_k}\}$?

2. Suppose that user data is mapped into the channel sequence in two steps as follows:

   (a) $P_k = D_k + P_{k-1} \mod 2$
   (b) $I_k = 2P_k - 1$

   Suppose again that the user sequence $\{D_k\}$ is the all-zero sequence, and the MLSE decodes a sequence $\{\widehat{I_k}\}$ with an error event starting at time n_0 and ending at time n_{l-1}. What is now the most likely incorrect sequence $\{\widehat{D_k}\}$?

Solution

1. The minimum (merged) distance on the bipolar channel with transfer function $(1 - D)$ is given by

   $$d^2_{<>} = \min_{\varepsilon(D)\neq 0} \|(1 - D)\varepsilon(D)\|^2,$$

   where the squared norm of a polynomial is defined as the sum of its squared coefficients, and $\varepsilon(D)$ is the polynomial corresponding to a finite length normalized error sequence:

   $$\varepsilon = \{\varepsilon_{n_0}, \varepsilon_1, \ldots, \varepsilon_{n_l-1}\} \text{ of length } l < \infty,\ \varepsilon_i \in \{-1, 0, 1\},\ \varepsilon_{n_0} = 1, \varepsilon_{n_l-1} \neq 0$$

   We have

   $$\begin{aligned} d^2_{<>} &= \min_{\varepsilon(D)\neq 0} \|(1 - D)\sum_{i=0}^{l-1} \varepsilon_{n_i} D^i\|^2 \\ &= \min_{\varepsilon(D)\neq 0} \|\varepsilon_{n_0} + \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}})D^i + \varepsilon_{n_{l-1}} D^l\|^2 \\ &= \min_{\varepsilon(D)\neq 0} (1 + \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}})^2 + 1) \\ &= 2 + \min_{\varepsilon(D)\neq 0} \sum_{i=1}^{l-1} (\varepsilon_{n_i} - \varepsilon_{n_{i-1}})^2 \geq 2 \end{aligned}$$

   where the equality holds iff

   $$\varepsilon(D) = \sum_{i=0}^{l-1} D^i.$$

   If the user sequence $\{D_k\}$ is the all-zero sequence, then the most likely incorrect sequence $\{\widehat{D_k}\}$ will have 1s starting at time n_0 and ending at time n_{l-1}.

2. If the user sequence $\{D_k\}$ is the all-zero sequence, sequence $\{P_k\}$ is also an all-zero sequence. From (a), we know that then the most likely incorrect sequence $\{\widehat{P_k}\}$ will have 1s starting at time n_0 and ending at time n_{l-1}. Since $\widehat{D_k} = \widehat{P_k} + \widehat{D_{k-1}} \mod 2$, the decoded sequence $\{\widehat{D_k}\}$ will have a 1 at time n_0 and a 1 at time n_l. The rest of $\{\widehat{D_k}\}$ is decoded correctly as 0.